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Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative.

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for a class of fractional differential equations with boundary conditions in the frame of Riesz-Caputo operators. We apply the methods of functional analysis such that the uniqueness result is established by using Banach's contraction principle, whereas Schaefer's and Krasnoslkii's fixed point theorems are applied to obtain existence results. Some examples are given to illustrate our acquired results.

Keywords: Riesz-Caputo derivative Boundary value problem fixed point theorem
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1. Introduction

Fractional calculus (FC) is a mathematical branch that investigates the properties of derivatives and integrals of non-integer order. The interested readers in the subject should refer to the books [34, 35, 36]. Fractional order models, which provide an excellent description of memory and genetic processes, are more accurate and appropriate than models with integer order. For the development of FC, there are sundry common definitions of fractional derivatives and integrals, such as Rimann-Liouville type, Caputo type, Hadamard type, Hilfer type, ψ-Caputo, ψ-Hilfer type, Caputo-Fabrizio type, Atangana-Baleanu type, conformable type, and Erdelyi-Kober type, etc, (see [11, 15, 16, 25, 32, 33, 37, 3]).

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Some recent contributions have been investigated the existence and uniqueness of solutions for different kinds of nonlinear fractional differential equations (FDEs) and inclusion (FDIs) by using various types of fixed point theorems, which can be found in [13, 6, 17, 7, 39, 38, 8, 19, 20, 21, 4, 5, 1, 2, 18], and the references cited therein. The study of FDEs or FDIs with anti-periodic boundary conditions, that are applied in numerous different fields, like chemical engineering, physics, economics, dynamics, etc., has received much attention recently, (see [23, 27, 40, 10, 26]) and the papers mentioned therein.
On the other hand, the authors in [31] investigated the existence results of the following FDEs

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) = \mathfrak{g}(t,\varkappa(t)), 0<\gamma\leq 1, 0\leq t\leq T, \\ \varkappa(0)=\varkappa_0, \varkappa(T)=\varkappa_T, \end{cases}$$

where ${}_0^{RC}D_T^{\vartheta}$ is the Riesz-Caputo derivative, $\mathfrak{g}:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $\varkappa_0$,$\varkappa_T$ are constants.
The positive solution of nonlinear FDEs with the Riesz space derivative

$$\begin{cases} {}_0^{RC}D_1^{\vartheta}\varkappa(t) = h(t,\varkappa(t)), 0<\vartheta\leq 1,\ t\in[0,1], \\ \varkappa(0)=\varkappa_0,\ \varkappa(1)=\varkappa_1,\quad \varkappa_0,\varkappa_1\geq 0, \end{cases}$$

has been studied by Yun Gu et al., in [24]. Also, Chen et al., in [27] discussed a class of FDEs with anti-periodic boundary condithions of the form

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) = \mathfrak{g}(t,\varkappa(t)), 0<\gamma\leq 1, t\in[0,T], \\ \varkappa(0)+\varkappa(T)=0,\quad \varkappa'(0)+\varkappa'(T)=0, \end{cases}$$

where ${}_0^{RC}D_T^{\vartheta}$ is the Riesz-Caputo derivative and $\mathfrak{g}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
Motivated by the above cited work, in this paper, we investigat the existence and uniqueness results of the following FDEs with the Riesz-Caputo derivative

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) + \mathfrak{F}(t,\varkappa(t), {}_0^{RC}D_T^{\varsigma}\varkappa(t)) = 0,\ t\in\mathcal{J}:=[0,T], \\ \varkappa(0)+\varkappa(T)=0,\quad \mu\varkappa'(0)+\sigma\varkappa'(T)=0, \end{cases} \tag{1}$$

where $1<\vartheta\leq 2$ and ,$0<\varsigma\leq 1$, ${}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa\in\{\vartheta,\varsigma\}$, $\mathfrak{F}:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, is a continuous function, and μ,σ are nonnegative constants with $\mu>\sigma$. We refer here to some very recent works that dealt with a similar analysis, see [28, 29, 30].
The paper is marshaled as follows. Section 2 has definitions and some of the most important basic concepts of the FC. In section 3, we prove the existence and uniqueness of solutions for the proposed problem with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems. Some illustrative examples associated with our suggested problem are provided in Section 4.

2. Preliminaries

In this section ,we recall some basic concepts, and preliminary facts. By $E=\mathfrak{C}(\mathcal{J},\mathbb{R})$ we denote the Banach space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$ as follows

$$E=\left\{\varkappa:\varkappa\in\mathfrak{C}([0,T]),\ {}^{RC}D^{\delta}\varkappa\in\mathfrak{C}([0,T])\right\},$$

endowed with the norm

$$\|\varkappa\|_E = \|\varkappa\| + \|{}^{RC}D^{\delta}\varkappa\|,$$

and

$$\|\varkappa\| = \sup_{t\in\mathcal{J}}|\varkappa(t)|, \|{}^{RC}D^{\delta}\varkappa\| = \sup_{t\in\mathcal{J}}|{}^{RC}D^{\delta}\varkappa(t)|.$$

We start with definitions.

Definition 2.1. *[9, 36] For $0 \leq t \leq T$, the classical Riesz–Caputo fractional derivative is defined by*

$$\begin{aligned} {}_0^{RC}D_T^{\vartheta}\mathfrak{F}(t) &= \frac{1}{\Gamma(n-\vartheta)}\int_0^T |t-\xi|^{n-\vartheta-1}\mathfrak{F}^{(n)}(\xi)d\xi \\ &= \frac{1}{2}({}_0^{C}D_t^{\vartheta} + (-1)^n\, {}_t^{C}D_T^{\vartheta})\mathfrak{F}(t), \end{aligned}$$

where ${}_0^{C}D_t^{\vartheta}$ and ${}_t^{C}D_T^{\vartheta}$ are the left and right Caputo derivative, respectively

$$\begin{aligned} {}_0^{C}D_t^{\vartheta}\mathfrak{F}(t) &= \frac{1}{\Gamma(n-\vartheta)}\int_0^t (t-\xi)^{n-\vartheta-1}\mathfrak{F}^{(n)}(\xi)d\xi, \\ {}_t^{C}D_T^{\vartheta}\mathfrak{F}(t) &= \frac{(-1)^n}{\Gamma(n-\vartheta)}\int_t^T (\xi-t)^{n-\vartheta-1}\mathfrak{F}^{(n)}(\xi)d\xi. \end{aligned}$$

Remark 2.2. *([27, 31]) In particular if $\mathfrak{F}(t) \in \mathfrak{C}([0,T])$ and $0 < \vartheta \leq 1$, then*

$${}_0^{RC}D_T^{\vartheta}\mathfrak{F}(t) = \frac{1}{2}({}_0^{C}D_t^{\vartheta} - {}_t^{C}D_T^{\vartheta})\mathfrak{F}(t),$$

if $\mathfrak{F}(t) \in \mathfrak{C}^2([0,T])$ and $1 < \vartheta \leq 2$, then

$${}_0^{RC}D_T^{\vartheta}\mathfrak{F}(t) = \frac{1}{2}({}_0^{C}D_t^{\vartheta} + {}_t^{C}D_T^{\vartheta})\mathfrak{F}(t).$$

Definition 2.3. *([31]) The Riemann-Liouville fractional integrals concepts of order ϑ are defined as*

$${}_0I_t^{\vartheta}\mathfrak{F}(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-\xi)^{\vartheta-1}\mathfrak{F}(\xi)d\xi,$$

$${}_tI_T^{\vartheta}\mathfrak{F}(t) = \frac{1}{\Gamma(\vartheta)}\int_t^T (\xi-t)^{\vartheta-1}\mathfrak{F}(\xi)d\xi,$$

$${}_0I_T^{\vartheta}\mathfrak{F}(t) = \frac{1}{\Gamma(\vartheta)}\int_0^T |\xi-t|^{\vartheta-1}\mathfrak{F}(\xi)d\xi.$$

Lemma 2.4. *([27]) If $\mathfrak{F}(t) \in \mathfrak{C}^n([0,T])$, then*

$${}_0I_t^{\vartheta}\,{}_0^{C}D_t^{\vartheta}\mathfrak{F}(t) = \mathfrak{F}(t) - \sum_{k=0}^{n-1}\frac{\mathfrak{F}^{(k)}(0)}{k!}(t-0)^k,$$

and

$${}_tI_T^{\vartheta}\,{}_t^{C}D_T^{\vartheta}\mathfrak{F}(t) = (-1)^n\left[\mathfrak{F}(t) - \sum_{k=0}^{n-1}\frac{(-1)^k\mathfrak{F}^{(k)}(T)}{k!}(T-t)^k\right].$$

From the above definitions and lemmas, we have

$$\begin{aligned} {}_0I_T^{\vartheta}\,{}_0^{RC}D_T^{\vartheta}\mathfrak{F}(t) &= \frac{1}{2}\left({}_0I_t^{\vartheta}\,{}_0^{C}D_t^{\vartheta} + {}_tI_T^{\vartheta}\,{}_0^{C}D_t^{\vartheta}\right)\mathfrak{F}(t) \\ &\quad + (-1)^n\frac{1}{2}\left({}_0I_t^{\vartheta}\,{}_t^{C}D_T^{\vartheta} + {}_tI_T^{\vartheta}\,{}_t^{C}D_T^{\vartheta}\right)\mathfrak{F}(t) \\ &= \frac{1}{2}\left({}_0I_t^{\vartheta}\,{}_0^{C}D_t^{\vartheta} + (-1)^n\,{}_tI_T^{\vartheta}\,{}_t^{C}D_T^{\vartheta}\right)\mathfrak{F}(t). \end{aligned}$$

In particular, if $1 < \vartheta \leq 2$ and $\mathfrak{F}(t) \in \mathfrak{C}^2([0,T])$, then

$${}_0I_T^{\vartheta}\,{}_0^{RC}D_T^{\vartheta}\mathfrak{F}(t) = \mathfrak{F}(t) - \frac{1}{2}(\mathfrak{F}(0) + \mathfrak{F}(T)) - \frac{1}{2}\mathfrak{F}'(0)t + \frac{1}{2}\mathfrak{F}'(T)(T-t). \tag{2}$$

3. Main Results

Lemma 3.1. *Assume that* $\mathfrak{g} \in \mathfrak{C}(\mathcal{J}, \mathbb{R})$ *and* $\varkappa \in \mathfrak{C}^2(\mathcal{J})$. *Then*

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) + \mathfrak{g}(t) = 0, t \in [0, T], 1 < \vartheta \leq 2, \\ \varkappa(0) + \varkappa(T) = 0, \mu\varkappa'(0) + \sigma\varkappa'(T) = 0, \end{cases} \tag{3}$$

is equivalent to the integral equation given by

$$\begin{aligned} \varkappa(t) = &\frac{(\sigma - \mu)t + \mu T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - s)^{\vartheta - 2}\mathfrak{g}(s)ds - \frac{1}{\Gamma(\vartheta)} \int_0^t (t - s)^{\vartheta - 1}\mathfrak{g}(s)ds \\ &- \frac{1}{\Gamma(\vartheta)} \int_t^T (s - t)^{\vartheta - 1}\mathfrak{g}(s)ds. \end{aligned} \tag{2}$$

Proof. Applying Lemma (2.4) on equation (3), we obtain

$$\begin{aligned} \varkappa(t) &= \frac{1}{2}(\varkappa(0) + \varkappa(T)) + \frac{1}{2}\varkappa'(0)t - \frac{1}{2}\varkappa_0'{}^{RC}I_T^{\vartheta}\mathfrak{g}(t) \\ &= \frac{1}{2}(\varkappa(0) + \varkappa(T)) + \frac{1}{2}\varkappa'(0)t - \frac{1}{2}\varkappa'(T)(T - t) - \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \xi)^{\vartheta - 1}\mathfrak{g}(\xi)d\xi \\ &- \frac{1}{\Gamma(\vartheta)} \int_t^T (\xi - t)^{\vartheta - 1}\mathfrak{g}(\xi)d\xi. \end{aligned} \tag{3}$$

Then

$$\begin{aligned} \varkappa'(t) = &\frac{1}{2}(\varkappa'(0) + \varkappa'(T)) - \frac{1}{\Gamma(\vartheta - 1)} \int_0^t (t - \xi)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi \\ &+ \frac{1}{\Gamma(\vartheta - 1)} \int_t^T (\xi - t)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi. \end{aligned}$$

By the boundary conditions of (3), we find that:

$$\begin{aligned} \varkappa(0) &= \frac{-\sigma T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi + \frac{1}{\Gamma(\vartheta)} \int_0^T (T - \xi)^{\vartheta - 1}\mathfrak{g}(\xi)d\xi, \\ \varkappa(T) &= \frac{\sigma T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi - \frac{1}{\Gamma(\vartheta)} \int_0^T (T - \xi)^{\vartheta - 1}\mathfrak{g}(\xi)d\xi, \\ \varkappa'(0) &= \frac{2\sigma T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi, \\ \varkappa'(T) &= \frac{-2\mu T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2}\mathfrak{g}(\xi)d\xi. \end{aligned}$$

Substituting the values of $\varkappa'(0)$and $\varkappa'(T)$ into (3), we obtain (2).
Let us introduce the following notations:

$$\begin{aligned} \Omega_1 &= \frac{\mu T^{\vartheta}}{(\mu + \sigma)\Gamma(\vartheta)} + \frac{2T^{\vartheta}}{\Gamma(\vartheta + 1)}, \\ \Omega_2 &= \frac{T^{\vartheta - \varsigma}(\mu - \sigma) + 2\mu T^{\vartheta}\Gamma(2 - \varsigma)}{2(\mu + \sigma)\Gamma(2 - \varsigma)\Gamma(\vartheta)} + \frac{2T^{\vartheta - \varsigma}}{\Gamma(\vartheta - \varsigma + 1)}, \\ k1 &= \frac{T^{\vartheta - \varsigma}}{\Gamma(\vartheta - \varsigma + 1)}, \\ k2 &= \frac{T^{\vartheta - \varsigma}}{\Gamma(\vartheta - \varsigma + 1)}. \end{aligned}$$

□

3.1. Uniqueness result via Banach's fixed point theorem

Theorem 3.2. *Let $\mathfrak{F} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that*

(H_1) *there exists nonnegative real numbers L_1, L_2 such that for all $(\xi, v), (\xi', v') \in \mathbb{R}^2$, we have*

$$|\mathfrak{F}(t, \xi, v) - \mathfrak{F}(t, \xi', v')| \leq L_1|\xi - \xi'| + L_2|v - v'|,$$

if

$$(\Omega_1 + \Omega_2)(L_1 + L_2) < 1.$$

Then the problem (1) has a unique solution on $\mathcal{J}$.

Proof. We transform BVP (1) into fixed point problem. Then we define the integral operator $\mathcal{H} : E \to E$ by

$$\begin{aligned}
\mathcal{H}\varkappa(t) &= \frac{(\sigma - \mu)t + \mu T}{(\mu + \sigma)\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2} \mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))d\xi \\
&- \frac{1}{(\Gamma)} \int_0^t (t - \xi)^{\vartheta - 1} \mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))d\xi \\
&- \frac{1}{(\Gamma)} \int_t^T (\xi - t)^{\vartheta - 1} \mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))d\xi.
\end{aligned}$$

Now, we prove that $\mathcal{H}$ is a contraction. For $\varkappa, \varpi \in E$ and for each $t \in \mathcal{J}$, we have

$$\begin{aligned}
&|\mathcal{H}\varkappa(t) - \mathcal{H}\varpi(t)| \\
&\leq \frac{(\sigma - \mu)t + \mu T|}{(\mu + \sigma)|\Gamma(\vartheta - 1)} \int_0^T (T - \xi)^{\vartheta - 2} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi, \varpi(\xi), {}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \xi)^{\vartheta - 1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi, \varpi(\xi), {}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)} \int_t^T (\xi - t)^{\vartheta - 1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi, \varpi(\xi), {}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&\leq \frac{\mu T^{\vartheta}}{(2\mu + \sigma)\Gamma(\vartheta)} \left(L_1\|\varkappa - \varpi\| + L_2\|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right) \\
&+ \frac{2T^{\vartheta}}{\Gamma(\vartheta + 1)} \left(L_1\|\varkappa - \varpi\| + L_2\|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right) \\
&\leq \left(\frac{\mu T^{\vartheta}}{(\mu + \sigma)\Gamma(\vartheta)} + \frac{2T^{\vartheta}}{\Gamma(\vartheta + 1)}\right) \left(L_1 + L_2)(\|\varkappa - \varpi\| + \|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right).
\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{H}\varkappa(t) - \mathcal{H}\varpi(t)\| \leq \Omega_1 \left(L_1 + L_2)(\|\varkappa - \varpi\| + \|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right). \tag{12}$$

On the other hand, we have

$$\begin{aligned}
&\left|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t) - {}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t)\right| \\
&\leq \frac{1}{\Gamma(\vartheta-\varsigma)}\int_0^t (t-\xi)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi,\varpi(\xi),{}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta-\varsigma)}\int_t^T (\xi-t)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi,\varpi(\xi),{}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&+ \frac{\left|\left(t^{1-\varsigma}-(T-t)^{1-\varsigma}\right)(\sigma-\mu)+2\mu\Gamma(2-\varsigma)T\right|}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta-1)} \\
&\int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi,(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi)) - \mathfrak{F}(\xi,\varpi(\xi),{}^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&\leq \frac{T^{\vartheta-\varsigma}(\sigma-\mu)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}\left(L_1\|\varkappa-\varpi\| + L_2\|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right) \\
&+ \frac{2T^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}\left(L_1\|\varkappa-\varpi\| + L_2\|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right).
\end{aligned}$$

Thus,

$$\|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t) - {}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t)\| \leq \Omega_2\left(L_1+L_2)(\|\varkappa-\varpi\| + \|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right) \tag{13}$$

From (12) and (13), we get

$$\|\mathcal{H}\varkappa(t) - \mathcal{H}\varpi(t)\|_E \leq (\Omega_1+\Omega_2)\left(L_1+L_2)(\|\varkappa-\varpi\| + \|{}^{RC}D^{\varsigma}\varkappa - {}^{RC}D^{\varsigma}\varpi\|\right).$$

Hence, $\mathcal{H}$ is a contraction. As a consequence of Banach contraction principle, the problem (1) has a unique solution on $\mathcal{J}$. □

3.2. Existence result via Shaefer fixed point theorem

Lemma 3.3. *Let E be a Banach space. Assume that $\mathcal{H}: E \to E$ be a completely continuous operator, and the set*

$$\omega(\mathcal{H}) = \{\varpi \in E : \varpi = \lambda\mathcal{H}\varpi, \lambda \in (0,1)\}.$$

$\omega(\mathfrak{F})$ is bounded. Then $\mathcal{H}$ has a fixed point in E.

Theorem 3.4. *Assume that there exists a positive $\mathfrak{M}$ such that*

$$|\mathfrak{F}(\xi,\varkappa,\varpi)| < \mathfrak{M} \quad \text{for} \quad t \in \mathcal{J}, \quad \varkappa,\varpi \in \mathbb{R}.$$

Then the problem (1) has at least one solution on $\mathcal{J}$.

Proof. We will use the Sheafer's fixed point theorem, to prove $\mathcal{H}$ has a fixed point on E, we subdivided the proof into several steps :

Step 1. $\mathcal{H}$ is continuous on E: in view of continuity of $\mathfrak{F}$, we conclude that operator $\mathcal{H}$ is continuous.

step 2. $\mathcal{H}$ maps bounded sets into bounded sets in E.

For each $\varkappa \in B_r = \{\varkappa \in E : \|\varkappa\|_E \leq r\}$ and $t \in \mathcal{J}$, we get

$$\begin{aligned}
|\mathcal{H}\varkappa(t)| &\leq \frac{|(\sigma-\mu)t+\mu T|}{(\sigma+\mu)\Gamma(\vartheta-1)}\int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)}\int_0^t (t-\xi)^{\vartheta-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&- \frac{1}{\Gamma(\vartheta)}\int_t^T (\xi-t)^{\vartheta-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\leq \frac{\mu T^{\vartheta}\mathfrak{M}}{(\sigma+\mu)\Gamma(\vartheta)} + \frac{2\mathfrak{M}T^{\vartheta}}{\Gamma(\vartheta+1)}.
\end{aligned}$$

Which implies that,

$$\|\mathcal{H}\varkappa(t)\| \leq \mathfrak{M}\Omega_1, \tag{14}$$

and,

$$\begin{aligned}
|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t)| &\leq \frac{1}{\Gamma(\vartheta-\varsigma)}\int_0^t (t-\xi)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta-\varsigma)}\int_t^T (t-\xi)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{\left|\left(t^{1-\varsigma}-(T-t)^{1-\varsigma}\right)(\sigma-\mu)+2\mu\Gamma(2-\varsigma)T\right|}{2(\sigma+\mu)|\Gamma(2-\varsigma)\Gamma(\vartheta-1)} \\
&\times \int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\leq \frac{T^{\vartheta-\varsigma}(\mu+\sigma)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2|\mu-\sigma|\Gamma(2-\varsigma)\Gamma(\vartheta)}\mathfrak{M} + \frac{2T^{\vartheta-\varsigma}}{2\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\
&\leq \frac{T^{\vartheta-\varsigma}(\sigma-\mu))+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}\mathfrak{M} + \frac{2T^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}\mathfrak{M}.
\end{aligned}$$

Which implies that,

$$\|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t)\| \leq \Omega_2\mathfrak{M}. \tag{15}$$

Adds side of inequality (14)and(15), we get

$$\|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t)\|_E < \mathfrak{M}\left(\Omega_1+\Omega_2\right)\infty.$$

Which implies that $\mathcal{H}$ maps bounded sets into bounded sets on E.

step 3. $\mathcal{H}$ maps bounded sets into equicontinuous sets in E.

Let B_r be a bounded set of E as in step 2, and let $\varkappa \in B_r$. For each $t_1, t_2 \in \mathcal{J}, t_1 < t_2$, we have

$$\begin{aligned}
|\mathcal{H}\varkappa(t_2)-\mathcal{H}\varkappa(t_1)| &\leq \frac{(\sigma-\mu)(t_2-t_1)}{(\sigma+\mu)\Gamma(\vartheta-1)}\int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)}\int_0^{t_1}\left|(t_1-\xi)^{\vartheta-1}-(t_2-\xi)^{\vartheta-1}\right||\mathfrak{F}(\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)}\int_{t_1}^{t_2}\left|(\xi-t_1)^{\vartheta-1}-(t_2-\xi)^{\vartheta-1}\right||\mathfrak{F}(\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)}\int_{t_2}^{T}\left|(\xi-t_1)^{\vartheta-1}-(\xi-t_2)^{\vartheta-1}\right||\mathfrak{F}(\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\leq \frac{\left|(t_1^{\vartheta}-t_2^{\vartheta})+(t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M} + \frac{(\sigma-\mu)|t_2-t_1|^{\vartheta}T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta+1)}\mathfrak{M} \\
&+ \frac{\left|\left((T-t_1)^{\vartheta}-(T-t_2)^{\vartheta}\right)+(t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M} \\
&\leq \frac{\left|(t_1^{\vartheta}-t_2^{\vartheta})+(t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M} + \frac{(\sigma-\mu)|t_2-t_1|^{\vartheta}T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta+1)}\mathfrak{M} \\
&+ \frac{\left|\left((T-t_1)^{\vartheta}-(T-t_2)^{\vartheta}\right)+(t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M},
\end{aligned}$$

and,

$$\begin{aligned}
|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t_2) -{}_0^{RC} D_T^{\varsigma}\mathcal{H}\varkappa(t_1)| &\leq \frac{(t_1^{\vartheta-\varsigma} - t_2^{\vartheta-\varsigma})}{2\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\
&+ \frac{\left((T-t_2)^{\vartheta-\varsigma} - (T-t_1)^{\vartheta-\varsigma}\right) + (t_2-t_1)^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\
&+ \frac{\left[(\sigma-\mu)(t_2^{1-\varsigma} - t_1^{1-\varsigma}) + ((T-t_2)^{1-\varsigma} - (T-t_1)^{1-\varsigma})\right] T^{\vartheta-1}\mathfrak{M}}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{H}\varkappa(t_2) - \mathcal{H}\varkappa(t_1)\|_{\varkappa} &\leq \frac{\left|(t_1^{\vartheta} - t_2^{\vartheta}) + (t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M} + \frac{(\sigma-\mu)|t_2-t_1|^{\vartheta}T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta+1)}\mathfrak{M} \\
&+ \frac{\left|\left((T-t_1)^{\vartheta} - (T-t_2)^{\vartheta}\right) + (t_2-t_1)^{\vartheta}\right|}{\Gamma(\vartheta+1)}\mathfrak{M} \\
&+ \frac{(t_1^{\vartheta-\varsigma} - t_2^{\vartheta-\varsigma})}{2\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\
&+ \frac{\left((T-t_2)^{\vartheta-\varsigma} - (T-t_1)^{\vartheta-\varsigma}\right) + (t_2-t_1)^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\
&+ \frac{\left[(\sigma-\mu)(t_2^{1-\varsigma} - t_1^{1-\varsigma}) + ((T-t_2)^{1-\varsigma} - (T-t_1)^{1-\varsigma})\right] T^{\vartheta-1}\mathfrak{M}}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}.
\end{aligned}$$

Which implies that $\|\mathcal{H}\varkappa(t_2) - \mathcal{H}\varkappa(t_1)\|_E \to 0$ as $t_2 \to t_1$.By Arzela-Ascoli theorem, we conclude that $\mathcal{H}$ is completely continuous operator.
step 4. We show that the set Δ defined by

$$\Delta = \{\varkappa \in E, \varkappa = \rho\mathcal{H}(\varkappa), 0 < \rho < 1\}$$

is bounded. Let $\varkappa \in \Delta$, for some $\rho \in (0,1)$. For each $t \in \mathcal{J}$, we have

$$\begin{aligned}
\frac{1}{\rho}|\varkappa(t)| &\leq \frac{|(\sigma-\mu)t + \mu T|}{(\sigma+\mu)\Gamma(\vartheta-1)} \int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\xi)^{\vartheta}|\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&+ \frac{1}{\Gamma(\vartheta)} \int_t^T (\xi-t)^{\vartheta}|\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\leq \frac{\mu T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta)}\mathfrak{M} + \frac{2T^{\vartheta}}{\Gamma(\vartheta+1)}\mathfrak{M}.
\end{aligned}$$

Therefore,

$$\|\varkappa\| \leq \rho\Omega_1\mathfrak{M}. \tag{16}$$

and,

$$\begin{aligned}\frac{1}{\rho}|{}_0^{RC}D_T^{\varsigma}\varkappa(t)| &\le \frac{1}{\Gamma(\vartheta-\varsigma)}\int_0^t (t-\xi)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\ &+ \frac{1}{\Gamma(\vartheta-\varsigma)}\int_t^T (\xi-t)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\ &+ \frac{\left|\left(t^{1-\varsigma}-(T-t)^{1-\varsigma}\right)(\sigma-\mu)+2\mu\Gamma(2-\varsigma)T\right|}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta-1)} \\ &\times \int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\ &\le \frac{T^{\vartheta-\varsigma}(\mu+\sigma)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2|\mu-\sigma|\Gamma(2-\varsigma)\Gamma(\vartheta)}\mathfrak{M} + \frac{2T^{\vartheta-\varsigma}}{2\Gamma(\vartheta-\varsigma+1)}\mathfrak{M} \\ &\le \frac{T^{\vartheta-\varsigma}(\sigma-\mu)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}\mathfrak{M} + \frac{2T^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}\mathfrak{M}.\end{aligned}$$

Therefore,

$$\|{}_0^{RC}D_T^{\varsigma}\varkappa(t)\| \le \rho\Omega_2\mathfrak{M}. \tag{17}$$

Adds side of inequality (16)and (17), we get

$$\|\varkappa\|_E \le \rho(\Omega_1+\Omega_2)\mathfrak{M}.$$

Hence,

$$\|\varkappa\|_E < \infty.$$

This shows that Δ is bounded.
As consequence of Schaefer's fixed point theorem, the problem (1) has at least one solution in $[0,T]$. □

3.3. Existence result via Karesnoslskii's fixed point theorem

Lemma 3.5. *(Karasnoselskii's fixed point theorem) Let M a closed bounded, convex and nonempty subset of a Banach space E, let A, Bbe operator, such that*

(a) $A\varkappa + B\varpi \in M$, *whenever,* $\varkappa, \varpi \in M$,

(b) A *is compact and continuous,*

(c) B *is a contraction mapping, then there exist* $z \in M$ *such that* $z = Az + Bz$.

Theorem 3.6. *Let $\mathfrak{F} : \mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, and let the conditions (H_1)-(H_2) hold. In addition, the function $\mathfrak{F}$ satisfying the assumptions :*

(H_3) *There exists a nonnegative function $\Omega \in \mathfrak{C}(\mathcal{J},\mathbb{R}^+)$ such that* $|\mathfrak{F}(t,\varkappa,\varpi)| \le \Omega(t)$ *for any* $(t,\varkappa,\varpi) \in \mathcal{J}\times\mathbb{R}\times\mathbb{R}$,

(H_4) $(k_1+k_2)(L_1+L_2) < 1$,

Then the problem (1) has a least one solution in $\mathcal{J}$.

Proof. We define two operators $\mathcal{H}_1\varkappa(t)$ and $\mathcal{H}_2\varkappa(t)$ as:

$$\begin{aligned}(\mathcal{H}_1\varkappa)(t) &= \frac{(\sigma-\mu)t)+\mu T}{(\sigma+\mu)\Gamma(\vartheta-1)}\int_0^T (T-\xi)^{\vartheta-2}\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))d\xi, \\ (\mathcal{H}_2\varkappa(t)) &= -\frac{1}{\Gamma(\vartheta)}\int_0^t (t-\xi)^{\vartheta-1}\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))d\xi \\ &- \frac{1}{\Gamma(\vartheta)}\int_t^T (\xi-t)^{\vartheta-1}\mathfrak{F}(\xi,\varkappa(\xi),{}^{RC}D^{\varsigma}\varkappa(\xi))d\xi.\end{aligned}$$

Choosing $d \geq (\Omega_1 + \Omega_2)(L_1 + L_2)\|\Omega\|$, and we consider $B_d = \{\varkappa \in E : \|\varkappa\|_E \leq d\}$.
Step1 We shall prove that $\mathcal{H}_1\varkappa(t) + \mathcal{H}_2\varkappa(t) \in B_d$.
For any $\varkappa, \varpi \in B_d$ and for each then $t \in \mathcal{J}$, we have

$$\begin{aligned} |\mathcal{H}_1\varkappa(t) + \mathcal{H}_2\varpi(t)| &\leq \frac{|(\sigma-\mu)t + \mu T|}{(\sigma+\mu)\Gamma(\vartheta-1)} \int_0^T (T-\xi)^{\vartheta-2} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\xi)^{\vartheta-1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &- \frac{1}{\Gamma(\vartheta)} \int_t^T (\xi-t)^{\vartheta-1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &\leq \frac{\mu T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta)} (L_1 + L_2) \|\Omega\| + \frac{2T^{\vartheta}}{\Gamma(\vartheta+1)} (L_1 + L_2)) \|\Omega\|. \end{aligned}$$

Then

$$\|\mathcal{H}_1\varkappa(t) + \mathcal{H}_2\varpi(t)\| \leq \Omega_1(L_1 + L_2)\|\Omega\| \tag{18}$$

On the other hand,

$$\begin{aligned} &|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t) + {}_0^{RC}D_T^{\varsigma}\mathcal{H}\varpi(t)| \leq \frac{1}{\Gamma(\vartheta-\varsigma)} \int_0^t (t-\xi)^{\vartheta-\varsigma-1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &+ \frac{1}{\Gamma(\vartheta-\varsigma)} \int_t^T (\xi-t)^{\vartheta-\varsigma-1} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &+ \frac{\left|\left(t^{1-\varsigma} - (T-t)^{1-\varsigma}\right)(\sigma-\mu) + 2\mu\Gamma(2-\varsigma)T\right|}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta-1)} \int_0^T (T-\xi)^{\vartheta-2} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &\leq \frac{T^{\vartheta-\varsigma}(\sigma-\mu) + 2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)} (L_1 + L_2) \|\Omega\| + \frac{2T^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)} (L_1 + L_2) \|\Omega\|. \end{aligned}$$

Hence

$$\|{}_0^{RC}D_T^{\varsigma}\mathcal{H}\varkappa(t) + {}_0^{RC}D_T^{\varsigma}\mathcal{H}\varpi(t)\| \leq \Omega_1(L_1 + L_2)\|\Omega\| \tag{19}$$

It follows from (18) and (19) that

$$\|\mathcal{H}_1\varkappa(t) + \mathcal{H}_2\varpi(t)\|_E \leq (\Omega_1 + \Omega_2)(L_1 + L_2)\|\Omega\| \leq d.$$

Hence, $\mathcal{H}_1\varkappa(t) + \mathcal{H}_2\varkappa(t) \in B_d$.
step2 We shall prove that $\mathcal{H}_1$is continuous and compact. The continuity of $\mathfrak{F}$ implies that the operator $\mathcal{H}_1$ is continuous.
Now, we prove that $\mathcal{H}_2$ maps bounded sets into bounded sets of E.
For $\varkappa \in B_d$,and for each $t \in \mathcal{J}$, we have

$$\begin{aligned} |\mathcal{H}_1\varkappa(t)| &\leq \frac{|(\sigma-\mu) + \mu T|}{(\sigma+\mu)\Gamma(\vartheta-1)} \int_0^T (T-\xi)^{\vartheta-2} |\mathfrak{F}(\xi, \varkappa(\xi), {}^{RC}D^{\varsigma}\varkappa(\xi))| d\xi \\ &\frac{\mu T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta-1)} (L_1 + L_2)\|\Omega\|. \end{aligned}$$

Hence

$$\|\mathcal{H}_1\varkappa(t)\| \leq \frac{\mu T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta-1)} (L_1 + L_2)\|\Omega\|, \tag{20}$$

and

$$
\begin{aligned}
|^{RC}D^{\varsigma}\mathcal{H}_1\varkappa(t)| &\leq \frac{\left|((T-t)^{1-\varsigma}-t^{1-\varsigma})(\sigma-\mu)+2\mu T\Gamma(2-\varsigma)\right|}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta-1)} \\
&\quad \times \int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\xi,\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\leq \frac{T^{\vartheta-\varsigma}(\sigma-\mu)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}(L_1+L_2)\|\Omega\|.
\end{aligned}
$$

Hence

$$
\|^{RC}D^{\varsigma}\mathcal{H}_1\varkappa(t)\| \leq \frac{T^{\vartheta-\varsigma}(\sigma-\mu)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}(L_1+L_2)\|\Omega\|. \tag{21}
$$

Combining (20) and (21), we get

$$
\|\mathcal{H}_1\varkappa(t)\|_E \leq \left(\frac{\mu T^{\vartheta}}{(\sigma-\mu)\Gamma(\vartheta-1)}+\frac{T^{\vartheta-\varsigma}(\sigma-\mu)+2\mu T^{\vartheta}\Gamma(2-\varsigma)}{2(\sigma+\mu)\Gamma(2-\varsigma)\Gamma(\vartheta)}\right)(L_1+L_2)\|\Omega\|,
$$

Consequently

$$
\|\mathcal{H}_1\varkappa(t)\|_E \leq \infty.
$$

Thus, it follows the above inequality that operator $\mathcal{H}_1$ is uniformly bounded.
The operator $\mathcal{H}_1$ maps bounded sets into equicontinuous sets of E. Let $t_1, t_2 \in \mathcal{J}, t_1 < t_2, \varkappa \in B_d$, then we have :

$$
\begin{aligned}
|\mathcal{H}_1\varkappa(t_2)-\mathcal{H}_1\varkappa(t_1)| &\leq \frac{(\sigma-\mu)(t_2-t_1)}{(\sigma+\mu)\Gamma(\vartheta-1)}\int_0^T (T-\xi)^{\vartheta-2}|\mathfrak{F}(\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))|d\xi \\
&\quad + \frac{(\sigma-\mu)|t_2-t_1|^{\vartheta}T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta+1)}(L_1+L_2)\|\varkappa\|.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&|^{RC}D^{\varsigma}\mathcal{H}_1\varkappa(t_2)-^{RC}D^{\varsigma}\mathcal{H}_1\varkappa(t_1)| \\
\leq\ & \frac{(\sigma-\mu)\left[(t_2^{1-\varsigma}-t_1^{1-\varsigma})+((T-t_2)^{1-\varsigma}-(T-t_1)^{1-\varsigma})\right]T^{\vartheta-1}(L_1+L_2)\|\Omega\|}{2(\sigma+\mu)|\Gamma(2-\varsigma)\Gamma(\vartheta)}
\end{aligned}
$$

It follows from (22) and (23) that

$$
\begin{aligned}
\|\mathcal{H}_1\varkappa(t_2)-\mathcal{H}_1\varkappa(t_1)\|_E &\leq \frac{(\sigma-\mu)|t_2-t_1|^{\vartheta}T^{\vartheta}}{(\sigma+\mu)\Gamma(\vartheta+1)}(L_1+L_2)\|\varkappa\| \\
&\quad + \frac{(\sigma-\mu)\left[(t_2^{1-\varsigma}-t_1^{1-\varsigma})+((T-t_2)^{1-\varsigma}-(T-t_1)^{1-\varsigma})\right]T^{\vartheta-1}(L_1+L_2)\|\Omega\|}{2(\sigma+\mu)|\Gamma(2-\varsigma)\Gamma(\vartheta).}
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of this inequality tends to zeros.
Then as a consequence od steps, we can conclude that $\mathcal{H}_1$ is continuous and compact.
Step3 Now, we prove that $\mathcal{H}_2$ is contraction mapping .
Let $\varkappa, \varpi \in E$. Then, for each $t \in \mathcal{J}$, we have

$$
\begin{aligned}
|\mathcal{H}_2\varkappa(t)-\mathcal{H}_2\varpi(t)| &\leq +\frac{1}{\Gamma(\vartheta)}\int_0^t (t-\xi)^{\vartheta-1}|\mathfrak{F}(\xi,\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))-\mathfrak{F}(\xi,\varpi(\xi),^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_t^T (\xi-t)^{\vartheta-1}|\mathfrak{F}(\xi,\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))-\mathfrak{F}(\xi,\varpi(\xi),^{RC}D^{\varsigma}\varpi(\xi))|d\xi \\
&\leq \frac{2T^{\vartheta}}{\Gamma(\vartheta+1)}(L_1+L_2)\left(\|\varkappa-\varpi\|+\|^{RC}D^{\varsigma}\varkappa-^{RC}D^{\varsigma}\varpi\|\right)
\end{aligned}
$$

Consequently we obtain

$$\|\mathcal{H}_2\varkappa(t)-\mathcal{H}_2\varpi(t)\| \leq k_1(L_1+L_2)(\|\varkappa-\varpi\|+\|^{RC}D^{\varsigma}\varkappa-^{RC}D^{\varsigma}\varpi\|) \tag{24}$$

and

$$\begin{aligned}
&|{}_0^{RC}D_T^{\varsigma}\mathcal{H}_2\varkappa(t)-{}_0^{RC}D_T^{\varsigma}\mathcal{H}_2\varkappa(t)|\\
&\leq \frac{1}{\Gamma(\vartheta-\varsigma)}\int_0^t (t-\xi)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))-\mathfrak{F}(\xi,\varpi(\xi),^{RC}D^{\varsigma}\varpi(\xi))|d\xi\\
&+\frac{1}{\Gamma(\vartheta-\varsigma)}\int_t^T (\xi-t)^{\vartheta-\varsigma-1}|\mathfrak{F}(\xi,\varkappa(\xi),^{RC}D^{\varsigma}\varkappa(\xi))-\mathfrak{F}(\xi,\varpi(\xi),^{RC}D^{\varsigma}\varpi(\xi))|d\xi\\
&\leq \frac{2T^{\vartheta-\varsigma}}{\Gamma(\vartheta-\varsigma+1)}(L_1+L_2)\left(\|\varkappa-\varpi\|+\|^{RC}D^{\varsigma}\varkappa-^{RC}D^{\varsigma}\varpi\|\right),
\end{aligned}$$

and

$$\|^{RC}D^{\varsigma}\mathcal{H}_2\varkappa(t)-^{RC}D^{\varsigma}\mathcal{H}_2\varpi(t)\| \leq k_2(L_1+L_2)(\|\varkappa-\varpi\|+\|^{RC}D^{\varsigma}\varkappa-^{RC}D^{\varsigma}\varpi\|) \tag{25}$$

It follows from (24) and (25) that

$$\|\mathcal{H}_2\varkappa(t)-\mathcal{H}_2\varpi(t)\|_E \leq (k_1+k_2)(L_1+L_2)\left(\|\varkappa-\varpi\|+\|^{RC}D^{\varsigma}\varkappa-^{RC}D^{\varsigma}\varpi\|\right)$$

Using the condition (H_4), we conclude that $\mathcal{H}_2$ is a contraction mapping. As consequence of a krasnosselski's fixed point theorem, we deduce that $\mathcal{H}$ has a fixed point which as solution of (1). □

Example 3.7. *Consider following nonlinear FDE with Riesz-Caputo derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{3}{2}}\varkappa(t)+\frac{(\sqrt{\pi}+1)|\varkappa(t)|}{\sqrt{t^2+144}(1+|\varkappa(t)|)}+\frac{1}{(1+e^{2\pi})^2}\cos(^{RC}D^{\frac{1}{3}}\varkappa(t))=0, t\in[0,1],\\ \varpi(0)+\varpi(1)=0, 2\varpi'(0)+\frac{1}{2}\varpi'(1)=0, \end{cases} \tag{26}$$

Here, $\vartheta=\frac{3}{2},\varsigma=\frac{1}{3},\mu=2,\sigma=\frac{1}{2}$, *and,*

$$\mathfrak{F}(t,\varkappa(t),^{RC}D^{\varsigma}\varkappa(t))=\frac{(\sqrt{\pi}+1)|\varkappa|}{\sqrt{t^2+144}(1+|\varkappa|)}+\frac{1}{(1+e^{2\pi})^2}\cos(^{RC}D^{\frac{1}{3}}\varkappa(t)).$$

We have

$$|\mathfrak{F}(\varkappa,\varpi)-\mathfrak{F}(\varkappa',\varpi')| \leq \frac{\sqrt{\pi}+1}{12}\|\varkappa-\varkappa'\|+\frac{1}{(1+e^{2\pi})^2}\|\varpi-\varpi'\|.$$

Then, the assumption $(\mathcal{H}_1)$ *is satisfied with* $L_1=\frac{\sqrt{\pi}+1}{12}, L_2=\frac{1}{(1+e^{2\pi})^2}$. *Using the Matlab program,* $\Omega_1=2,1493,\Omega_2=1,9057$.
Therefore, $(L_1+L_2)(\Omega_1+\Omega_2)=0,9369<1$. *By using the theorem (3.2), the problem (26) has a unique solution on* $[0,1]$.

Example 3.8. *Consider following nonlinear FDE with Riesz-Caputo derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{5}{3}}\varkappa(t)+\frac{e^{-2t}\sin(\frac{|\varkappa|}{1+|\varkappa|})}{(e^{2\pi}+2)}+\frac{|^{RC}D^{\frac{1}{2}}\varkappa(t)|(\sqrt{\pi}+1)}{(|^{RC}D^{\frac{1}{2}}\varkappa(t)|+1)(\pi+2)^2}=0, t\in[0,1],\\ \varpi(0)+\varpi(1)=0, \frac{3}{5}\varpi'(0)+\frac{2}{3}\varpi'(1)=0, \end{cases} \tag{27}$$

Here, $\vartheta=\frac{5}{3},\varsigma=\frac{1}{2},\mu=\frac{3}{5},\sigma=\frac{2}{3},L_1=\frac{\sqrt{\pi}+1}{12},L_2=\frac{1}{(1+e^{2\pi})^2}$, *and*

$$\mathfrak{F}(t,\varkappa(t),^{RC}D^{\varsigma}\varkappa(t))=\frac{e^{-2t}\sin(\frac{|\varkappa|}{1+|\varkappa|})}{(e^{2\pi}+2)}+\frac{|^{RC}D^{\frac{1}{2}}\varkappa(t)|(\sqrt{\pi}+1)}{(|^{RC}D^{\frac{1}{2}}\varkappa(t)|+1)(\pi+2)^2}.$$

Moreover,

$$|\mathfrak{F}(\varkappa,\varpi)-\mathfrak{F}(\varkappa',\varpi')|\leq\frac{1}{(e^{2\pi}+2)}\|\varkappa-\varkappa'\|+\frac{\sqrt{\pi}+1}{(1+\pi)^2}\|\varpi-\varpi'\|.$$

Therefor,

$$|\mathfrak{F}(t,\varkappa(t),^{RC}D^{\varsigma}\varkappa(t))|\leq\frac{e^{-2t}}{e^{2\pi+2}}+\frac{\sqrt{\pi}+1}{(\pi+2)^2}=\Omega(t).$$

$\Omega_1=2,1493,\Omega_2=1,9057,K_1=1,5045,k_2=0,9239$.
Then $(k_1+k_2)(L_1+L_2)=0,2406<1$, $(\mathcal{H}_4)$ *is satisfied, by using the theorem (3.6), the problem (27) has at least one solution on* $[0,1]$.

4. Conclusion

We have effectively achieved several necessary conditions describing the the existence and uniqueness of solutions for a class of fractional differential equations with boundary conditions involving Riesz-Caputo fractional derivatives. Under some fixed point theorems such as Banach, Schaefer, and Krasnoselskii, the necessary results have been investigated. Moreover, by giving appropriate examples, all the main results have been testified. In future such type of analysis can be established for more general type fractional differential equations involving ψ-Riesz-Caputo fractional derivatives.

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