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OPERATOR INEQUALITIES IN REPRODUCING KERNEL HILBERT SPACES

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Abstract. In this paper, by using some classical Mulholland type inequality, Berezin symbols and reproducing kernel technique, we prove the power inequalities for the Berezin number $ber(A)$ for some self-adjoint operators A on $\mathcal{H}(\Omega)$. Namely, some Mulholland type inequality for reproducing kernel Hilbert space operators are established. By applying this inequality, we prove that $(ber(A))^n \leq C_1 ber(A^n)$ for any positive operator A on $\mathcal{H}(\Omega)$.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} \frac{1}{m} a_m^p < +\infty$ and $0 < \sum_{n=2}^{\infty} \frac{1}{n} b_n^q$, then the Mulholland's inequality [13, 20] is given by

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=2}^{\infty} \frac{1}{n} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty} \frac{1}{n} b_n^q\right\}^{\frac{1}{q}} \tag{1}$$

and an equivalent form is

$$\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} \frac{a_m}{m \ln mn}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=2}^{\infty} \frac{1}{n} a_n^p, \tag{2}$$

where the constants $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible.

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The integral analogues of (1) and (2) are as follows:

$$\int_1^\infty \int_1^\infty \frac{f(x)\, g(y)}{xy \ln xy} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_1^\infty \frac{f^p(x)}{x} dx \right)^{1/p} \left(\int_1^\infty \frac{g^q(y)}{y} dy \right)^{1/q}, \tag{3}$$

$$\int_1^\infty \left(\int_1^\infty \frac{f(x)}{xy \ln xy} dx \right) dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_1^\infty \frac{f^p(x)}{x} dx.$$

Inequalities (1) and (3) are called the Mulholland's inequality and Mulholland's integral inequality, respectively (see [13, 20]). Some generalizations of these type inequalities are given in [5, 8, 11, 12, 14, 29].

Denote by $\mathcal{F}(\Omega)$ the set of all complex valued functions on some set Ω. A reproducing kernel Hilbert space (RKHS for short) on the set Ω is a Hilbert space $\mathcal{H} \subset \mathcal{F}(\Omega)$ with a function $k_\lambda : \Omega \times \Omega \to \mathcal{H}$, which is called the reproducing kernel enjoying the reproducing property $k_\lambda := k(., \lambda) \in \mathcal{H}$ for all $\lambda \in \Omega$ and

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$$

holds for all $\lambda \in \Omega$ and all $f \in \mathcal{H}$ (see [1, 23]).

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For any bounded linear operator A on $\mathcal{H}$, the Berezin symbol of A is the function $\widetilde{A}$ defined by (see [4])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

Recall that the Berezin set and the Berezin number for an operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$ were introduced in [15, 16] as follows:

$$Ber(A) := Range(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \quad \text{(Berezin set)}.$$

$$ber(A) := \sup \left\{ \left| \widetilde{A}(\lambda) \right| : \lambda \in \Omega \right\} \quad \text{(Berezin number)}.$$

Clearly, $Ber(A) \subset W(A) := \{\langle Ax, x\rangle : \|x\|_{\mathcal{H}} = 1\}$ (numerical range) and $ber(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : \|x\|_{\mathcal{H}} = 1\}$ (numerical radius). More information about numerical range and numerical radius can be found in [6, 7, 9, 18, 19, 21].

Using the Hardy-Hilbert type inequalities and some well-known inequalities, some important results about the Berezin number inequalities were obtained in [2, 3, 10, 22, 24–28].

In the present paper, by using inequalities (1), (2) and some ideas of paper [17], we will estimate Berezin number (which is a new numerical value of the bounded linear operators on RKHS) of operators acting in the reproducing kernel Hilbert spaces.

2. Mulholland Type Inequalities and Berezin Number of Some Operators

In the following result, we prove an analog of inequality (1) for some self-adjoint operators on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 1. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be two continuous functions defined on an interval $\Delta \subset (0, +\infty)$ and $f, g \geq 0$. Then the following is true:*

$$\begin{aligned}
&\frac{8\widetilde{f(A)g(A)}(\lambda) + \widetilde{f(C)\,g(C)}(\xi)}{32\ln 4} + \frac{\widetilde{f(A)}(\lambda)\widetilde{g(B)}(\mu) + \widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)}{6\ln 6} \qquad (4)\\
&+ \frac{\widetilde{f(A)}(\lambda)\widetilde{g(C)}(\xi) + \widetilde{f(C)}(\xi)\,\widetilde{g(A)}(\lambda)}{8\ln 8} + \frac{\frac{1}{2}\widetilde{f(B)g(B)}(\mu)}{9\ln 9}\\
&+ \frac{\widetilde{f(B)}(\mu)\widetilde{g(C)}(\xi) + \widetilde{f(C)}(\xi)\,\widetilde{g(B)}(\mu)}{12\ln 12}\\
&< \frac{\pi}{\sin(\pi/p)}\left\langle \left(\frac{f^p(A)}{2} + \frac{f^p(B)}{3} + \frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2} + \frac{g^q(B)}{3} + \frac{g^q(C)}{4}\right)^{1/q}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle,
\end{aligned}$$

for all self-adjoint operators $A, B, C \in \mathcal{B}(\mathcal{H}(\Omega))$ with spectra contained in Δ and for all μ, λ, $\xi \in \Omega$.

Proof. Let $a_2, a_3, a_4, b_2, b_3, b_4$ be positive scalars. Then using inequality (1), we have

$$\begin{aligned}
&\frac{8a_2b_2 + a_4b_4}{32\ln 4} + \frac{a_2b_3 + a_3b_2}{6\ln 6} + \frac{a_2b_4 + a_4b_2}{8\ln 8} + \frac{a_3b_3}{9\ln 9} + \frac{a_3b_4 + a_4b_3}{12\ln 12} \qquad (5)\\
&< \frac{\pi}{\sin(\pi/p)}\left(\frac{a_2^p}{2} + \frac{a_3^p}{3} + \frac{a_4^p}{4}\right)^{1/p}\left(\frac{b_2^q}{2} + \frac{b_3^q}{3} + \frac{b_4^q}{4}\right)^{1/q}.
\end{aligned}$$

Let $x, y, z \in \Delta$. By the hypothyses of the theorem $f(x) \geq 0$, $g(x) \geq 0$ for all $x \in \Delta$. If we put $a_2 = f(x)$, $a_3 = f(y)$, $a_4 = f(z)$, $b_2 = g(x)$, $b_3 = g(y)$, $b_4 = g(z)$ in (5), then we have

$$\begin{aligned}
&\frac{8f(x)g(x) + f(z)\,g(z)}{32\ln 4} + \frac{f(x)g(y) + f(y)g(x)}{6\ln 6} \qquad (6)\\
&+ \frac{f(x)g(z) + f(z)\,g(x)}{8\ln 8} + \frac{f(y)g(y)}{9\ln 9} + \frac{f(y)g(z) + f(z)\,g(y)}{12\ln 12}\\
&< \frac{\pi}{\sin(\pi/p)}\left(\frac{f^p(x)}{2} + \frac{f^p(y)}{3} + \frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(x)}{2} + \frac{g^q(y)}{3} + \frac{g^q(z)}{4}\right)^{1/q}
\end{aligned}$$

for all $x, y, z \in \Delta$. Let A be a self-adjoint operator. Then, by using functional calculus and inequality (6), we get

$$\frac{8f(A)g(A) + f(z)\,g(z)}{32\ln 4} + \frac{f(A)g(y) + f(y)g(A)}{6\ln 6}$$

$$+\frac{f(A)g(z)+f(z)g(A)}{8\ln 8}+\frac{f(y)g(y)}{9\ln 9}+\frac{f(y)g(z)+f(z)g(y)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left(\frac{f^p(A)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(y)}{3}+\frac{g^q(z)}{4}\right)^{1/q},$$

and therefore, we have that

$$\frac{8\left\langle f(A)g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+f(z)g(z)}{32\ln 4}+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(y)+f(y)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{6\ln 6}$$
$$+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(z)+f(z)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{8\ln 8}+\frac{f(y)g(y)}{9\ln 9}+\frac{f(y)g(z)+f(z)g(y)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left\langle\left(\frac{f^p(A)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(y)}{3}+\frac{g^q(z)}{4}\right)^{1/q}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

for all $\lambda\in\Omega$ and any $y,z\in\Delta$.

Using the functional calculus once more to the self-adjoint operators B and C, we get

$$\frac{8\left\langle f(A)g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+f(C)g(C)}{32\ln 4}+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(B)+f(B)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{6\ln 6}$$
$$+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(C)+f(C)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{8\ln 8}+\frac{f(B)g(B)}{9\ln 9}+\frac{f(B)g(C)+f(C)g(B)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left[\left(\frac{f^p(A)}{2}+\frac{f^p(B)}{3}+\frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(B)}{3}+\frac{g^q(C)}{4}\right)^{1/q}\right]^{\sim}(\lambda).\tag{7}$$

Hence, we have from (7) that

$$\frac{8\widetilde{f(A)g(A)}(\lambda)+\widetilde{f(C)g(C)}(\xi)}{32\ln 4}+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(B)}(\mu)+\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)}{6\ln 6}$$
$$+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(A)}(\lambda)}{8\ln 8}+\frac{\widetilde{f(B)g(B)}(\mu)}{9\ln 9}$$
$$+\frac{\widetilde{f(B)}(\mu)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(B)}(\mu)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left\langle\left(\frac{f^p(A)}{2}+\frac{f^p(B)}{3}+\frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(B)}{3}+\frac{g^q(C)}{4}\right)^{1/q}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

for all self-adjoint operators $A,B,C\in\mathcal{B}(\mathcal{H}(\Omega))$ and for all $\lambda,\mu,\xi\in\Omega$. This proves the theorem. □

Corollary 1. *$(ber\,(f\,(A)))^2 < C_1 ber\left(f\,(A)^2\right)$ for any self-adjoint operator $A \in \mathcal{B}(H(\Omega))$ with spectrum contained in Δ; in particular,*

$$(ber\,(A))^2 < C_1 ber\left(A^2\right),$$

where $C_1 = \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)$.

Proof. Indeed, for $C = B = A$, $g = f$ and $\xi = \mu = \lambda$, we have from inequality (4) that

$$\frac{9\widetilde{f^2(A)}(\lambda)}{32\ln 4} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{6\ln 6} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{8\ln 8} + \frac{\widetilde{f^2(A)}(\lambda)}{18\ln 3} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{12\ln 12}$$
$$< \frac{13\pi}{12\sin(\pi/p)}\widetilde{f^2(A)}(\lambda),$$

or equivalently

$$\left(\frac{4\log_6 e + 3\log_8 e + 2\log_{12} e}{12}\right)\left[\widetilde{f(A)}(\lambda)\right]^2$$
$$< \left(\frac{13\pi}{12\sin(\pi/p)} - \frac{81\log_4 e + 16\log_3 e}{288}\right)\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. Since $\left[\widetilde{f(A)}(\lambda)\right]^2 \geq 0$ and $\widetilde{f(A)^2}(\lambda) \geq 0$, we have that

$$\sup_{\lambda\in\Omega}\left[\widetilde{f(A)}(\lambda)\right]^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)\sup_{\lambda\in\Omega}\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. This obviously implies that

$$ber(f(A))^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)ber(f^2(A));$$

in particular, for $f(x) = x$, we have that

$$ber(A)^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)ber(A^2).$$

□

Our more general result is the following theorem which gives a sharper estimate than Corollary 1.

Theorem 2. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let f be a continuous function defined on an interval $\triangle \subset (0, +\infty)$ and $f \geq 0$. Let $A : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ be a positive operator on a RKHS $\mathcal{H}(\Omega)$ with spectrum contained in $\triangle$. Then there exists a constant $C_1 > 1$ such that*

$$[ber(f(A))]^p < C_1 ber(f^p(A));$$

in particular, $ber(A)^p < C_1 ber(A^p)$*, where* $C_1 = 1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^p$.

Proof. Let a_2, a_3, a_4 be positive numbers. Then using (2), we have that

$$\begin{aligned}&\frac{1}{2}\left(\frac{a_2}{2\ln 4}+\frac{a_3}{3\ln 6}+\frac{a_4}{4\ln 8}\right)^p+\frac{1}{3}\left(\frac{a_2}{2\ln 6}+\frac{a_3}{3\ln 9}+\frac{a_4}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{a_2}{2\ln 8}+\frac{a_3}{3\ln 12}+\frac{a_4}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{a_2^p}{2}+\frac{a_3^p}{3}+\frac{a_4^p}{4}\right).\end{aligned} \tag{8}$$

Let $x, y, z \in \Delta$. Since $f(x) \geq 0$ for all $x \in \Delta$, by putting $a_2 = f(x), a_3 = f(y)$ and $a_4 = f(z)$ in (8), we have

$$\begin{aligned}&\frac{1}{2}\left(\frac{f(x)}{2\ln 4}+\frac{f(y)}{3\ln 6}+\frac{f(z)}{4\ln 8}\right)^p+\frac{1}{3}\left(\frac{f(x)}{2\ln 6}+\frac{f(y)}{3\ln 9}+\frac{f(z)}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{f(x)}{2\ln 8}+\frac{f(y)}{3\ln 12}+\frac{f(z)}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{f^p(x)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right).\end{aligned} \tag{9}$$

So, by using the same functional calculus arguments as in the proof of Theorem 1, finally we get from (9) that

$$\begin{aligned}&\frac{1}{2}\left(\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2\ln 4}+\frac{\left\langle f(B)\widehat{k}_\mu,\widehat{k}_\mu\right\rangle}{3\ln 6}+\frac{\left\langle f(C)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{4\ln 8}\right)^p \\ &+\frac{1}{3}\left(\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2\ln 6}+\frac{\left\langle f(B)\widehat{k}_\mu,\widehat{k}_\mu\right\rangle}{3\ln 9}+\frac{\left\langle f(C)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2\ln 8}+\frac{\left\langle f(B)\widehat{k}_\mu,\widehat{k}_\mu\right\rangle}{3\ln 12}+\frac{\left\langle f(C)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{\left\langle f^p(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}+\frac{\left\langle f^p(B)\widehat{k}_\mu,\widehat{k}_\mu\right\rangle}{3}+\frac{\left\langle f^p(C)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{4}\right).\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 4}+\frac{\widetilde{f(B)}(\mu)}{3\ln 6}+\frac{\widetilde{f(C)}(\xi)}{4\ln 8}\right)^p \\ &+\frac{1}{3}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 6}+\frac{\widetilde{f(B)}(\mu)}{3\ln 9}+\frac{\widetilde{f(C)}(\xi)}{4\ln 12}\right)^p\end{aligned}$$

$$+\frac{1}{4}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 8}+\frac{\widetilde{f(B)}(\mu)}{3\ln 12}+\frac{\widetilde{f(C)}(\xi)}{4\ln 16}\right)^p$$
$$<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{\widetilde{f^p(A)}(\lambda)}{2}+\frac{\widetilde{f^p(B)}(\mu)}{3}+\frac{\widetilde{f^p(C)}(\xi)}{4}\right).$$

for all positive operators A, B, C which spectrum contained in $\triangle$ and all $\lambda, \mu, \xi \in \Omega$. Now by replacing $C = B = A$ and $\xi = \mu = \lambda$, we have from the latter equality that

$$\left[\frac{129\log_2 e+32\log_3 e+192\log_6 e+96\log_{12} e}{576}\right]\left[\widetilde{f(A)}(\lambda)\right]^p<\frac{13}{12}\left[\frac{\pi}{\sin(\pi/p)}\right]^p\left[\widetilde{f^p(A)}(\lambda)\right],$$

for all $\lambda \in \Omega$. Since $\widetilde{f^p(A)}(\lambda) \geq 0$ for all $\lambda \in \Omega$ and for all $p > 1$, the last inequality shows that

$$\sup_{\lambda\in\Omega}\left[\widetilde{f(A)}(\lambda)\right]^p<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^p\sup_{\lambda\in\Omega}\left[\widetilde{f^p(A)}(\lambda)\right]$$

for all $\lambda \in \Omega$ and $p > 1$. This implies that

$$[ber(f(A))]^p<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^p ber(f^p(A)),$$

in particular,

$$[ber(A)]^p<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^p ber(A^p).$$

This proves the theorem. □

Declaration of Competing Interests The author declare that they have no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

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