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Intuitionistic Fuzzy Hypersoft Separation Axioms

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Abstract. In the present paper, we introduce the notion of T_i $(i = 0, 1, 2, 3, 4)$ separation axioms in intuitionistic fuzzy hypersoft topological spaces and discuss some of its properties. By using this notions, we also give some basic theorems of separation axioms in intuitionistic fuzzy hypersoft topological spaces. Finally, we present hereditary property of intuitionistic fuzzy hypersoft topological spaces.

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1. Introduction

A broad variety of various paradigms have been built over the last decades and have been committed to the analysis of uncertainty. Among them, three non-classical set theories for working with uncertainties are fuzzy sets [24], rough sets [13], and soft sets [8]. These soft computing methods excel in capturing core ambiguity characteristics based on independent viewpoints, such as graduality, granularity and parametrization. Zadeh's theory of fuzzy sets [24] is one of those theories which is known to be a mathematical way of tackling a range of challenging issues containing a variety of complexities in various fields of mathematical science. But these hypotheses are unable to successfully solve these problems due to the inadequacy of the parametrization method. This deficiency is resolved by the soft set theory of Molodtsov [8], which is free of all such impediments and has arisen as a new parameterized family of subsets of the universe of discourse.

Soft set theory draws many authors because it has a wide variety of applications in many fields such as functional smoothness, decision making, probability theory, data processing, estimation theory, forecasting and operational science. Nowadays, several scholars are trying to hybridize various models of soft sets and have obtained success in a variety of relevant theories. Maji describes a fuzzy soft set and an intuitionistic fuzzy soft set [9,10]. Further extensions of soft sets are then added, such as the generalized fuzzy soft set [12], the interval-valued fuzzy soft set [18], the soft rough set [2], the vague soft set [17], the trapezoidal fuzzy soft set [16], the neutrosophic soft set [11], the intuitionistic neutrosophic soft set [5], the multi-fuzzy soft set [19] and the hesitant fuzzy soft set [15].

Smarandache [14] expanded the definition of soft sets to hypersoft sets by replacing the F function of a single parameter with a multi-parameter (sub-attributes) function specified by the cartesian product of n different attributes. The developed hypersoft set is more versatile than soft sets and is more suited for decision-making environments. He also introduced further extensions of hypersoft set, such as crisp hypersoft set, fuzzy hypersoft set, intuitionistic fuzzy

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hypersoft set, neutrosophic hypersoft set, and plithogenic hypersoft set. The hypersoft set theory and its extensions are rapidly advancing nowadays, and several researchers have developed numerous operators and properties based on hypersoft set and its extensions [21].

Topological structures of all these structures have been studied by different researchers. Many researchers have studied the structures of fuzzy topology [6], intuitionistic fuzzy topology [7], fuzzy soft topology [9], intuitionistic fuzzy soft topology [10], neutrosophic soft topology [4], fuzzy hypersoft topology [23], intuitionistic fuzzy hypersoft topology [22] and various forms of these structures continue to be studied.

The present paper is structured as follows: Section 2 is the preliminary section where some notions and properties of intuitionistic fuzzy soft sets, hypersoft set, intuitionistic fuzzy hypersoft set (IFH) and IFH topology are presented. In Section 3, we introduce intuitionistic fuzzy hypersoft neighborhood and we give some properties of IFH points. In Section 4, the concept of intuitionistic fuzzy hypersoft separation axioms is given. Some properties of IFH T_i-spaces $(i = 0, 1, 2, 3, 4)$ and some relations between them are examined.

2. Preliminaries

Definition 2.1 ([3]). An intuitionistic fuzzy set $G = \{(u, \theta_G(u), \sigma_G(u)) : u \in U\}$, where $\theta_G : U \to [0, 1]$, $\sigma_G : U \to [0, 1]$ with the condition $0 \leq \theta_G(u) + \sigma_G(u) \leq 1$, $\forall u \in U$. $\theta_G, \sigma_G \in [0, 1]$ denote the degree of membership and non-membership of u to G, respectively. The set of all intuitionistic fuzzy sets over U will be denoted by $IFP(U)$.

Definition 2.2 ([8]). Let U be an initial universe and E be a set of parameters. A pair (G, E) is called a soft set over U, where G is a mapping $G : E \to \mathcal{P}(U)$. In other words, the soft set is a parameterized family of subsets of the set U.

Definition 2.3 ([10]). Let U be an initial universe and E be a set of parameters. A pair (G, E) is called an intuitionistic fuzzy soft set over U, where G is a mapping given by, $G : E \to IFP(U)$.

In general, for every $e \in E$, $G(e)$ is an intuitionistic fuzzy set of U and it is called intuitionistic fuzzy value set of parameter e. Clearly, $G(e)$ can be written as a intuitionistic fuzzy set such that $G(e) = \{(u, \theta_G(u), \sigma_G(u)) : u \in U\}$.

Definition 2.4 ([14]). Let U be the universal set and $P(U)$ be the power set of U. Consider $e_1, e_2, e_3, ..., e_n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attribute values are resspectively the sets $E_1, E_2, ..., E_n$ with $E_i \cap E_j = \emptyset$, for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$, then the pair $(G, E_1 \times E_2 \times ... \times E_n)$ is said to be Hypersoft set over U, where

$$H : E_1 \times E_2 \times ... \times E_n \to P(U).$$

Definition 2.5 ([20]). Let U be the universal set and $IFP(U)$ be the intuitionistic fuzzy power set of U. Consider $e_1, e_2, e_3, ..., e_n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attribute values are respectively the sets $E_1, E_2, ..., E_n$ with $E_i \cap E_j = \emptyset$, for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$. Let A_i be the nonempty subset of E_i for each $i = 1, 2, ..., n$. An intuitionistic fuzzy hypersoft set defined as the pair $(G, A_1 \times A_2 \times ... \times A_n)$, where $G : A_1 \times A_2 \times ... \times A_n \to IFP(U)$ and

$$G(A_1 \times A_2 \times ... \times A_n) = \{< \alpha, (\frac{u}{\theta_{G(\alpha)}(u), \sigma_{G(\alpha)}(u)}) >: u \in U, \alpha \in A_1 \times A_2 \times ... \times A_n \subseteq E_1 \times E_2 \times ... \times E_n\},$$

where θ and σ are the membership and non-membership value, respectively such that $0 \leq \theta_{H(\alpha)}(u) + \sigma_{H(\alpha)}(u) \leq 1$ and $\theta_{H(\alpha)}(u), \sigma_{H(\alpha)}(u) \in [0, 1]$. For sake of simplicity, we write the symbols Δ for $E_1 \times E_2 \times ... \times E_n$, Ω for $A_1 \times A_2 \times ... \times A_n$ and α for an element of the set Γ.

Definition 2.6 ([20]). i) An intutionistic fuzzy hypersoft set (G, Δ) over the universe U is said to be null intuitionistic fuzzy hypersoft set and denoted by $0_{(U_{IFH}, \Delta)}$ if for all $u \in U$ and $\alpha \in \Delta$, $\theta_{H(\alpha)}(u) = 0$ and $\sigma_{H(\alpha)}(u) = 1$.

ii) An intutionistic fuzzy hypersoft set (G, Δ) over the universe U is said to be absolute intuitionistic fuzzy hypersoft set and denoted by $1_{(U_{IFH}, \Delta)}$ if for all $u \in U$ and $\alpha \in \Delta$, $\theta_{H(\alpha)}(u) = 1$ and $\sigma_{H(\alpha)}(u) = 0$.

Definition 2.7 ([20]). Let U be an initial universe set and $(H, \Omega_1), (G, \Omega_2)$ be two intuitionistic fuzzy hypersoft sets over the universe U. We say that (G_1, Ω_1) is an intuitionistic fuzzy hypersoft subset of (G_2, Ω_2) and denote $(G_1, \Omega_1) \widetilde{\subseteq} (G_2, \Omega_2)$, if

i) $\Omega_1 \subseteq \Omega_2$,

ii) For any $\alpha \in \Omega_1$, $G_1(\alpha) \subseteq G_2(\alpha)$.

That is, for all $u \in U$ and $\alpha \in \Omega_1$, $\theta_{G_1(\alpha)}(u) \leq \theta_{G_2(\alpha)}(u)$ and $\sigma_{G_1(\alpha)}(u) \geq \sigma_{G_2(\alpha)}(u)$.

Definition 2.8 ([20]). The complement of intutionistic fuzzy hypersoft set (G,Ω) over the universe U is denoted by $(G,\Omega)^c$ and defined as $(G,\Omega)^c = (G^c,\Omega)$, where $G^c : (E_1 \times E_2 \times ... \times E_n) = \Delta \to IFP(U)$ and $G^c(\Omega) = (G(\Omega))^c$ for all $\Omega \subseteq \Delta$. Thus, if $(G,\Omega) = \{< \alpha, (\frac{u}{\theta_{G(\alpha)}(u),\sigma_{G(\alpha)}(u)}) >: u \in U, \alpha \in \Omega)\}$, then $(G,\Omega)^c = \{< \alpha, (\frac{u}{\sigma_{G(\alpha)}(u),\theta_{G(\alpha)}(u)}) >: u \in U, \alpha \in \Omega)\}$.

Definition 2.9 ([20]). Let U be an initial universe set, $\Omega_1, \Omega_2 \subseteq \Delta$ and $(G_1,\Omega_1), (G_2,\Omega_2)$ be two intuitionistic fuzzy hypersoft sets over the universe U. The union of (G_1,Ω_1) and (G_2,Ω_2) is denoted by $(G_1,\Omega_1)\tilde{\cup}(G_2,\Omega_2) = (K,\Omega_3)$, where $\Omega_3 = \Omega_1 \cup \Omega_2$ and

$$\theta_{K(\alpha)}(u) = \begin{cases} G_1(\alpha) & \text{if } \alpha \in \Omega_1 - \Omega_2 \\ G_2(\alpha) & \text{if } \alpha \in \Omega_2 - \Omega_1 \\ \max(G_1(\alpha),(G_2(\alpha)) & \text{if } \alpha \in \Omega_1 \cap \Omega_2, \end{cases}$$

$$\sigma_{K(\alpha)}(u) = \begin{cases} G_1(\alpha) & \text{if } \alpha \in \Omega_1 - \Omega_2 \\ G_2(\alpha) & \text{if } \alpha \in \Omega_2 - \Omega_1 \\ \min(G_1(\alpha),(G_2(\alpha)) & \text{if } \alpha \in \Omega_1 \cap \Omega_2. \end{cases}$$

Definition 2.10. Let U be an initial universe set, $\Omega_1, \Omega_2 \subseteq \Delta$ and $(H,\Omega_1), (G,\Omega_2)$ be two intuitionistic fuzzy hypersoft sets over the universe U. The intersection of (H,Ω_1) and (G,Ω_2) is denoted by $(H,\Omega_1)\tilde{\cap}(G,\Omega_2) = (K,\Omega_3)$, where $\Omega_3 = \Omega_1 \cap \Omega_2$ and

$$(K,\Omega_3) = \left\{ < \xi, \left(\frac{u}{(\min\{\theta_{H(\xi)}(u), \theta_{G(\xi)}(u)\}, \max\{\theta_{H(\xi)}(u), \theta_{G(\xi)}(u)\})} \right) >: u \in U, \xi \in \Omega \right\}.$$

Definition 2.11 ([22]). Let $IFHS(U,\Delta)$ be the set of all intuitionistic fuzzy hypersoft subsets over the universe U and $\widetilde{\tau} \subseteq IFHS(U,\Delta)$. Then, $\widetilde{\tau}$ is called a intuitionistic fuzzy hypersoft topology on U if the following condition hold.

(1) $0_{(U_{IFH},\Delta)}, 1_{(U_{IFH},\Delta)}$ belong to $\widetilde{\tau}$,
(2) $(G_1,\Omega_1), (G_2,\Omega_2) \in \widetilde{\tau}$ implies $(G_1,\Omega_1)\widetilde{\cap}(G_2,\Omega_2)$,
(3) $\{(G_i,\Omega_i) : i \in I\} \subseteq \widetilde{\tau}$ implies $\widetilde{\cup}_{i\in I}(G_i,\Omega_i) \in \widetilde{\tau}$.

Then $(U,\widetilde{\tau},\Delta)$ is called an intuitionistic fuzzy hypersoft topological space over U. The members of $\widetilde{\tau}$ are said to be intuitionistic fuzzy hypersoft open sets in U.

An intuitionistic fuzzy hypersoft set (G,Ω) over U is said to be an intuitionistic fuzzy hypersoft closed set if its complement $(G,\Omega)^c$ belongs to $\widetilde{\tau}$.

Definition 2.12 ([22]). Let $IFHS(U,\Delta)$ be the set of all intuitionistic fuzzy hypersoft subsets over the universe U. Then,

(1) If $\widetilde{\tau} = \{0_{(U_{IFH},\Delta)}, 1_{(U_{IFH},\Delta)}\}$, then $\widetilde{\tau}$ is called to be intuitionistic fuzzy hypersoft indiscrete topology and $(U,\widetilde{\tau},\Delta)$ is called to be intuitionistic fuzzy hypersoft indiscrete topological space over the universe U.
(2) If $\widetilde{\tau} = IFHS(U,\Delta)$, then $\widetilde{\tau}$ is called to be intuitionistic fuzzy hypersoft discrete topology and $(U,\widetilde{\tau},\Delta)$ is called to be intuitionistic fuzzy hypersoft discrete topological space over the universe U.

Definition 2.13 ([22]). Let $(U,\widetilde{\tau},\Delta)$ be an intuitionistic fuzzy hypersoft topological spaces over U and (G,Ω) be a intuitionistic fuzzy hypersoft set. The intuitionistic fuzzy hypersoft interior of (G,Ω), denoted by int_{IFH} (G,Ω), is defined by the intuitinistic fuzzy hypersoft union of all intuitionistic fuzzy hypersoft open subsets of (G,Ω).

Clearly, $int_{IFH}(G,\Omega)$ is the largest intuitionistic fuzzy hypersoft open set that is contained in (H,Ω).

Definition 2.14 ([22]). Let $(U,\widetilde{\tau},\Delta)$ be an intuitionistic fuzzy hypersoft topological spaces over U and (G,Ω) be a intuitionistic fuzzy hypersoft set. The intuitionistic fuzzy hypersoft closure of (G,Ω), denoted by cl_{IFH} (G,Ω), is defined by the intuitinistic fuzzy hypersoft intersection of all intuitionistic fuzzy hypersoft closed supersets of (G,Ω).

Clearly, $cl_{IFH}(G,\Omega)$ is the smallest intuitionistic fuzzy hypersoft closed set which contain (G,Ω).

Definition 2.15 ([22]). Let $(U,\widetilde{\tau},\Delta)$ be a intuitionistic fuzzy hypersoft topological space over U and $\tilde{B} \subseteq \tilde{\tau}$. $\tilde{B}$ is called a intuitionistic fuzzy hypersoft basis for the intuitionistic fuzzy hypersoft topology $\tilde{\tau}$ if every element of $\tilde{\tau}$ can be written as the intuitionistic fuzzy hypersoft union of elements of $\tilde{B}$.

Definition 2.16 ([22]). Let $(U,\widetilde{\tau},\Delta)$ be a intuitionistic fuzzy hypersoft topological space over U and (G,Ω) be a intuitionistic fuzzy hypersoft set over U. Then the intuitionistic fuzzy hypersoft topology $\widetilde{\tau}_{(H,\Omega)} = \{(G,\Omega)\widetilde{\cap}(G_i,\Gamma_i) :$

$(G_i, \Gamma_i) \in \widetilde{\tau}$ for $i \in I\}$ is called intuitionistic fuzzy hypersoft subspace topology and $((G,\Omega), \widetilde{\tau}_{(G,\Omega)}, \Omega)$ is called a intuitionistic fuzzy hypersoft subspace of $(U, \widetilde{\tau}, \Delta)$.

Definition 2.17 ([1]). Let $\Omega \subseteq \Delta$, $\alpha \in \Omega$ and $u \in U$. A IFH set (G,Ω) is said to be an IFH point if $G(\alpha')$ is a null IFH set for every $\alpha' \in \Omega \backslash \{\alpha\}$ and $G(\alpha)(v) = (0,1)$ for all $u \neq v$. We will denote (G,Ω) simply by $P_{IFH}^{(\alpha,u)}$ and denote all the IFH points over U simply by $IFHP(U,\Delta)$.

Definition 2.18 ([1]). An IFH point $P_{IFH}^{(\alpha,u)}$ is said to belong to an IFH set (G,Ω) if $P_{IFH}^{(\alpha,u)} \widetilde{\subseteq} (G,\Omega)$. Then, we write it as $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G,\Omega)$. It is clear that, IFH union of IFH points of a (G,Ω) returns the (G,Ω), that is,

$$(G,\Omega) = \widetilde{\cup} \Big\{ P_{IFH}^{(\alpha,u)} : P_{IFH}^{(\alpha,u)} \in (G,\Omega) \Big\}.$$

3. Some Properties of IFH Points

Definition 3.1. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space over U. Then, an IFH set (G_1,Ω_1) in $IFHS(U,\Delta)$ is called an IFH neighborhood of the IFH point $P_{IFH}^{(\alpha,u)} \widetilde{\subseteq} (G_1,\Omega_1)$, if there exists an IFH open set (G_2,Ω_2) such that $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_2,\Omega_2) \widetilde{\subseteq} (G_1,\Omega_1)$. The IFH neighborhood system of an IFH point $P_{IFH}^{(\alpha,u)}$, denoted by $\aleph\left(P_{IFH}^{(\alpha,u)}\right)$ is the family of all its IFH neighborhoods.

Theorem 3.2. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and (G,Ω) be an IFH set over U. Then, (G,Ω) is an IFH open set iff (G,Ω) is an IFH neighborhood of its each IFH points.*

Proof. Let (G_1,Ω_1) be IFH open set and $P_{IFH}^{(\alpha,u)} \widetilde{\subseteq} (G_1,\Omega_1)$. Then $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_1,\Omega_1) \widetilde{\subseteq} (G_1,\Omega_1)$. Therefore, (G_1,Ω_1) is an IFH neighborhood of $P_{IFH}^{(\alpha,u)}$.

Conversely, suppose that (G_1,Ω_1) be an IFH neighborhood of its each IFH points and $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_1,\Omega_1)$. Then, there exist $(G_2,\Omega_2) \widetilde{\in} \widetilde{\tau}$ such that $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_2,\Omega_2) \widetilde{\subseteq} (G_1,\Omega_1)$. Since $(G_1,\Omega_1) = \widetilde{\cup} \left\{ P_{IFH}^{(\alpha,u)} : P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_1,\Omega_1) \right\}$ it follows that (G_1,Ω_1) is a union of IFH points and hence, (G_1,Ω_1) is an IFH open set. □

Theorem 3.3. *The neighborhood system $\aleph\left(P_{IFH}^{(\alpha,u)}\right)$ at $P_{IFH}^{(\alpha,u)}$ in an IFH topological space $(U, \widetilde{\tau}, \Delta)$ has the following properties:*

(1) *If $(G_1,\Omega_1) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$, then $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_1,\Omega_1)$,*
(2) *If $(G_1,\Omega_1) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$, and $(G_1,\Omega_1) \widetilde{\subseteq} (G_2,\Omega_2)$, then $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$,*
(3) *If $(G_1,\Omega_1), (G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$, then $(G_1,\Omega_1) \widetilde{\cap} (G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$,*
(4) *If $(G_1,\Omega_1) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$, then there exist a $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$ such that $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\beta,v)}\right)$ for each $\left(P_{IFH}^{(\beta,v)}\right) \widetilde{\in}$ (G_2,Ω_2).*

Proof. We will prove only (4). By the definiton of IFH neighborhood, (1), (2) and (3) are clear.

(4) If $(G_1,\Omega_1) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$ then there exist an IFH open set (G_2,Ω_2) such that $P_{IFH}^{(\alpha,u)} \widetilde{\in} (G_2,\Omega_2) \widetilde{\subseteq} (G_1,\Omega_1)$. Therefore, $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$, so for each $P_{IFH}^{(\beta,v)} \widetilde{\in} (G_2,\Omega_2)$, $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\beta,v)}\right)$ is obtained. □

Definition 3.4. Let $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$ be two IFH points over U. Then, we say that the IFH points are disjoint IFH points if $P_{IFH}^{(\alpha,u)} \widetilde{\cap} P_{IFH}^{(\beta,v)} = 0_{(U_{IFH},\Delta)}$.

It is clear that, $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$ are disjoint IFH points if and only if $u \neq v$ or $\alpha \neq \beta$.

Definition 3.5. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space over U. Let (G_1,Ω_1) be an IFH set and $P_{IFH}^{(\alpha,u)}$ be an IFH over U. Then,

(1) $P_{IFH}^{(\alpha,u)}$ is an IFH interior point of (G_1,Ω_1), if $(G_2,\Omega_2) \widetilde{\subseteq} (G_1,\Omega_1)$ for some $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$,
(2) $P_{IFH}^{(\alpha,u)}$ is an IFH adherent point of (G_1,Ω_1), if $(G_1,\Omega_1) \widetilde{\cap} (G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$ for any $(G_2,\Omega_2) \widetilde{\in} \aleph\left(P_{IFH}^{(\alpha,u)}\right)$.

Theorem 3.6. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and (G,Ω) be an IFH set over U. Then,*

(1) $int_{IFH}(G,\Omega) = \widetilde{\cup} \left\{ P_{IFH}^{(\alpha,u)} : P_{IFH}^{(\alpha,u)} \text{ is an } IFH \text{ interior point of } (G,\Omega) \right\}$,
(2) $cl_{IFH}(G,\Omega) = \widetilde{\cap} \left\{ P_{IFH}^{(\alpha,u)} : P_{IFH}^{(\alpha,u)} \text{ is an } IFH \text{ adherent point of } (G,\Omega) \right\}$.

Proof. Straightforward. □

Theorem 3.7. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and (G, Ω) be an IFH set over U. Let (G_1, Ω_1) be an IFH set over U and $\widetilde{B}$ be a basis for $(U, \widetilde{\tau}, \Delta)$. Then,*

$$(G_1, \Omega_1)\widetilde{\in}\widetilde{\tau} \Leftrightarrow \forall P_{IFH}^{(\alpha,u)}\widetilde{\in} IFH(U, \Delta), \exists (G_2, \Omega_2)\widetilde{\in}\widetilde{B} \text{ such that } P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2, \Omega_2)\widetilde{\subseteq}(G_1, \Omega_1).$$

Proof. ($\Rightarrow$) Suppose that $(G_1, \Omega_1)\widetilde{\in}\widetilde{\tau}$ and $P_{IFH}^{(\alpha,u)}\widetilde{\in} IFH(U, \Delta)$. Since $\widetilde{B}$ is a basis for $(U, \widetilde{\tau}, \Delta)$, there exist $\widetilde{B}'\widetilde{\subseteq}\widetilde{B}$ such that $(G_1, \Omega_1) = \widetilde{\cup}\left\{(G_2, \Omega_2) : (G_2, \Omega_2)\widetilde{\in}\widetilde{B}'\right\}$ such that $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2, \Omega_2)$ for $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1, \Omega_1)$. Hence, $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2, \Omega_2)\widetilde{\subseteq}(G_1, \Omega_1)$.

($\Leftarrow$) Assume that sufficient conditions of the theorem are provided. So,

$$(G_1, \Omega_1) = \left\{P_{IFH}^{(\alpha,u)} : P_{IFH}^{(\alpha,u)}\widetilde{\subseteq}(G_1, \Omega_1)\right\}\widetilde{\subseteq}\left\{(G_2, \Omega_2) : P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2, \Omega_2)\widetilde{\subseteq}(G_1, \Omega_1)\right\}\widetilde{\subseteq}\widetilde{\cup}(G_2, \Omega_2).$$

Thus, $(G_1, \Omega_1)\widetilde{\in}\widetilde{\tau}$. □

4. IFH Separation Axioms

Definition 4.1. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and for every $P_{IFH}^{(\alpha,u)}$, $P_{IFH}^{(\beta,v)}$ be IFH points over U such that $P_{IFH}^{(\alpha,u)} \neq P_{IFH}^{(\beta,v)}$. If there exist at least one IFH open set (G_1, Ω_1) or (G_2, Ω_2) such that

$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1, \Omega_1)$ and $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(G_2, \Omega_2) = 0_{(U_{IFH}, \Delta)}$ or $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2, \Omega_2)$ and $P_{IFH}^{(\beta,v)}\widetilde{\cap}(G_1, \Omega_1) = 0_{(U_{IFH}, \Delta)}$,

then $(U, \widetilde{\tau}, \Delta)$ is called an IFH T_0-space.

Definition 4.2. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and for every $P_{IFH}^{(\alpha,u)}$, $P_{IFH}^{(\beta,v)}$ be IFH points over U such that $P_{IFH}^{(\alpha,u)} \neq P_{IFH}^{(\beta,v)}$. If there exist IFH open set (G_1, Ω_1) and (G_2, Ω_2) such that

$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1, \Omega_1)$ and $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(G_2, \Omega_2) = 0_{(U_{IFH}, \Delta)}$ and $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2, \Omega_2)$ and $P_{IFH}^{(\beta,v)}\widetilde{\cap}(G_1, \Omega_1) = 0_{(U_{IFH}, \Delta)}$,

then $(U, \widetilde{\tau}, \Delta)$ is called an IFH T_1-space.

Definition 4.3. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and for every $P_{IFH}^{(\alpha,u)}$, $P_{IFH}^{(\beta,v)}$ be IFH points over U such that $P_{IFH}^{(\alpha,u)} \neq P_{IFH}^{(\beta,v)}$. If there exist IFH open set (G_1, Ω_1) and (G_2, Ω_2) such that

$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1, \Omega_1)$, $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2, \Omega_2)$ and $(G_1, \Omega_1)\widetilde{\cap}(G_2, \Omega_2) = 0_{(U_{IFH}, \Delta)}$,

then $(U, \widetilde{\tau}, \Delta)$ is called an IFH T_2-space.

Example 4.4. Let $\{u_1, u_2\}$ be a universe set. Suppose that

$P_{IFH}^{(\gamma_1,u_1)} = \left\{< (\alpha_1, \alpha_3), \left\{\frac{x_1}{(0.3,0.5)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_1,u_2)} = \left\{< (\alpha_1, \alpha_3), \left\{\frac{x_2}{(0.1,0.6)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_2,u_1)} = \left\{< (\alpha_1, \alpha_4), \left\{\frac{x_1}{(0.2,0.4)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_2,u_2)} = \left\{< (\alpha_1, \alpha_4), \left\{\frac{x_2}{(0.5,0.3)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_3,u_1)} = \left\{< (\alpha_2, \alpha_3), \left\{\frac{x_1}{(0.7,0.1)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_3,u_2)} = \left\{< (\alpha_2, \alpha_3), \left\{\frac{x_2}{(0.5,0.9)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_4,u_1)} = \left\{< (\alpha_2, \alpha_4), \left\{\frac{x_1}{(0.2,0.8)}\right\} >\right\}$,
$P_{IFH}^{(\gamma_4,u_2)} = \left\{< (\alpha_2, \alpha_4), \left\{\frac{x_2}{(0.9,0.4)}\right\} >\right\}$

such that $\gamma_1 = (\alpha_1, \alpha_3)$, $\gamma_2 = (\alpha_1, \alpha_4)$, $\gamma_3 = (\alpha_2, \alpha_3)$, $\gamma_4 = (\alpha_2, \alpha_4)$, for $\gamma_i \in E_1 \times E_2 = \Delta$. The IFH topology that accepts the family $\widetilde{B}$,

$$\widetilde{B} = \left\{P_{IFH}^{(\gamma_1,u_1)},\ P_{IFH}^{(\gamma_1,u_2)},\ P_{IFH}^{(\gamma_2,u_1)},\ P_{IFH}^{(\gamma_2,u_2)},\ P_{IFH}^{(\gamma_3,u_1)},\ P_{IFH}^{(\gamma_3,u_2)},\ P_{IFH}^{(\gamma_4,u_1)}\right\}$$

as the basis is

$$\widetilde{\tau} = \{0_{(U_{IFH}, \Delta)},\ 1_{(U_{IFH}, \Delta)},\ (G_1, \Delta),\ (G_2, \Delta),\ (G_3, \Delta),\ \ldots,\ (G_{128}, \Delta)\},$$

where $(G_1, \Delta) = \left\{P_{IFH}^{(\gamma_1,u_1)}\right\}$, $(G_2, \Delta) = \left\{P_{IFH}^{(\gamma_1,u_2)}\right\}$, $(G_3, \Delta) = \left\{P_{IFH}^{(\gamma_2,u_1)}\right\}$, $(G_4, \Delta) = \left\{P_{IFH}^{(\gamma_2,u_2)}\right\}$, $(G_5, \Delta) = \left\{P_{IFH}^{(\gamma_3,u_1)}\right\}$, $(G_6, \Delta) = \left\{P_{IFH}^{(\gamma_3,u_2)}\right\}$, $(G_7, \Delta) = \left\{P_{IFH}^{(\gamma_4,u_1)}\right\}$, $(G_8, \Delta) = (G_1, \Delta)\widetilde{\cup}(G_2, \Delta)$, ... , $(G_{128}, \Delta) = (G_1, \Delta)\widetilde{\cup}(G_2, \Delta)\widetilde{\cup}...\widetilde{\cup}(G_7, \Delta)$. Then $\widetilde{\tau}$ is an IFH topology over U. It is clear $(U, \widetilde{\tau}, \Delta)$ is IFH T_0-space but not an IFH T_1-space. Because, there does not exist each IFH open sets consisting $P_{IFH}^{(\gamma_4,u_2)}$ and other IFH points.

Example 4.5. We consider Example 4.4. The IFH topology that that accepts the family $\widetilde{B}$,

$$\widetilde{B} = \left\{P_{IFH}^{(\gamma_1,u_1)},\ P_{IFH}^{(\gamma_1,u_2)},\ P_{IFH}^{(\gamma_2,u_1)},\ P_{IFH}^{(\gamma_2,u_2)},\ P_{IFH}^{(\gamma_3,u_1)},\ P_{IFH}^{(\gamma_3,u_2)},\ P_{IFH}^{(\gamma_4,u_1)},\ P_{IFH}^{(\gamma_4,u_2)}\right\},$$

as the basis is

$$\widetilde{\tau} = \{0_{(U_{IFH},\Delta)},\ 1_{(U_{IFH},\Delta)},\ (G_1,\Delta),\ (G_2,\Delta),\ (G_3,\Delta),\ \dots,(G_{256},\Delta)\},$$

where $(G_1,\Delta) = \left\{P_{IFH}^{(\gamma_1,u_1)}\right\}$, $(G_2,\Delta) = \left\{P_{IFH}^{(\gamma_1,u_2)}\right\}$, $(G_3,\Delta) = \left\{P_{IFH}^{(\gamma_2,u_1)}\right\}$, $(G_4,\Delta) = \left\{P_{IFH}^{(\gamma_2,u_2)}\right\}$, $(G_5,\Delta) = \left\{P_{IFH}^{(\gamma_3,u_1)}\right\}$, $(G_6,\Delta) = \left\{P_{IFH}^{(\gamma_3,u_2)}\right\}$, $(G_7,\Delta) = \left\{P_{IFH}^{(\gamma_4,u_1)}\right\}$, $(G_8,\Delta) = \left\{P_{IFH}^{(\gamma_4,u_2)}\right\}$, $(G_9,\Delta) = (G_1,\Delta)\widetilde{\cup}(G_2,\Delta)$, ... ,
$(G_{256},\Delta) = (G_1,\Delta)\widetilde{\cup}(G_2,\Delta)\widetilde{\cup}...\widetilde{\cup}(G_8,\Delta)$. Then, $\widetilde{\tau}$ is an IFH topology over U. It is clear that, $(U,\widetilde{\tau},\Delta)$ is IFH T_1-space and IFH T_2-space.

Theorem 4.6. *Let $(U,\widetilde{\tau},\Delta)$ be an IFH topological space over U. Then, $(U,\widetilde{\tau},\Delta)$ is an IFH T_1-space if and only if each IFH point is a IFH closed set.*

Proof. Suppose that $(U,\widetilde{\tau},\Delta)$ is an IFH T_1-space and $P_{IFH}^{(\alpha,u)}$ be an arbitrary IFH point over U. We should show that $\left(P_{IFH}^{(\alpha,u)}\right)^c$ is an IFH open set. Let $P_{IFH}^{(\beta,v)}\widetilde{\in}\left(P_{IFH}^{(\alpha,u)}\right)^c$, then $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$ are disjoint IFH points. Thus, $\alpha \neq \beta$ or $u \neq v$. Since $(U,\widetilde{\tau},\Delta)$ is an IFH T_1-space, there exists IFH open sets (G_1,Ω_1), (G_2,Ω_2) such that

$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)$ and $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$, $\quad P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2,\Omega_2)$ and $P_{IFH}^{(\beta,v)}\widetilde{\cap}(G_1,\Omega_1) = 0_{(U_{IFH},\Delta)}$.

Then, $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$. We have $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2,\Omega_2)\widetilde{\subseteq}\left(P_{IFH}^{(\alpha,u)}\right)^c$. Therefore, $\left(P_{IFH}^{(\alpha,u)}\right)^c$ is an IFH open set then $P_{IFH}^{(\alpha,u)}$ is an IFH closed set.

Conversely, let each IFH point $P_{IFH}^{(\alpha,u)}$ is an IFH closed set. Then, $\left(P_{IFH}^{(\alpha,u)}\right)^c$ is an IFH open set. Suppose that $P_{IFH}^{(\alpha,u)}\widetilde{\cap}P_{IFH}^{(\beta,v)} = 0_{(U_{IFH},\Delta)}$, then $P_{IFH}^{(\alpha,u)}\widetilde{\cap}\left(P_{IFH}^{(\alpha,u)}\right)^c = 0_{(U_{IFH},\Delta)}$ and $P_{IFH}^{(\beta,v)}\widetilde{\in}\left(P_{IFH}^{(\alpha,u)}\right)^c$. So, $(U,\widetilde{\tau},\Delta)$ is an IFH T_1-space. □

Theorem 4.7. *Let $(U,\widetilde{\tau},\Delta)$ be an IFH topological space over U. $(U,\widetilde{\tau},\Delta)$ is an IFH T_2-space if and only if for disjoint IFH points $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$, there exist an IFH open set (G,Ω) containing $P_{IFH}^{(\alpha,u)}$ but not $P_{IFH}^{(\beta,v)}$ such that $P_{IFH}^{(\beta,v)} \neq cl_{IFH}(G,\Omega)$.*

Proof. Let $(U,\widetilde{\tau},\Delta)$ be an IFH T_2-space and $P_{IFH}^{(\alpha,u)}$, $P_{IFH}^{(\beta,v)}$ be two IFH point over U. Then, there exist disjoint IFH open sets (G_1,Ω_1) and (G_2,Ω_2) such that $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)$ and $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2,\Omega_2)$. Since $P_{IFH}^{(\alpha,u)}\widetilde{\cap}P_{IFH}^{(\beta,v)} = 0_{(U_{IFH},\Delta)}$ and $(G_1,\Omega_1)\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$, $P_{IFH}^{(\beta,v)} \notin (G_1,\Omega_1)$. It implies that, $P_{IFH}^{(\beta,v)} \notin cl_{IFH}(G_1,\Omega_1)$.

Conversely, suppose that for distinct IFH points $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$ there exists an IFH open set (G,Ω) containing $P_{IFH}^{(\alpha,u)}$ but not $P_{IFH}^{(\beta,v)}$ such that $P_{IFH}^{(\beta,v)} \notin cl_{IFH}(G,\Omega)$. Then, $P_{IFH}^{(\beta,v)} \in (cl_{IFH}(G,\Omega))^c$, i.e. (G,Ω) and $(cl_{IFH}(G,\Omega))^c$ are disjoint IFH open sets containing $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$, respectively. □

Theorem 4.8. *Let $(U,\widetilde{\tau},\Delta)$ be an IFH T_1-space for every IFH point $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)\widetilde{\in}\widetilde{\tau}$. If there exist an IFH open set (G_2,Ω_2) in $(U,\widetilde{\tau},\Delta)$ such that $P_{IFH}^{(\alpha,u)} \notin (G_2,\Omega_2)\widetilde{\subseteq}cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_1,\Omega_1)$, then $(U,\widetilde{\tau},\Delta)$ is an IFH T_2-space.*

Proof. Suppuse that $P_{IFH}^{(\alpha,u)}\widetilde{\cap}P_{IFH}^{(\beta,v)} = 0_{(U_{IFH},\Delta)}$. Since $(U,\widetilde{\tau},\Delta)$ is an IFH T_1-space, $P_{IFH}^{(\alpha,u)}$ and $P_{IFH}^{(\beta,v)}$ are IFH closed sets in $(U,\widetilde{\tau},\Delta)$. Thus, $P_{IFH}^{(\alpha,u)}\widetilde{\in}\left(P_{IFH}^{(\beta,v)}\right)^c\widetilde{\in}\widetilde{\tau}$. Then, there exist an IFH open set $(G_2,\Omega_2)\widetilde{\in}\widetilde{\tau}$ such that

$$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2,\Omega_2)\widetilde{\subseteq}cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}\left(P_{IFH}^{(\beta,v)}\right)^c.$$

So, we have $P_{IFH}^{(\beta,v)}\widetilde{\in}(cl_{IFH}(G_2,\Omega_2))^c$, $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2,\Omega_2)$ and $(G_2,\Omega_2)\widetilde{\cap}(cl_{IFH}(G_2,\Omega_2))^c = 0_{(U_{IFH},\Delta)}$, i.e. $(U,\widetilde{\tau},\Delta)$ is an IFH T_2-space. □

Definition 4.9. Let $(U,\widetilde{\tau},\Delta)$ be an IFH topological space over U, (F,Υ) be an IFH closed set in $(U,\widetilde{\tau},\Delta)$ and $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(F,\Upsilon) = 0_{(U_{IFH},\Delta)}$. If there exist IFH open sets (G_1,Ω_1) and (G_2,Ω_2) such that $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)$, $(F,\Upsilon)\widetilde{\subseteq}(G_2,\Omega_2)$ and $(G_1,\Omega_1)\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$, then $(U,\widetilde{\tau},\Delta)$ is called an IFH regular space. $(U,\widetilde{\tau},\Delta)$ is said to be an IFH T_3-space if it is an IFH regular and IFH T_1-space.

Theorem 4.10. *Let $(U,\widetilde{\tau},\Delta)$ be an IFH topological space over U. $(U,\widetilde{\tau},\Delta)$ is an IFH T_3-space if and only if for every $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)\widetilde{\in}\widetilde{\tau}$, there exists $(G_2,\Omega_2)\widetilde{\in}\widetilde{\tau}$ such that*

$$P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2,\Omega_2)\widetilde{\subseteq}cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_1,\Omega_1).$$

Proof. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH T_3-space and $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)\widetilde{\in}\widetilde{\tau}$. Since $(U, \widetilde{\tau}, \Delta)$ is an IFH T_3-space for the IFH point $P_{IFH}^{(\alpha,u)}$ and IFH closed set $(G_1,\Omega_1)^c$, there exist IFH open set (G_2,Ω_2), (G_3,Ω_3) such that $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2,\Omega_2)$, $(G_1,\Omega_1)^c\widetilde{\subseteq}(G_3,\Omega_3)$ and $(G_2,\Omega_2)\widetilde{\cap}(G_3,\Omega_3) = 0_{(U_{IFH},\Delta)}$. Thus, we have $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_2,\Omega_2)\widetilde{\subseteq}(G_3,\Omega_3)^c\widetilde{\subseteq}(G_1,\Omega_1)$.
Since $(G_3,\Omega_3)^c$ is an IFH closed set, so $cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_3,\Omega_3)^c$.

Conversely , suppose that $P_{IFH}^{(\alpha,u)}\widetilde{\cap}(F,\Upsilon) = 0_{(U_{IFH},\Delta)}$ and (F,Υ) is an IFH closed set in $(U, \widetilde{\tau}, \Delta)$. Thus, $P_{IFH}^{(\alpha,u)}\widetilde{\in}(F,\Upsilon)^c$ and from the condition of the theorem, we have $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G,\Omega)\widetilde{\subseteq}cl_{IFH}(G,\Omega)\widetilde{\subseteq}(F,\Upsilon)^c$.
Then, $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G,\Omega)$, $(F,\Upsilon)\widetilde{\subseteq}(cl_{IFH}(G,\Omega))^c$ and $(G,\Omega)\widetilde{\cap}(cl_{IFH}(G,\Omega)) = 0_{(U_{IFH},\Delta)}$ are satisfied, i.e., $(U, \widetilde{\tau}, \Delta)$ is an IFH T_3-space. □

Definition 4.11. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space over U. If for every two non empty IFH closed sets (F_1,Υ_1), (F_2,Υ_2) such that $(F_1,\Upsilon_1)\widetilde{\cap}(F_2,\Upsilon_2) = 0_{(U_{IFH},\Delta)}$, there exists IFH open sets (G_1,Ω_1), (G_2,Ω_2) such that $(G_1,\Omega_1)\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$ and $(F_1,\Upsilon_1)\widetilde{\subseteq}(G_1,\Omega_1)$, $(F_2,\Upsilon_2)\widetilde{\subseteq}(G_2,\Omega_2)$ then $(U, \widetilde{\tau}, \Delta)$ is called an IFH normal space. $(U, \widetilde{\tau}, \Delta)$ is said to be an IFH T_4-space if it is an IFH normal and IFH T_1-space.

Theorem 4.12. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space over U. Then, $(U, \widetilde{\tau}, \Delta)$ is an IFH T_4-space if and only if for each IFH closed set (F,Υ) and IFH open set (G_1,Ω_1) with $(F,\Upsilon)\widetilde{\subseteq}(G_1,\Omega_1)$, there exist an IFH open set (G_2,Ω_2) such that*

$$(F,\Upsilon)\widetilde{\subseteq}(G_2,\Omega_2)\widetilde{\subseteq}cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_1,\Omega_1).$$

Proof. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH T_4-space, (F,Υ) be an IFH closed set in $(U, \widetilde{\tau}, \Delta)$ and $(F,\Upsilon)\widetilde{\subseteq}(G_1,\Omega_1)\widetilde{\in}\widetilde{\tau}$. Then, $(G_1,\Omega_1)^c$ is an IFH closed set and $(F,\Upsilon)\widetilde{\cap}(G_1,\Omega_1)^c = 0_{(U_{IFH},\Delta)}$. Since $(U, \widetilde{\tau}, \Delta)$ is an IFH T_4-space, there exists IFH open sets (G_2,Ω_2), (G_3,Ω_3) such that $(F,\Upsilon)\widetilde{\subseteq}(G_2,\Omega_2)$, $(G_1,\Omega_1)^c\widetilde{\subseteq}(G_3,\Omega_3)$ and $(G_2,\Omega_2)\widetilde{\cap}(G_3,\Omega_3) = 0_{(U_{IFH},\Delta)}$. This implies that,

$$(F,\Upsilon)\widetilde{\subseteq}(G_2,\Omega_2)\widetilde{\subseteq}(G_3,\Omega_3)^c\widetilde{\subseteq}(G_1,\Omega_1).$$

$(G_3,\Omega_3)^c$ an IFH closed set and $cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_3,\Omega_3)^c$ is satisfied. Thus,

$$(F,\Upsilon)\widetilde{\subseteq}(G_2,\Omega_2)\widetilde{\subseteq}cl_{IFH}(G_2,\Omega_2)\widetilde{\subseteq}(G_1,\Omega_1)$$

is obtained.

Conversly, suppose that (F_1,Υ_1), (F_2,Υ_2) be two non empty disjoint IFH closed sets in $(U, \widetilde{\tau}, \Delta)$.
Then, $(F_1,\Upsilon_1)\widetilde{\subseteq}(F_2,\Upsilon_2)^c$.From the condition of theorem, the exist an IFH open set (G,Ω) such that

$$(F_1,\Upsilon_1)\widetilde{\subseteq}(G,\Omega)\widetilde{\subseteq}cl_{IFH}(G,\Omega)\widetilde{\subseteq}(F_2,\Upsilon_2)^c.$$

Therefore, (G,Ω), $(cl_{IFH}(G,\Omega))^c$ are IFH open sets and $(F_1,\Upsilon_1)\widetilde{\subseteq}(G,\Omega)$, $(F_2,\Upsilon_2)\widetilde{\subseteq}(cl_{IFH}(G,\Omega))^c$ and $(G,\Omega)\widetilde{\cap}(cl_{IFH}(G,\Omega))^c = 0_{(U_{IFH},\Delta)}$ are obtained. So, $(U, \widetilde{\tau}, \Delta)$ is an IFH T_4-space. □

Theorem 4.13. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space and (G,Ω) be an IFH set over U. If $(U, \widetilde{\tau}, \Delta)$ is an IFH T_i-space, then the IFH topological subspace $((G,\Omega), \widetilde{\tau}_{(G,\Omega)}, \Omega)$ is an IFH T_i-space for $i = 0, 1, 2, 3$.*

Proof. Let $P_{IFH}^{(\alpha,u)}$, $P_{IFH}^{(\beta,v)} \in ((G,\Omega), \widetilde{\tau}_{(G,\Omega)}, \Omega)$ such that $P_{IFH}^{(\alpha,u)}\widetilde{\cap}P_{IFH}^{(\beta,v)} = 0_{(U_{IFH},\Delta)}$. Hence, there exist IFH open sets (G_1,Ω_1), (G_2,Ω_2) satisfying the conditions of IFH T_i-space such that $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)$ and $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2,\Omega_2)$. Then, $P_{IFH}^{(\alpha,u)}\widetilde{\in}(G_1,\Omega_1)\widetilde{\cap}(G,\Omega)$ and $P_{IFH}^{(\beta,v)}\widetilde{\in}(G_2,\Omega_2)\widetilde{\cap}(G,\Omega)$. Also, IFH open sets $(G_1,\Omega_1)\widetilde{\cap}(G,\Omega)$, $(G_2,\Omega_2)\widetilde{\cap}(G,\Omega)$ in $\widetilde{\tau}_{(G,\Omega)}$ satisfiying the conditions of IFH T_i-space for $i = 0, 1, 2, 3$. □

Theorem 4.14. *Let $(U, \widetilde{\tau}, \Delta)$ be an IFH topological space over U. If $(U, \widetilde{\tau}, \Delta)$ is IFH T_4-space and (F,Υ) is an IFH closed set in $(U, \widetilde{\tau}, \Delta)$, then $((F,\Upsilon), \widetilde{\tau}_{(F,\Upsilon)}, \Upsilon)$ is an IFH T_4-space.*

Proof. Let $(U, \widetilde{\tau}, \Delta)$ be an IFH T_4-space and (F,Υ) be an IFH closed set in $(U, \widetilde{\tau}, \Delta)$. Let (F_1,Υ_1) and (F_2,Υ_2) be two IFH closed set in $((F,\Upsilon), \widetilde{\tau}_{(F,\Upsilon)}, \Upsilon)$ such that $(F_1,\Upsilon_1)\widetilde{\cap}(F_2,\Upsilon_2) = 0_{(U_{IFH},\Delta)}$. When (F,Υ) is an IFH closed set in $(U, \widetilde{\tau}, \Delta)$, (F_1,Υ_1) and (F_2,Υ_2) are IFH closed sets in $(U, \widetilde{\tau}, \Delta)$. Since $(U, \widetilde{\tau}, \Delta)$ is an IFH T_4-space, there exist IFH open sets (G_1,Ω_1), (G_2,Ω_2) such that $(F_1,\Upsilon_1)\widetilde{\subseteq}(G_1,\Omega_1)$, $(F_2,\Upsilon_2)\widetilde{\subseteq}(G_2,\Omega_2)$ and $(G_1,\Omega_1)\widetilde{\cap}(G_2,\Omega_2) = 0_{(U_{IFH},\Delta)}$. Then, $(F_1,\Upsilon_1)\widetilde{\subseteq}(G_1,\Omega_1)\widetilde{\cap}(F,\Upsilon)$, $(F_2,\Upsilon_2)\widetilde{\subseteq}(G_2,\Omega_2)\widetilde{\cap}(F,\Upsilon)$ and $\left[(G_1,\Omega_1)\widetilde{\cap}(F,\Upsilon)\right]\widetilde{\cap}\left[(G_2,\Omega_2)\widetilde{\cap}(F,\Upsilon)\right] = 0_{(U_{IFH},\Delta)}$. This implies that, $((F,\Upsilon), \widetilde{\tau}_{(F,\Upsilon)}, \Upsilon)$ is IFH T_4-space. □

5. Conclusion

In 2020, Abbas et.al. introduced IFH points and some properties of them. In this paper, we have continued to study the concept of hypersoft sets. Some concepts and properties such as IFH neighborhood, interior point, adherent point, related to IFH point are explored. We defined IFH T_i-space ($i = 0, 1, 2, 3, 4$) with respect to IFH points and studied their basic properties in IFH topological spaces. We also extented these separation axioms to different results. These separation axioms would be useful for the growth of IFH topology. We hope that, these results in this paper will help the researchers for strengthening the toolbox of IFH topological spaces.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Abbas, M., Murtaza, G., Smarandache, F., *Basic operation on hypersoft sets and hypersoft point*, Neutrosophic sets and system, **35**(2020), 407–421.
[2] Ali, M.I., *A note on soft sets, rough soft sets and fuzzy soft sets*, Appl. Soft Comput., **11**(2011), 3329–3332.
[3] Atanassov, K., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1986), 87–96
[4] Bera, T., Mahapatra, N.K., *Introduction to neutrosophic soft topological space*, Opsearch, **4**(2017), 841–867.
[5] Broumi, S., Smarandache, F., *Intuitionistic neutrosophic soft set*, J. Inf. Comput. Sci., **8**(2013), 130–140.
[6] Chang, C.L., *Fuzzy topological spaces*, Journal of mathematical Analysis and Applications, **1**(1968), 182–190.
[7] Coker, D., *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems, **1**(1997), 81–89.
[8] Molodtsov, D., *Soft set theory—first results*, Computers & Mathematics with Applications, **37**(1999), 19–31.
[9] Maji, P.K., Biswas, R., Roy, A.R., *Fuzzy soft sets*, J. Fuzzy Math., **9**(2001), 589–602.
[10] Maji, P.K., Biswas, R., Roy, A., *Intuitionistic fuzzy soft sets*, Journal of Fuzzy Mathematics **3**(2001), 677–692.
[11] Maji, P.K., *Neutrosophic soft set*, Ann. Fuzzy Math. Inform., **5**(2013), 57–168.
[12] Majumdar, P. Samanta, S.K., *Generalised fuzzy soft sets*, Comput. Math. Appl., **59**(2010), 1425–1432
[13] Pawlak, Z., *Rough sets*, Int. J. Inf. Comput. Sci., **11**(1982), 341–356.
[14] Smarandache, F., *Extension of soft set to hypersoft set, and then to plithogenic hypersoft set*, Neutrosophic Sets and System, **22**(2018), 168-170.
[15] Wang, F., Li, X., Chen, X., *Hesitant fuzzy soft set and its applications in multicriteria decision making*, J. Appl. Math., **2014**(2014).
[16] Xiao, Z., Xia, S., Gong, K., Li, D., *The trapezoidal fuzzy soft set and its application in MCDM*, Appl. Math. Model., **36**(2012), 5844–5855.
[17] Xu, W., Ma, J. Wang, S. Hao, G., *Vague soft sets and their properties*, Comput. Math. Appl., **59**(2010), 787–794.
[18] Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y., Yu, D.Y., *Combination of interval-valued fuzzy set and soft set*, Comput. Math. Appl., **58**(2009), 521–527.
[19] Yang, Y., Tan, X., Meng, C.C., *The multi-fuzzy soft set and its application in decision making*, Appl. Math. Model., **37**(2013), 4915–4923.
[20] Yolcu, A., Smarandache F., Ozturk, T.Y, *Intuitionistic fuzzy hypersoft sets*, Communications Faculty of Sciences University of Ankara Series A Mathematics and Statistics, **70**(2021), 443–455.
[21] Yolcu, A., Ozturk, T.Y., *Fuzzy hypersoft sets and its application to decision-making*, Theory and Application of Hypersoft Set, **50**(2021).
[22] Yolcu, A., *Intuitionistic fuzzy hypersoft topology and its application to multi-criteria decision making*, Sigma Journal of Mathematics, In press.
[23] Yolcu, A., Ozturk T.Y., *An introduction to fuzzy hypersoft topological spaces*, Caucasian Journal of Science, In press.
[24] Zadeh, L.A, *Fuzzy sets*, Inf. Control, **8**(1965), 338–353.