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ON GLM TYPE INTEGRAL EQUATION FOR SINGULAR STURM-LIOUVILLE OPERATOR WHICH HAS DISCONTINUOUS COEFFICIENT

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Abstract. In this study, we derive Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for singular Sturm-Liouville equation which has discontinuous coefficient. Then we prove the unique solvability of the main integral equation.

1. Introduction

We consider boundary value problem L as follows:

$$-y''+\left[\frac{A}{x}+q(x)\right]y=\lambda^2\rho(x)y,\quad x\in I=(0,d)\cup(d,\pi)\,, \tag{1}$$

$$U(y):=y(0)=0, V(y):=\ y(\pi)=0 \tag{2}$$

where λ is spectral parameter, $A\in\mathbb{R}^+$, $\rho(x)=\left\{\begin{array}{ll}1, & 0\leq x\leq d\\ \alpha^2, & d<x\leq\pi\end{array}\right.$, $\alpha\in R$, $\alpha\neq1$, $\alpha>0$, $d\in\left(\frac{\pi}{2},\pi\right)$, $q(x)$ is a real valued bounded function and $q(x)\in L_2(0,\pi)$.

Boundary value problems with discontinuous coefficient often appear in applied mathematics, geophysics, mechanics, electromagnetics, elasticity and other branches of engineering and physics. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences. For example, torodial vibrations and free vibrations of the earth, reconstructing the

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discontinuous material properties of a nonabsorbing media, as a rule leads to direct and inverse problems or the Sturm-Liouville equation which has discontinuous coefficient. (see [1]- [7]) Discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [6]. After reducing corresponding mathematical model we come to boundary value problem L where $q(x)$ must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium [7], [1]. Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth [2]. Here, the main discontinuity is cased by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behaviour of solutions for such nonlinear equations. We also note that inverse problem considered here appears in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

Sturm-Liouville operators with singular potential were studied in [8]- [10]. In [10], Sturm-Liouville operators generated by the differential expression $-y''+q(x)\,y$ were considered. Here $q(x)$ is a distribution of first order, i.e., $\int q(x)\,dx \in L_2[0,\pi]$. The minimal and maximal operators corresponding to potentials of this type on a finite interval were constructed in [8]. All self-adjoint extensions of the minimal operator were described and the asymptotics of the eigenvalues of these extensions were found there.

The authors in [11]- [14] study asymptotics of eigenvalue, eigenfunctions and normalizing numbers and solve the inverse spectral problems of recovering the singular potential $q \in W_2^{-1}(0,1)$ of Sturm-Liouville operators by two spectra. The reconstruction algorithm is presented and necessary and sufficient conditions on two sequences to be spectral data Sturm-Liouville operators under consideration are given. Unlike these studies, the proposed method in our work is more practical and more feasible.

In this study, we derive the Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for singular Sturm-Liouville equation which has discontinuous coefficient. Then we prove the unique solvability of the main integral equation.

In [15] and [16] , we defined $y_1(x) = y(x)\,, y_2(x) = (\Gamma y)(x) = y'(x) - u(x)\,y(x)\,,$ $u(x) = A\ln x$ and got the expression of left hand side of the equation (1) as follows

$$\ell(y) = -\left[(\Gamma y)(x)\right]' - u(x)\,(\Gamma y)(x) - u^2(x)\,y + q(x)\,y = \lambda^2\rho(x)y, \tag{3}$$

then the equation (1) reduced to the system;

$$\begin{cases} y_1' - y_2 = u(x)\,y_1 \\ y_2' + \lambda^2\rho(x)y_1 = -u(x)\,y_2 - u^2(x)\,y_1 + q(x)\,y_1 \end{cases} \tag{4}$$

with the boundary conditions

$$y_1(0) = 0, y_1(\pi) = 0. \tag{5}$$

Matrix form of system (4)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda^2\rho(x) - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{6}$$

or $y' = Ay$ such that $A = \begin{pmatrix} u(x) & 1 \\ -\lambda^2\rho(x) - u^2(x) + q(x) & -u(x) \end{pmatrix}$, $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

$x = 0$ is a regular-singular end point for equation (4) and Theorem 2 in [17] (see Remark 1-2, p.56) extends to interval $[0, \pi]$. For this reason, by [17], there exists only one solution of the system (2) which satisfies the initial conditions $y_1(\xi) = \upsilon_1, y_2(\xi) = \upsilon_2$ for each $\xi \in [0, \pi], \upsilon = (\upsilon_1, \upsilon_2)^T \in C^2$, especially the initial conditions $y_1(0) = 1, y_2(0) = i\lambda$.

Definition 1. *The first component of the solution of the system (4) which satisfies the initial conditions* $y_1(\xi) = \upsilon_1, y_2(\xi) = (\Gamma y)(\xi) = \upsilon_2$ *is called the solution of the equation (1) which satisfies these same initial conditions.*

It was obtained in [3] by the successive approximations method that (see [18], [19]) the following theorem is true.

Theorem 1. *[3] For each solution of system (6) satisfying the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i\lambda \end{pmatrix}$ *the following expression is true:*

$$\begin{cases} y_1 = e^{i\lambda x} + \int\limits_{-x}^{x} K_{11}(x,t) e^{i\lambda t} dt \\ y_2 = i\lambda e^{i\lambda x} + b(x) e^{i\lambda x} + \int\limits_{-x}^{x} K_{21}(x,t) e^{i\lambda t} dt + i\lambda \int\limits_{-x}^{x} K_{22}(x,t) e^{i\lambda t} dt \end{cases}, \quad x < d$$

$$\begin{cases} y_1 = \alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)} + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{11}(x,t) e^{i\lambda t} dt \\ y_2 = i\lambda\alpha \left(\alpha^+ e^{i\lambda\mu^+(x)} - \alpha^- e^{i\lambda\mu^-(x)}\right) \\ \qquad + b(x) \left[\alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)}\right] \\ \qquad + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{21}(x,t) e^{i\lambda t} dt + i\lambda\alpha \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{22}(x,t) e^{i\lambda t} dt \end{cases}, \quad x > d$$

where

$$b(x) = -\frac{1}{2}\int_0^x \left[u^2(s) - q(s)\right] e^{-\frac{1}{2}\int_s^x u(t)dt} ds,$$

$$K_{11}(x,x) = \frac{\alpha^+}{2} u(x),$$

$$K_{21}(x,x) = b'(x) - \frac{1}{2}\int_0^x \left[u^2(s) - q(s)\right] K_{11}(s,s)\,ds - \frac{1}{2}\int_0^x u(s) K_{21}(s,s)\,ds,$$

$$K_{22}(x,x) = -\frac{\alpha^+}{2}\left[u(x) + 2b(x)\right],$$

$$K_{11}(x, 2d - x + 0) - K_{11}(x, 2d - x - 0) = \frac{\alpha^-}{2} u(x),$$

$$\frac{\partial K_{ij}(x,.)}{\partial x}, \frac{\partial K_{ij}(x,.)}{\partial t} \in L_2(0,\pi), i,j = 1,2,$$

$$\alpha^{\pm}(x) = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{\rho(x)}}\right), \quad \mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + d\left(1 \pm \sqrt{\rho(x)}\right).$$

2. The Main Equation of the Inverse Problem

Assume that $s(x,\lambda)$ is solution of the equation (1) with initial condition

$$s(0,\lambda) = \begin{pmatrix} 0 \\ i\lambda \end{pmatrix}.$$

We have

$$s(x,\lambda) = s_0(x,\lambda) + \int_{\mu^-(x)}^{\mu^+(x)} K_{11}(x,t)\sin\lambda t dt,$$

where

$$s_0(x,\lambda) = \alpha^+(x)\sin\lambda\mu^+(x) + \alpha^-(x)\sin\lambda\mu^-(x).$$

Also, let us define α_n, α_n^0 and $\Phi_N(x,t)$ as follows:

$$\alpha_n = \int_0^\pi \rho(x)\, s^2(x,\lambda_n)\,dx,$$

$$\alpha_n^0 = \int_0^\pi \rho(x)\, s_0^2\left(x,\lambda_n^0\right)dx,$$

$$\Phi_N(x,t) = \Phi_{N_1}(x,t) + \Phi_{N_2}(x,t) + \Phi_{N_3}(x,t) + \Phi_{N_4}(x,t),$$

$$\Phi_N(x,t)=\sum_{n=0}^{N}\left(\frac{s(x,\lambda_n)\,s(t,\lambda_n)}{\alpha_n}-\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\right),$$

$$\Phi_{N_1}(x,t)=\sum_{n=0}^{N}\left(\frac{s_0(x,\lambda_n)\,s_0(t,\lambda_n)}{\alpha_n}-\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\right),$$

$$\Phi_{N_2}(x,t)=\int_0^{\mu^+(x)}K_{11}(x,\xi)\sum_{n=0}^{N}\frac{s_0\left(x,\lambda_n^0\right)\sin\lambda_n^0\xi}{\alpha_n^0}d\xi,$$

$$\Phi_{N_3}(x,t)=\int_0^{\mu^+(x)}K_{11}(x,\xi)\sum_{n=0}^{N}\left(\frac{s_0(x,\lambda_n)\sin\lambda_n\xi}{\alpha_n}-\frac{s_0\left(x,\lambda_n^0\right)\sin\lambda_n^0\xi}{\alpha_n^0}\right)d\xi,$$

$$\Phi_{N_4}(x,t)=\int_0^{\mu^+(x)}K_{11}(x,\xi)\sum_{n=0}^{N}\frac{s(x,\lambda_n)\sin\lambda_n\xi}{\alpha_n}d\xi.$$

Here, using

$$s_0(\xi,\lambda)=\begin{cases}\sin\lambda_n\xi, & \xi\le d\\ \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\sin\lambda\mu^+(\xi)+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\sin\lambda\mu^+(\xi), & \xi>d\end{cases},$$

we have

$$s_0(\xi,\lambda)=\alpha^+\sin\lambda\mu^+(\xi)+\alpha^-s_0\lambda\left(2d-\mu^+(\xi)\right),\quad \xi>d.$$

Also, since $2d-\mu^+(\xi)<d$ we obtain

$$\sin\lambda\mu^+(\xi)=\frac{1}{\alpha^+}s_0(\xi,\lambda)-\frac{\alpha^-}{\alpha^+}s_0\left(2d-\mu^+(\xi),\lambda\right).$$

Substituting $\mu^+(\xi)\to\xi$, we get

$$\sin\lambda_n\xi=\begin{cases}s_0(\xi,\lambda), & \xi\le d\\ \frac{1}{\alpha^+}s_0(\xi,\lambda)-\frac{\alpha^-}{\alpha^+}s_0\left(2d-\mu^+(\xi),\lambda\right), & \xi>d\end{cases}. \tag{7}$$

Now, define $F_0(x,t)$ and $F(x,t)$ as follows:

$$F_0(x,t)=\sum_{n=0}^{\infty}\left[\frac{s_0(t,\lambda_n)\sin\lambda_nx}{\alpha_n}-\frac{s_0\left(t,\lambda_n^0\right)\sin\lambda_n^0x}{\alpha_n^0}\right] \tag{8}$$

and

$$F(x,t)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)F_0(\mu^+(x),t)+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)F_0(\mu^-(x),t). \tag{9}$$

We can write

$$F(x,t)= \sum_{n=0}^{\infty}\left[\frac{s_0\left(x,\lambda_n\right)s_0\left(t,\lambda_n\right)}{\alpha_n}-\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\right]. \tag{10}$$

Let $f\in AC[0,\pi]$, using Theorem 6 in [4],

$$f(x)=\sum_{n=0}^{\infty}\int_0^{\pi} f(t)\rho\left(t\right)\frac{s\left(x,\lambda_n\right)s\left(t,\lambda_n\right)}{\alpha_n}dt \tag{11}$$

and

$$f(x)=\sum_{n=0}^{\infty}\int_0^{\pi} f(t)\rho\left(t\right)\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}dt, \tag{12}$$

we get

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\int_0^{\pi} f(t)\rho\left(t\right)\Phi_N(x,t)dt$$

$$\le \lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho\left(t\right)\sum_{n=0}^{\infty}\frac{s\left(x,\lambda_n\right)s\left(t,\lambda_n\right)}{\alpha_n}dt-f(x)\right|,$$

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho\left(t\right)\sum_{n=0}^{\infty}\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}dt-f(x)\right|=0.$$

Furthermore, uniformly for $x\in[0,\pi]$,

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho\left(t\right)\Phi_{N_1}(x,t)dt=\int_0^{\pi} f(t)\rho\left(t\right)F(x,t)dt. \tag{13}$$

Similarly, we have

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho\left(t\right)\Phi_{N_2}(x,t)dt=\int_0^{d} f(t)K_{11}(x,t)dt+$$

$$\frac{1}{\alpha^+}\int_d^{x} f(t)K_{11}(x,\mu^+\left(t\right))dt-\frac{\alpha^-}{\alpha^+}\int_d^{x} f(t)K_{11}(x,\mu^+\left(2d-t\right))dt.$$

Because for $2d-t>\mu^+\left(x\right)$, $K(x,\mu^+\left(2d-t\right)\equiv 0$, we have

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_2}(x,t)dt =$$

$$\int_0^{x} f(t)K_{11}(x,\mu^+(t))\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}dt+\int_0^{x} f(t)K_{11}(x,\mu^+(2d-t))\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}dt$$

uniformly in $x\in[0,\pi]$.

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_3}(x,t)dt=\int_0^{\pi}\rho(t)\,f(t)\left(\int_0^{\mu^+(x)}K_{11}(x,\xi)\,F_0(\xi,t)d\xi\right)dt.$$

Using residue theorem, we get

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_4}(x,t)dt =$$

$$=2\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\frac{1}{2\pi i}\oint_{\Gamma_n}\left(\frac{\lambda}{\Delta(\lambda)}\int_0^{\mu^+(x)}K_{11}(x,\xi)\sin\lambda\xi\right)d\lambda dt$$

here $\Gamma_n=\{\lambda:|\lambda|=N\}$.

$$s(x,\lambda)=O\left(e^{|\mathrm{Im}\,\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}\right)$$

and

$$|\Delta(\lambda)|\geq C_\delta\,|\lambda|\,e^{\left|\mathrm{Im}\,\lambda\mu^+(x)\right|},\quad \lambda\in G_\delta$$

where $C_\delta>0$, $G_\delta=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta\right\}$, for all $\lambda\in G_\delta$, we get

$$\left|\frac{\lambda}{\Delta(\lambda)}\right|\leq\widetilde{C_\delta}e^{-|\mathrm{Im}\,\lambda|\left(\mu^+(x)-\mu^+(t)\right)}$$

where $\widetilde{C_\delta}>0$ is a constant. Using $\mu(t)<\mu^+(x)$, we have

$$\lim_{|\lambda|\to\infty}\max_{0\leq x\leq\pi}\frac{\lambda}{\Delta(\lambda)}=0.$$

By the way, due to Riemann-Lebesgue lemma, we can write

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_4}(x,t)dt=0.$$

If we use the last equations we obtain

$$\int_0^\pi f(t)\rho(t)\,F(x,t)dt+\int_0^x f(t)K_{11}(x,\mu^+(t))\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}dt+$$

$$\int_0^x f(t)K_{11}(x,\mu^+(2d-t))\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}dt+\int_0^\pi f(t)\rho(t)\int_0^{\mu^+(x)}K_{11}(x,\xi)\,F_0(\xi,t)d\xi dt=0.$$

Since $f\in AC[0,\pi]$ is arbitrary, the following theorem could be proved:

Theorem 2. *For every fix $x\in(0,\pi)$, the kernel function $K_{11}(x,t)$ of the integral representation of the solution $\varphi(x,\lambda)$ satisfies the following linear-functional integral equation.*

$$\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{11}(x,\mu^+(t))+\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(x,\mu^+(2d-t))+F(x,t)+$$

$$+\int_0^{\mu^+(x)}K_{11}(x,\xi)\,F_0(\xi,t)d\xi dt=0, \tag{14}$$

where the functions $F_0(x,t)$ and $F(x,t)$ are defined by the formulas (8) and (9) respectively.

Theorem 3. *For every fix $x\in(0,\pi)$, the equation (14) has a unique solution $K_{11}(x,t)$,which belongs to $L_2(0,\pi)$.*

Proof. For $x\le d$, equation (14) is written as follows:

$$K_{11}(x,t)+F(x,t)+\int_0^x K_{11}(x,\xi)\,F_0(\xi,t)d\xi=0 \tag{15}$$

which is a Fredholm integral equation and equivalent to the equation of type

$$(I+B)\,f=g \tag{16}$$

where I is the unit operator, B is a compact operator in the space $L_2(0,\pi)$, $f,g\in L_2(0,\pi)$. Let us prove that in the case $x>d$ the equation (14) is also equivalent to an equation of type (16).

If $x>d$, the equation (14) can be written as

$$L_xK_{11}(x,.)+M_xK_{11}(x,.)=-F(x,.),$$

where

$$(L_xf)(t)=\frac{2}{1+\sqrt{\rho(t)}}f\left(\mu^+(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}f(2d-t),\quad 0<t<x, \tag{17}$$

$$(M_x f)(t) = \int_0^{\mu^+(x)} f(\xi) F_0(\xi, t) d\xi, \quad 0 < t < x. \tag{18}$$

It was shown in [5] that the operator L_x has a bounded inverse in the space $L_2(0, \pi)$ and

$$L_x^{-1} f(t) = \begin{cases} f(t) - \frac{1-\alpha}{2} f\left(\frac{-t+\alpha d+d}{\alpha}\right), & t < d \\ \frac{1+\alpha}{2} f\left(\frac{t+\alpha d-d}{\alpha}\right), & t > d \end{cases} \tag{19}$$

Therefore the equation (14) is equivalent to the equation

$$K_{11}(x, .) + L_x^{-1} M_x K_{11}(x, .) = -L_x^{-1} F(x, .). \tag{20}$$

Because L_x^{-1} is a bounded and M_x is a compact operator in $L_2(0, \pi)$, then the operator $B_x = L_x^{-1} M_x$ is compact in $L_2(0, \pi)$. The right hand side of (20) also belongs to $L_2(0, \pi)$, since M_x is invertible in $L_2(0, \pi)$. Consequently, the equation (20) is a Fredholm integral equation type (16) and it is sufficient to prove that the homogeneous equation

$$L_x K_{11}(x, .) + \int_0^{\mu^+(x)} K_{11}(x, \xi) F_0(\xi, t) d\xi = 0 \tag{21}$$

has only trivial solution $K_{11}(x, t) = 0$. Let $K(t) := K_{11}(x, t)$ be solution of equation (21). Then

$$\int_0^x \rho(t) [L_x K(t)]^2 dt + \int_0^x \rho(t) L_x K(t) dt \int_0^{\mu^+(x)} K(\xi) F_0(\xi, t) d\xi dt = 0. \tag{22}$$

Using for $\xi < \mu^-(x)$, $K(2d - \xi) = 0$ and the formulas (7) and (8), we have

$$L_x K(t) = \int_0^{\mu^+(x)} K(\xi) F_0(\xi, t) d\xi = \int_0^x \rho(\xi) L_x K(\xi) F(\xi, t) d\xi.$$

Therefore (20) can be written as

$$\int_0^x \rho(t) [L_x K(t)]^2 dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^x \rho(t) s_0(t, \lambda_n) L_x K(t) dt \right)^2 \tag{23}$$

$$- \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) s_0\left(t, \lambda_n^0\right) L_x K(t) dt \right)^2 = 0. \tag{24}$$

Now if we use Parseval's equality [4],

$$\int_0^x \rho(t) f^2(t)\,dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) f(t) s_0\left(t, \lambda_n^0\right) dt \right)^2,$$

for

$$f(t) = \begin{cases} L_x K(t), & 0 < t < x, \\ 0, & t > x \end{cases}$$

which belongs to $L_2(0,\pi)$, we have

$$\int_0^x \rho(t)\left(L_x K(t)\right) s_0(t, \lambda_n)\,dt = 0, \quad n \geq 0.$$

Since the system of function $\{s_0(t,\lambda_n)\}_{n\geq 0}$ is complete in $L_2(0,\pi)$ by the theorem in [3], we get $L_x K(t) = 0$. Since the operator L_x has inverse in the space $L_2(0,\pi)$, we obtain $K(t) \equiv K(x,.)$. It means that, the theorem is proved. □

Using Theorem 1 and the fact that the functions $\{s_0(t,\lambda_n)\}_{n\geq 0}$ is a Riesz basis of the space $L_2(0,\pi)$ (see [3]), we get the following theorem:

Theorem 4. *The spectral data* $\left\{\lambda_n^2, \alpha_n\right\}_{n\geq 0}$ *uniquely determines the boundary value problem* L.

The integral equation (14) is called main integral equation of GLM (Gelfand-Levitan-Marchenko) type for the problem L.

3. Properties of the Functions $F_0(x,t)$, $F(x,t)$, $K_{11}(x,t)$.

Lemma 1. *Denote*

$$B(x) = \sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x}{\alpha_n} - \frac{\sin \lambda_n^0 x}{\alpha_n^0} \right). \tag{25}$$

Then, $B(x) \in W_2^1(0,2\pi), F_0(x,x) \in W_2^1(0,2\pi), F(x,x) \in W_2^1(0,2\pi)$.

Proof.

$$\sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x}{\alpha_n} - \frac{\sin \lambda_n^0 x}{\alpha_n^0} \right) = \sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x - \sin \lambda_n^0 x}{\alpha_n} + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \sin \lambda_n^0 x \right) \tag{26}$$

If we denote $\varepsilon_n := \lambda_n - \lambda_n^0$ and using asymptotic formulas of λ_n as follows:(see [3])

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, d_n \text{ is a bounded squence, } \{k_n\} \in \ell_2, \tag{27}$$

then

$$\sin \lambda_n x - \sin \lambda_n^0 x = \varepsilon_n x \cos \lambda_n^0 x + (\sin \varepsilon_n x - \varepsilon_n x) \cos \lambda_n^0 x - 2 \sin^2 \frac{\varepsilon_n x}{2} \sin \lambda_n^0 x. \tag{28}$$

$$B(x) = B_1(x) + B_2(x),$$

where

$$B_1(x) = \sum_{n=0}^{\infty} \frac{d_n x \cos \lambda_n^0 x}{\alpha_n^0 \lambda_n^0} \tag{29}$$

$$\begin{aligned} B_2(x) = \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \sin \lambda_n^0 x + \left(\frac{\sin \lambda_0 x - \sin \lambda_0^0 x}{\alpha_n^0} \right) \\ - \sum_{n=1}^{\infty} \frac{k_n x \cos \lambda_n^0 x}{\alpha_n^0 n} - \sum_{n=1}^{\infty} \frac{\cos \lambda_n^0 x}{\alpha_n^0 n} \left(\sin \varepsilon_n x - \varepsilon_n x \right) \\ - 2\sum_{n=1}^{\infty} \frac{\sin \lambda_n^0 x}{\alpha_n^0} \sin^2 \frac{\varepsilon_n x}{2}. \end{aligned} \tag{30}$$

Using

$$\alpha_n^0 = \int_0^{\pi} \rho(x) s_0^2 \left(x, \lambda_n^0 \right) dx, \tag{31}$$

where

$$s_0(x, \lambda_n) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) \sin \lambda \mu^+(x) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) \sin \lambda \mu^+(x) \tag{32}$$

and asymptotic behaviour of α_n we obtain $B_1(x), B_2(x) \in W_2^1(0, 2\pi)$ i.e., $B(x) \in W_2^1(0, 2\pi)$.

It is easy to verify that

$$\begin{aligned} F_0(x,t) = \frac{1}{4} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) \left[B\left(x - \mu^+(t) \right) + B\left(x + \mu^+(t) \right) \right] + \\ \frac{1}{4} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) \left[B\left(x - \mu^-(t) \right) + B\left(x + \mu^-(t) \right) \right]. \end{aligned} \tag{33}$$

So, $F_0(x,x) \in W_2^1(0,2\pi)$ and by formula (9) we have $F(x,x) \in W_2^1(0,2\pi)$. □

Now using the main integral equation (14), the formulas (15), (16), (18), (33) and (9) we obtain the following theorem.

Theorem 5. *The kernel function $K(x,t)$ of the main integral equation and the functions $F_0(x,t), F(x,t)$ satisfy the following relations:*

$$\rho(t) \frac{\partial^2 F_0(x,t)}{\partial t^2} = \rho(x) \frac{\partial^2 F_0(x,t)}{\partial x^2}, \ \rho(t) \frac{\partial^2 F(x,t)}{\partial t^2} = \rho(x) \frac{\partial^2 F(x,t)}{\partial x^2}, \tag{34}$$

$$F_0(x,t) \,|_{t=0} = B(x),$$

$$F(x,t) \,|_{t=0} = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) B\left(\mu^+(t) \right) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) B\left(\mu^+(t) \right), \tag{35}$$

$$\frac{\partial F_0(x,t)}{\partial t}\mid_{t=0}=0, \frac{\partial F(x,t)}{\partial t}\mid_{t=0}=0, \tag{36}$$

$$\frac{\partial F_0(\mu^{\pm}(x),t)}{\partial x}=\pm\rho(x)\frac{\partial F_0(\xi,t)}{\partial \xi}\mid_{t=\mu^{\pm}(x)}, \tag{37}$$

$$\frac{\partial K(x,0)}{\partial x}=0, \tag{38}$$

$$\frac{\sqrt{\rho(x)}-1}{\sqrt{\rho(x)}+1}K_{11}(x,\mu^{+}(x))=\frac{d}{dx}\left[K_{11}(x,\mu^{-}(x)+0)-K_{11}(x,\mu^{-}(x)-0)\right]. \tag{39}$$

4. Solution of the Inverse Problem

In this section the following theorem has been proved for the necessary and sufficient condition for solvability of the inverse problem with respect to the spectral data.

The following asymptotic relations were obtained in [3]:

Let $\left\{\lambda_n^2,\alpha_n\right\}_{n\geq 0}$ to be the spectral data for a certain boundary value problem $L=L(q(x),A)$ with $q(x)\in L_2(0,\pi)$, then

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{k_n}{n},(k_n)\in\ell_2, \tag{40}$$

$$\alpha_n=\alpha_n^0+\frac{t_n}{n},(t_n)\in\ell_2, \tag{41}$$

where λ_n^0 are zeros of the characteristic function $\Delta_0(\lambda)=s_0(\pi,\lambda),(d_n)$ is the bounded sequence

$$d_n=\frac{\alpha^{+}\sin\lambda_n^0\mu^{+}(\pi)-\alpha^{-}\sin\lambda_n^0\mu^{-}(\pi)}{\dot{\Delta}_0\left(\lambda_n^0\right)}$$

$$\alpha_n^0=\int_0^{\pi}\rho(x)\,s_0^2(x,\lambda_n)\,dx.$$

Let real numbers $\{\lambda_n,\alpha_n\}_{n\geq 0}$ be given. We construct function $F_0(x,t)$, $F(x,t)$ by the formulas (8) and (9) of the section 2 and consider the main integral equation (14). Let the function $K_{11}(x,t)$ is the solution of (14). We construct the function $\varphi(x,\lambda)$ by the formula

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^{\mu^{+}(x)}K_{11}(x,t)\sin\lambda t dt. \tag{42}$$

To prove the theorem we need some lemmas.

Lemma 2. *The following relations hold:*

$$- s''(x,\lambda) + \left[\frac{A}{x} + q(x)\right] s(x,\lambda) = \lambda\rho(x) s(x,\lambda) \tag{43}$$

$$s(0,\lambda) = 0,\ \ s(\pi,\lambda) = 0. \tag{44}$$

Proof. Assume that $B(x) \in W_2^2(0,\pi)$, where $B(x)$ is defined in equation (25). Differentiating the identity

$$G(x,t) := \frac{2}{1+\sqrt{\rho(t)}} K_{11}\big(x,\mu^+(x)\big) + \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} +$$

$$F(x,t) + \int\limits_0^{\mu^+(x)} K_{11}(x,\xi) F_0(\xi,t) d\xi = 0, 0 < t < x \tag{45}$$

we calculate

$$G_t(x,t) := \frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}} \frac{\partial K_{11}(x,\mu^+(t))}{\partial t} - \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} \frac{\partial K_{11}(x,\mu^+(2d-t))}{\partial t}$$

$$+F_t(x,t) + \int\limits_0^{\mu^+(x)} K_{11}(x,\xi) \frac{\partial F_0(\xi,t)}{\partial t} d\xi = 0,$$

$$G_{tt}(x,t) := \frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}} \frac{\partial^2 K_{11}(x,\mu^+(t))}{\partial t^2} + \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} \frac{\partial^2 K_{11}(x,\mu^+(2d-t))}{\partial t^2}$$

$$+ \frac{\partial^2 F(x,t)}{\partial t^2} + \int\limits_0^{\mu^+(x)} K_{11}(x,\xi) \frac{\partial^2 F_0(\xi,t)}{\partial t^2} d\xi = 0, \tag{46}$$

$$G_x(x,t) := \frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}} \frac{\partial K_{11}(x,\mu^+(t))}{\partial x} + \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} \frac{\partial K_{11}(x,\mu^+(2d-t))}{\partial x}$$

$$+\frac{\partial F(x,t)}{\partial x} + \sqrt{\rho(x)} K_{11}\big(x,\mu^+(x)\big) F_0(\mu^+(x),t) + \int\limits_0^{\mu^+(x)} K_{11}(x,\xi) F_0(\xi,t) d\xi$$

$$+\sqrt{\rho(x)} \left[K_{11}\big(x,\mu^-(x)+0\big) - K_{11}\big(x,\mu^-(x)-0\big)\right] F_0(\mu^-(x),t) = 0,$$

$$G_{xx}(x,t) := \frac{2}{1+\sqrt{\rho(t)}} \frac{\partial^2 K_{11}(x,\mu^+(t))}{\partial x^2} + \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} \frac{\partial^2 K_{11}(x,\mu^+(2d-t))}{\partial x^2}$$

$$+\frac{\partial^2 F(x,t)}{\partial x^2} + \int\limits_0^{\mu^+(x)} \frac{\partial^2 K_{11}(x,\xi)}{\partial x^2} F_0(\xi,t) d\xi + \sqrt{\rho(x)} F_0(\mu^+(x),t) \frac{\partial K_{11}(x,\xi)}{\partial x} \Big|_{\xi=\mu^+(x)}$$

$$+\sqrt{\rho(x)} F_0(\mu^+(x),t) \left[\frac{\partial K_{11}(x,\xi)}{\partial x} \Big|_{\xi=\mu^+(x)+0} - \frac{\partial K_{11}(x,\xi)}{\partial x} \Big|_{\xi=\mu^+(x)-0}\right]$$

$$+\sqrt{\rho(x)}F_0(\mu^+(x),t)\frac{d}{dx}K_{11}\left(x,\mu^+(x)\right)+\sqrt{\rho(x)}K_{11}\left(x,\mu^+(x)\right)\frac{\partial F_0(\mu^+(x),t)}{\partial x}$$
$$+\sqrt{\rho(x)}\left[K_{11}\left(x,\mu^-(x)+0\right)-K_{11}\left(x,\mu^-(x)-0\right)\right]\frac{\partial F_0(\mu^-(x),t)}{\partial x}$$
$$+\sqrt{\rho(x)}F_0(\mu^-(x),t)\frac{d}{dx}\left[K_{11}\left(x,\mu^-(x)+0\right)-K_{11}\left(x,\mu^-(x)-0\right)\right]=0. \quad (47)$$

Using (34) we can write the last equation as follows:

$$G_{tt}(x,t):=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{tt}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{tt}\left(x,\mu^+(2d-t)\right)$$
$$+\frac{\partial^2}{\partial t^2}F(x,t)+\rho(t)\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial^2 F_0(\xi,t)}{\partial\xi^2}d\xi=0. \quad (48)$$

Then using the formula(15) we have

$$\frac{1}{\rho(t)}\frac{\partial^2}{\partial t^2}G(x,t):=\frac{2\rho(t)}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{1}{\rho(t)}$$
$$\times\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^+(2d-t)\right)+\frac{1}{\rho(t)}\frac{\partial^2}{\partial t^2}F(x,t)+\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial^2 F_0(\xi,t)}{\partial\xi^2}d\xi=0. \quad (49)$$

By integrating in parts we obtain

$$\int_0^{\mu^+(x)}K_{11}(x,\xi)\tfrac{\partial^2 F_0(\xi,t)}{\partial\xi^2}d\xi=\left[K_{11}(x,\mu^-(x)+0)-K_{11}(x,\mu^-(x)-0)\right]\frac{\partial F_0(\xi,t)}{\partial\xi}\Big|_{\xi=\mu^-(x)}$$
$$+K_{11}\left(x,\mu^+(x)\right)\frac{\partial}{\partial\xi}F_0(\xi,t)\Big|_{\xi=\mu^+(x)}-F_0\left(x,\mu^-(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^-(x)-0}$$
$$+F_0(x,0)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\Big|_{\xi=0}-F_0\left(x,\mu^+(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^+(x)-0}$$
$$+F_0\left(x,\mu^-(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\Big|_{\xi=\mu^-(x)+0}+\int_0^{\mu^+(x)}\frac{\partial^2 K_{11}(x,\xi)}{\partial^2\xi}F_0(\xi,t)d\xi. \quad (50)$$

Therefore

$$G_{tt}(x,t)=\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}(x,\mu^+(t))+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial t^2}K_{11}(x,2d-t)+$$
$$\frac{1}{\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}F(x,t)$$
$$+\left[K_{11}\left(x,\mu^-(x)-0\right)-K_{11}\left(x,\mu^-(x)+0\right)\right]\frac{\partial}{\partial\xi}F_0(\xi,t)\Big|_{\xi=\mu^-(x)}$$

$$+K_{11}\left(x,\mu^{+}(x)\right)\frac{\partial}{\partial\xi}F_0(\xi,t)\,|_{\xi=\mu^{+}(x)}\,-F_0\left(x,\mu^{-}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{-}(x)-0}$$

$$+F_0(x,0)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=0}\,-F_0\left(x,\mu^{+}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{+}(x)-0}$$

$$+F_0\left(x,\mu^{-}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{-}(x)+0}+\int_0^{\mu^{+}(x)}\frac{\partial^2 K_{11}(x,\xi)}{\partial^2\xi}F_0(\xi,t)d\xi. \tag{51}$$

It follows from (45), (46), and (50), the identity

$$G_{xx}(x,t)-\rho(x)\,G_{tt}(x,t)-\left[\frac{A}{x}+q(x)\right]G(x,t)\equiv 0.$$

Using the identity acording to formulas (9),(16)- (22), we get

$$\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial x^2}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial x^2}K_{11}(x,2d-t)$$

$$-\rho(x)\left[\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial t^2}K_{11}(2d-t)\right]$$

$$-\left[\frac{A}{x}+q(x)\right]\left[\frac{2}{1+\sqrt{\rho(t)}}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(2d-t)\right]$$

$$\int_0^{\mu^{+}(x)}\left\{K_{11xx}(x,t)-\rho(x)\,K_{11tt}(x,t)-\left[\frac{A}{x}+q(x)\right]K_{11}(x,t)\right\}F_0(\xi,t)d\xi=0. \tag{52}$$

By the Theorem 3 in the first section, the equation (52) has only trivial solution i.e.,

$$K_{11xx}(x,t)-\rho(x)\,K_{11tt}(x,t)-\left[\frac{A}{x}+q(x)\right]K_{11}(x,t)\equiv 0,\quad 0<t<x. \tag{53}$$

Now differentiating equation (42) twice, we have

$$s'(x,\lambda)=s_0'(x,\lambda)+\int_0^{\mu^{+}(x)}(K_{11})_x(x,t)\sin\lambda t dt+\sqrt{\rho(x)}K_{11}\left(x,\mu^{+}(x)\right)\sin\lambda\mu^{+}(x)$$

$$+\sqrt{\rho(x)}\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]\sin\lambda\mu^{-}(x)$$

$$s''(x,\lambda)=s_0''(x,\lambda)+\int_0^{\mu^{+}(x)}(K_{11})_{xx}(x,t)\sin\lambda t dt$$

$$
\begin{aligned}
&+\sqrt{\rho(x)}\sin\lambda\mu^{+}(x)\left[\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)\right)\right]_{t=\mu^{-}(x)}\\
&+\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)\left[\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)+0\right)|_{t=\mu^{-}(x)+0}\right.\\
&\left.-\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)-0\right)|_{t=\mu^{-}(x)-0}\right]\\
&+\sqrt{\rho(x)}\sin\lambda\mu^{+}(x)\frac{\partial}{\partial x}\left[K_{11}\left(x,\mu^{-}(x)\right)\right]\\
&+\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)\frac{\partial}{\partial x}\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]\\
&+\lambda\rho(x)K_{11}\left(x,\mu^{+}(x)\right)\cos\lambda\mu^{+}(x)\\
&+\lambda\rho(x)\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]\cos\lambda\mu^{-}(x)
\end{aligned}
$$

□

Lemma 3. *For each function $g(x)\in L_2(0,\pi)$ the following relation holds:*

$$\int_0^\pi \rho(x)\,g^2(x)\,dx=\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \rho(t)\,s(t,\lambda_n)\,dt\right)^2 \tag{54}$$

Proof. Using the formulas (7), (8), (9) of the previous section it is easy to transform solution

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^x \rho(x)\,s(x,t)\,s_0(t,\lambda)\,dt \tag{55}$$

and the main integral equation (14) form the previous section to the form

$$w(x,t)+F(x,t)+\int_0^x \rho(\xi)\,w(x,\xi)\,F(\xi,t)\,d\xi=0, \tag{56}$$

where

$$w(x,t)=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{11}(x,\mu^{+}(t))+\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(x,\mu^{+}(2d-t)).$$

Solving the equation (55) with respect to $s_0(x,\lambda)$ we obtain
For $x<d$

$$s_0(x,\lambda)=s(x,\lambda)+\int_0^x \rho(t)\,H(x,t)\,s(x,\lambda)\,dt. \tag{57}$$

By the standart method (see [20]) it can be proved that

$$H(x,t) = F(x,t) + \int_0^t \rho(\xi)\, w(t,\xi)\, F(x,\xi)\, d\xi, 0 \le t \le x. \tag{58}$$

Denote $Q(\lambda) = \int_0^\pi \rho(t)\, g(t)\, \varphi(t,\lambda)\, dt$. Then using (55) we have

$$Q(\lambda) = \int_0^\pi \rho(t)\, h(t)\, s_0(t,\lambda)\, dt$$

where

$$h(t) = g(t) + \int_t^\pi \rho(\xi)\, g(\xi)\, w(t,\xi)\, d\xi. \tag{59}$$

By the similar way, using the formula (57) we obtain

$$g(t) = h(t) + \int_t^\pi \rho(\xi)\, h(\xi)\, H(\xi,t)\, d\xi. \tag{60}$$

Now according to equation (59) we have

$$\begin{aligned} &\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt \\ &= \int_0^x \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, W(\xi)\, F(x,\xi)\, d\xi \right] dt \\ &+ \int_x^\pi \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, W(\xi)\, F(x,\xi)\, d\xi \right] dt. \end{aligned} \tag{61}$$

Consequently, by the formulas (56) and (58) we obtain

$$\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt = \int_0^\pi \rho(t)\, g(t)\, H(x,t)\, dt - \int_x^\pi \rho(t)\, g(t)\, W(x,t)\, dt \tag{62}$$

From the Parseval equality we have

$$\int_0^\pi \rho(t)\, h^2(t)\, dt + \int_0^\pi \rho(t)\, \rho(x)\, h(t)\, h(x)\, F(x,t)\, dxdt$$

$$= \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^{\pi} h(t)\, g(t)\, s_0(t, \lambda_n)\, \varphi(t, \lambda_n)\, dt \right)^2 = \sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n}.$$

Using (61) we get

$$\sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n} = \int_0^{\pi} \rho(t)\, h^2(t)\, dt + \int_0^{\pi} \rho(t)\, g(t) \left[\int_t^{\pi} \rho(x)\, h(x)\, H(x,t)\, dx \right] dt$$
$$- \int_0^{\pi} \rho(x)\, h(x) \left[\int_x^{\pi} \rho(t)\, g(t)\, W(t,x)\, dx \right] dt.$$

Finally, from (59) and (60) we get

$$\sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n} = \int_0^{\pi} \rho(t)\, h^2(t)\, dt + \int_0^{\pi} \rho(t)\, g(t)\, (g(t) - h(t))\, dt$$
$$- \int_0^{\pi} \rho(x)\, g(x)\, (g(x) - h(x))\, dx = \int_0^{\pi} \rho(t)\, g^2(t)\, dt.$$

The lemma is proved. □

Corollary 1. *For arbitrary functions $f, g \in L_2(0, \pi)$, the following relation holds:*

$$\int_0^{\pi} \rho(x)\, f(x)\, g(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \int_0^{\pi} f(t)\, s(t, \lambda_n)\, dt \int_0^{\pi} g(t)\, s(t, \lambda_n)\, dt. \tag{63}$$

Lemma 4. *The following relations hold:*

$$\int_0^{\pi} s(t, \lambda_k)\, s(t, \lambda_n)\, dt = \begin{cases} 0, n \neq k \\ \alpha_n, n = k \end{cases} \tag{64}$$

Proof. Let $f(x) \in W_2^2(0, \pi)$, consider the series

$$f^*(x) = \sum_{n=0}^{\infty} c_n s(x, \lambda_n), \tag{65}$$

where

$$c_n = \frac{1}{\alpha_n} \int_0^{\pi} f(x)\, s(x, \lambda_n)\, dx. \tag{66}$$

Using Lemma 1 and integrating by parts we calculate:

$$c_n = \frac{1}{\alpha_n \lambda_n^2} \Big(h f(0) - f'(0) + s(\pi, \lambda_n)\, f'(\pi) - \varphi(\pi, \lambda_n)\, f(\pi)$$

$$+\int_0^\pi s(x,\lambda_n)\left[-f''(x)+g(x)\right]dx\Bigg)$$

From the asymptotic formulas for the $\varphi(x,\lambda)$ and λ_n in [3], we get

$$c_n = O\left(\frac{1}{n^2}\right), s(x,\lambda_n) = O(1)$$

uniformly for $x \in [0,\pi]$. Therefore the series (64) converges absolutely and uniformly on $[0,\pi]$. Using (62) and (65) we obtain

$$\int_0^\pi \rho(x) f(x) g(x)\, dx = \int_0^\pi g(t) \sum_{n=0}^{\infty} c_n s(x,\lambda_n)\, dt = \int_0^\pi g(t) f^*(t)\, dt.$$

Since $g(x)$ is arbitrary, we get

$$f^*(x) = f(x) = \sum_{n=0}^{\infty} c_n s(x,\lambda_n). \tag{67}$$

Now, for fix $k \geq 0$ and take $f(x) = \varphi(x,\lambda_k)$, then since (66)

$$s(x,\lambda_k) = \sum_{n=0}^{\infty} c_{n_k} s(x,\lambda_n), c_{n_k} = \frac{1}{\alpha_n} \int_0^\pi s(x,\lambda_k) s(x,\lambda_n)\, dx.$$

Moreover, the system $\{s_0(x,\lambda_n)\}_{n\geq 0}$ is minimal in $L_2(0,\pi)$, (see theorem 2 of the previous section), and consequently, in view of (42) the system $\{\varphi(x,\lambda_n)\}_{n\geq 0}$ is also minimal in $L_2(0,\pi)$. Therefore $c_{n_k} = \delta_{n_k}$ (δ_{n_k} is a Kronecker symbol). The lemma is proved. □

Now we can give the algorithm to construct the problem $L(q(x))$ using the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ as follows:

1- Use formulas (8) and (9), to construct the functions $F_0(x,t)$ and $F(x,t)$.

2- Construct the function $K(x,t)$ as the unique solution of the main integral equation.

3- Calculate the function $q(x)$ and coefficient A by the formulas in Theorem 1.

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