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Estimation of Risk Measures for Transmuted Weibull Distribution

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Abstract

In this paper, we tackle a problem of the estimation of some risk measures for transmuted Weibull distribution. In this regard, the maximum likelihood method is used to estimate the risk measures. We also obtain asymptotic confidence intervals based on the asymptotic distributions of maximum likelihood estimators of risk measures. Then, we consider a comprehensive Monte Carlo simulation study to assess the performances of these estimators at different sample sizes and parameter settings.

Keywords: Risk measures; Transmuted Weibull distribution; Point estimation; Interval estimation.

Dönüştürülmüş Weibull Dağılımı için Risk Ölçülerinin Tahmini

Öz

Bu çalışmada dönüştürülmüş Weibull dağılımı için bazı risk ölçülerinin tahmini problemini ele aldık. Bu bağlamda risk ölçülerini tahmin edebilmek için en çok olabilirlik yöntemi kullanıldı. Ayrıca risk ölçülerinin en çok olabilirlik tahmin edicilerinin asimptotik dağılımlarına dayalı yaklaşık güven aralıkları elde ettik. Sonrasında, bu tahmin edicilerin farklı örnek hacimleri ve parametre değerlerinde performanslarını değerlendirmek için geniş bir Monte Carlo benzetim çalışması tasarladık.

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Anahtar Kelimeler: Risk ölçüleri; Dönüştürülmüş Weibull dağılımı; Nokta tahmini; Aralık tahmini.

1. Introduction

Transmuted Weibull distribution is suggested by [1] via quadratic transmutation map (QRTM). The QRTM is proposed by [2], and it is summarized by

$$F(x) = G(x)\left[1 + \lambda\left(1 - G(x)\right)\right], \tag{1}$$

where $\lambda \in [-1,1]$, $G(x)$ refers the cumulative distribution function (CDF) of baseline distribution, and $F(x)$ denotes the CDF referring transmuted distribution which newly generated by the QTRM. Consider the baseline distribution Weibull distribution with CDF $G(x;\alpha,\beta) = 1 - exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}$ and the probability density function (PDF) $g(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}$ then, the PDF and CDF of transmuted Weibull distribution are

$$F(x;\alpha,\beta,\lambda) = \left[1 - exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right]\left[1 + \lambda\, exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right], \tag{2}$$

and

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\left[1 - \lambda + 2\lambda\, exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right], \tag{3}$$

respectively, where $\beta > 0$ is a scale parameter, $\alpha > 0$ shape parameter and $\lambda \in [-1,1]$ [1]. In this study, the transmuted Weibull distribution is briefly denoted by $TW(\alpha,\beta,\lambda)$. The $TW(\alpha,\beta,\lambda)$ distribution has a potential to model the data sets in many fields such as, agriculture, biology, economics, actuarial sciences. Aryal and Tsokos [1] described some characteristic properties such as moments, variance, quantile function, reliability function, hazard function, order statistics of $TW(\alpha,\beta,\lambda)$ distribution. They emphasized that the hazard function can be increasing, decreasing or constant for $TW(\alpha,\beta,\lambda)$ distribution in [1]. In this case, it can be said that due to the flexible of the hazard function, the $TW(\alpha,\beta,\lambda)$ distribution has the potential to model many datasets having different hazard functions. Khan et al. [3] examined some statistical properties such as geometric mean, harmonic mean, entropies, mean deviation, L-moments of $TW(\alpha,\beta,\lambda)$ distribution. They also provided the log-transmuted Weibull regression model and its applications in [3]. For more details about $TW(\alpha,\beta,\lambda)$ distribution please see [1, 3].

Recently, many actuaries and insurance practitioners have focused on the measurement of financial risk. The risk measures manifest themselves in many different types of insurance problems including the determination of capital, and the estimation of possible maximum losses [4]. Therefore, we focus on the estimation of risk measures for $TW(\alpha,\beta,\lambda)$ distribution.

The main purpose of this paper to tackle the problem of point and interval estimation of risk measures for the $TW(\alpha, \beta, \lambda)$ distribution. We estimate the risk measures such as value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance Premium (TVP) for the $TW(\alpha, \beta, \lambda)$ distribution. The rest of this study is organized as follows: In Section 2, we describe the risk measures for the $TW(\alpha, \beta, \lambda)$ distribution. Then, the maximum likelihood estimators (MLEs) of these risk measures and asymptotic confidence intervals based on MLEs are derived in Section 3. In Section 4, an extensive Monte Carlo simulation study designed to evaluate the performances of these estimators according to mean squares errors (MSEs) and bias.

2. Risk Measures

2.1. VaR measure

The VaR is one of the popular risk measures, and it quantifies maximum loss for investments. It is also known quantile risk measure. The VaR is generally used by firms and regulators in the financial sector in order to determine the amount of assests required to cover potential losses. The VaR of a random variable X is the *qth* quantile of its cdf, denoted by VaRq, and it is defined by $VaR_q = Q(q)$ [5-7].

Let X be a random variable from $TW(\alpha, \beta, \lambda)$distribution. The VaR is defined as follows:

$$VaR_q = \beta \left[-log\left(1 - \frac{\lambda+1-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)\right]^{\frac{1}{\alpha}}, \tag{4}$$

where $q \in (0,1)$.

2.2. TVaR measure

TVaR, also known as tail conditional expectation is important risk measure. It measures the expected value of the loss given that an event outside a given probability level has occurred [6, 8, 9]. The TVaR of $TW(\alpha, \beta, \lambda)$ distribution is

$$TVaR_q = \frac{1}{1-q} \int_{VaR_q}^{1} xf\,(x)\,dx$$

$$= \frac{1}{1-q} \Gamma\left(1 + \frac{1}{\alpha}, \left(\frac{VaR_q}{\beta}\right)^{\alpha}\right), \tag{5}$$

where $\Gamma(., x)$ is incomplete gamma function, and VaR_q is given in Eqn. (4).

2.3. TV measure

The TV is important risk measure suggested by [10]. The TV of $TW(\alpha, \beta, \lambda)$ distribution is given by

$$TV_q\, X = E(\, X^2 \mid X > x_q) - \{TVaR_q\}^2$$

$$= \frac{1}{1-q} \int_{VaR_q}^{1} x^2 f\,(x)\, dx - \{TVaR_q\}^2$$

$$= \frac{\beta^2}{1-q} \Gamma\left(1 + \frac{2}{\alpha}, \left(\frac{VaR_q}{\beta}\right)^{\alpha}\right) - \{TVaR_q\}^2\,, \qquad (6)$$

where $TVaR_q$ is given in Eqn. (5).

2.4. TVP measure

The TVP is one of the significant measures of risk which play a crucial role in insurance sciences [9]. The TVP of $TW(\alpha, \beta, \lambda)$distribution is

$$TVP_q = TVaR_q + \theta TV_q, \qquad (7)$$

where $0 < \theta < 1$, $TVaR_q$ and TV_q are defined in Eqn. (5) and Eqn. (6), respectively.

3. Estimation of Risk Measures

3.1. Maximum Likelihood Estimation of risk measures

In order to obtain the MLEs of examined risk measures, we first derive MLEs of α, β and λ.

Let $X_1, X_2, \ldots, X_n$ be a random sample from $TW(\alpha, \beta, \lambda)$ distribution. Then log-likelihood function is given by [1, 2]

$$\boldsymbol{\ell}(\Psi) = n\, log\left(\frac{\alpha}{\beta}\right) - \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\alpha} + \sum_{i=1}^{n} log\left(\frac{x_i}{\beta}\right)^{\alpha-1} + \sum_{i=1}^{n\sum\left[1-\lambda+2\lambda\, exp\left(-\frac{x_i}{\beta}\right)^{\alpha}\right]} log \qquad (8)$$

where $\Psi = (\alpha, \beta, \lambda)$. The MLE of Ψ is given by

$$\widehat{\Psi} = \underset{\Psi}{argmax}\{\boldsymbol{\ell}(\Psi)\} \qquad (9)$$

By using Eqns. (4)-(7) and invariant property of MLE, we can compute the MLEs of mentioned risk measures of VaR, TVaR, TV, and TVP by

$$\hat{V}aR_q = \hat{\beta}\left[-log\left(1-\frac{\hat{\lambda}+1-\sqrt{(1+\hat{\lambda})^2-4\hat{\lambda}q}}{2\hat{\lambda}}\right)\right]^{\frac{1}{\hat{\alpha}}}, \tag{10}$$

$$\hat{T}VaR_q = \frac{1}{1-q}\Gamma\left(1+\frac{1}{\hat{\alpha}},\left(\frac{\hat{V}aR_q}{\hat{\beta}}\right)^{\hat{\alpha}}\right), \tag{11}$$

$$\hat{T}V_q = \frac{\hat{\beta}^2}{1-q}\Gamma\left(1+\frac{2}{\hat{\alpha}},\left(\frac{\hat{V}aR_q}{\hat{\beta}}\right)^{\hat{\alpha}}\right)-\left\{\hat{T}VaR_q\right\}^2, \tag{12}$$

and

$$\hat{T}VP_q = \hat{T}VaR_q + \theta\hat{T}V_q, \tag{13}$$

respectively.

3.2. Asymptotic confidence interval

In this subsection, we provide the asymptotic variances and covariances of the MLEs $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$ by entries of the inverse of the observed Fisher information matrix is given by

$$I^{-1}(\hat{\Psi}) = \begin{pmatrix} -\frac{\partial^2\ell(\Psi)}{\partial\alpha^2} & -\frac{\partial^2\ell(\Psi)}{\partial\alpha\partial\beta} & -\frac{\partial^2\ell(\Psi)}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2\ell(\Psi)}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell(\Psi)}{\partial\beta^2} & -\frac{\partial^2\ell(\Psi)}{\partial\beta\partial\lambda} \\ -\frac{\partial^2\ell(\Psi)}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\Psi)}{\partial\lambda\partial\beta} & -\frac{\partial^2\ell(\Psi)}{\partial\lambda^2} \end{pmatrix}.$$

Now, we can obtain the variance of $Var(\hat{R})$ using delta method as $Var(\hat{R}) = B'I^{-1}(\hat{\Psi})B$ where R denotes one of the risk measures (VaR, TVaR, TV, TVP), $\hat{R}$is the MLE of R and $B' = \left(\frac{\partial R}{\partial\alpha},\frac{\partial R}{\partial\beta},\frac{\partial R}{\partial\lambda}\right)$. By using the MLEs of α,β and λ $Var(\hat{R})$can be estimated. The asymptotic $100(1-\eta)\%$ confidence interval of risk measures by

$$\left(\hat{R}-z_{1-\frac{\eta}{2}}\sqrt{Var(\hat{R})}, \hat{R}-z_{1-\frac{\eta}{2}}\sqrt{Var(\hat{R})}\right)$$

where $z_\eta 100\ \eta^{th}$ percentile of $N(0,1)$.

4. Simulation Study

In this section, we design a comprehensive Monte Carlo simulation study to assess the performances of MLEs, of risk measures according to biases and MSEs. The simulation study is performed based on 5000 repetitions. We consider the sample size 25, 50, 100, 200, 500 and two parameter settings as follows:$(\alpha = 0.5, \beta = 1.5, \lambda = 0.2), (\alpha = 1, \beta = 2, \lambda = 0.5)$. The results of simulation study are presented in Tables 1-2. Table 1 provides average of biases and MSEs of risk measures such as VaR, TVaR, TV, and TVP. Also, Table 2 presents the average of lengths and coverage probabilities (CPs) of these risk measures.

Table 1: Average biases and MSEs of risk measures

						bias				MSE			
n	α	β	λ	Sig.level	θ	VaR	TVaR	TV	TVP	VaR	TVaR	TV	TVP
25						-0.0629	-0.0096	-0.006	-0.012	0.0883	0.0018	0.0004	0.0026
50						-0.0406	-0.0043	-0.0035	-0.0061	0.0433	0.0008	0.0001	0.0011
100	0.5	1.5	0.2	0.5	0.5	-0.0195	-0.0006	-0.0021	-0.0017	0.0188	0.0003	0.00008	0.0005
200						-0.0116	0.0005	-0.0019	-0.0004	0.0093	0.0002	0.00007	0.0003
500						-0.006	0.001	-0.0013	0.0003	0.0037	0.0001	0.00004	0.0001
25						-0.0463	-0.0405	0.0485	-0.0162	1.0149	0.0841	0.0505	0.0407
50						0.0116	0.002	0.0004	0.003	0.4698	0.0368	0.0117	0.0234
100	0.5	1.5	0.2	0.75	0.5	0.0089	0.0208	-0.0152	0.0132	0.231	0.0199	0.0047	0.014
200						0.0218	0.028	-0.0201	0.018	0.119	0.0127	0.0029	0.0093
500						0.0172	0.0249	-0.0201	0.014	0.0485	0.0069	0.0018	0.0054
25						-0.03	-0.0151	-0.0013	-0.016	0.0574	0.0019	0.001	0.0034
50						-0.0151	-0.0096	0.0022	-0.0082	0.0253	0.0008	0.0004	0.0014
100	1	2	0.5	0.4	0.6	-0.006	-0.006	0.0029	-0.0042	0.0149	0.0004	0.0002	0.0007
200						-0.0056	-0.0047	0.0027	-0.003	0.0075	0.0002	0.0001	0.0003
500						-0.0049	-0.0029	0.0014	-0.002	0.0024	0.0001	0.0001	0.0001
25						0.0448	-0.1257	0.7922	0.3495	0.7267	0.4067	3.8147	0.4188
50						0.0327	-0.0684	0.501	0.2321	0.2071	0.2191	1.8846	0.2019
100	1	2	0.5	0.8	0.6	0.0095	-0.0226	0.2892	0.1508	0.6506	0.1288	1.0662	0.1183
200						0.01287	0.0029	0.1564	0.0967	0.0663	0.0841	0.7361	0.083
500						-0.0022	0.0064	0.0941	0.0628	0.0196	0.0487	0.5343	0.0601

Table 2: Average lengths and CPs of risk measures

						length				CP			
n	α	β	λ	Sig.level	θ	VaR	TVaR	TV	TVP	VaR	TVaR	TV	TVP
25						1.0773	0.1596	0.075	0.19	0.9138	0.923	0.84	0.9208
50						0.7571	0.1129	0.0506	0.134	0.9316	0.9364	0.848	0.939
100	0.5	1.5	0.2	0.5	0.5	0.5284	0.0804	0.036	0.0958	0.943	0.9428	0.8404	0.9424
200						0.374	0.0596	0.0278	0.0716	0.9494	0.945	0.8436	0.944
500						0.2387	0.0405	0.0204	0.0494	0.949	0.9304	0.8322	0.9264
25						3.741	1.1613	0.8475	0.8683	0.8836	0.9104	0.8476	0.9192
50						2.6325	0.8108	0.4831	0.6535	0.9074	0.926	0.865	0.9344
100	0.5	1.5	0.2	0.75	0.5	1.8941	0.595	0.3286	0.4978	0.9292	0.9316	0.8786	0.9404
200						1.3459	0.4449	0.242	0.3815	0.9356	0.9262	0.8838	0.933
500						0.8693	0.3003	0.1645	0.2651	0.9452	0.9342	0.8932	0.9316
25						0.7477	0.1629	0.1262	0.2025	0.9172	0.9494	0.8898	0.9352
50	1	2	0.5	0.4	0.6	0.535	0.1113	0.0915	0.1362	0.9358	0.9582	0.9038	0.9454
100						0.3797	0.0779	0.0687	0.0934	0.9394	0.9498	0.9094	0.9472

200						0.2704	0.057	0.053	0.0658	0.9438	0.9458	0.9142	0.953
500						0.1716	0.0379	0.0374	0.0412	0.9496	0.9172	0.9126	0.9554
25						1.928	2.7358	8.9104	2.8183	0.898	0.9482	0.9524	0.9646
50						1.4025	2.0489	6.4002	1.9501	0.9094	0.9544	0.9558	0.9606
100	1	2	0.5	0.8	0.6	1.0116	1.5629	4.8082	1.4423	0.9282	0.9498	0.9314	0.8688
200						0.7256	1.2158	3.756	1.1273	0.9384	0.9286	0.881	0.7732
500						0.4699	0.8916	2.8454	0.8698	0.9442	0.8804	0.8248	0.7528

From Tables 1-2, It is seen that as the sample size increases, the MSEs and biases of risk measures decrease and approach zero. Also, we observed that the lengths decrease and CPs approach 0.95 as expected. In the case of high significance level (it is defined in Eqn. (4) as q), the MSEs and biases of the risk measures are larger than in other cases.

5. Conclusion

In this study, we provide some risk measures such as VaR, TVaR, TV, and TVP for $TW(\alpha,\beta,\lambda)$ distribution. We use the maximum likelihood method to estimate these risk measures. Then, we obtain MLEs of examined risk measures using the invariant property of MLE. Not only point estimates of risk measures but also interval estimates are discussed. Approximate confidence intervals based on the asymptotic distribution of MLE were obtained. Monte Carlo simulations are performed to observe the performance of the estimators according to MSE and bias. From the results of the simulation study, it is observed that the MLEs of risk measures provided the estimation procedures.

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