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Sequential fractional pantograph differential equations with nonlocal boundary conditions: Uniqueness and Ulam-Hyers-Rassias stability

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Abstract

In the current manuscript, we study the uniqueness and Ulam-stability of solutions for sequential fractional pantograph differential equations with nonlocal boundary conditions. The uniqueness of solutions is established by Banach's fixed point theorem. We also define and study the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of mentioned problem. An example is presented to illustrate the main results.

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1. Introduction

Differential equations of arbitrary order have recently been studied by many researchers, these equations will be used to describe phenomena of real world problems. For more details, see the works [2, 14, 21, 27] and the references therein. Many interesting and important area concerning of research for differential equations with fractional calculus are devoted to the existence theory and stability analysis of the solutions, for instance, for instance, see papers [4, 7, 9, 12, 17, 24]. Recently, several scholars have discussed the existence, uniqueness and different types of Ulam-stability of solutions for some classes of differential equations involving fractional derivatives, for instance, see [6, 13, 15, 16, 26] and the references cited therein. Considerable attention has been given to the study of the existence, uniqueness and Ulam stability of solutions for sequential fractional differential equations, we refer the reader to the monographs [20, 22, 29] and the reference therein. In the present work, we shall be concerned with a very special delay differential equation that has many applications

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in different fields of pure and applied mathematics. The equation is called pantograph equation. For more information and some applications on pantograph equation, we refer the reader to the works [8, 10, 11, 23]. It is to note that the standard form of pantograph equation has the form

$$u^{'}(t) = Au(t) + Bu(\eta t), \ t \in [0, T], \ 0 < \eta < 1,$$

$$u(0) = u_0.$$

In recent years, many researchers have discussed the existence, uniqueness and different types of Ulam-Hyers stability of the above equation, For more details, see the monographs [1, 3, 18, 25, 28] and the references therein. In [5] the authors considered the following problem of the pantograph type

$$^{C}D^{\alpha}u(t) = \phi(t, u(t), u(\eta t)), t \in [0, T], \ 0 < \alpha, \eta < 1,$$

$$u(0) = u_0,$$

where $^{C}D^{q}$ is the Caputo fractional derivative. The uniqueness results were obtained by applying Banach's contraction mapping principle. In this paper, we discuss the uniqueness, Ulam-Hyers stability and Ulam-Hyers-Rassias stability of solutions for the following sequential fractional pantograph equation

$$\left[D^{\alpha} + kD^{\beta}\right] u(t) = \phi\left(t, u(t), u(\eta t), D^{\beta}u(\eta t)\right), t \in [0, T], \tag{1}$$

under the conditions

$$u(0) = f(u), \qquad u(T) = \theta, \ \theta \in \mathbb{R}, \tag{2}$$

where $k \in \mathbb{R}^{+}, 0 < \eta < 1, 1 < \alpha \leq 2, 0 < \beta \leq 1$, D^{α}, D^{β} are the Caputo type fractional derivatives, $\phi : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f : C([0, T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions. The operator D^{ϑ} is the fractional derivative in the sense of Caputo, defined by

$$\begin{aligned} D^{\vartheta}\varphi(t) &= \frac{1}{\Gamma(n-\vartheta)} \int_0^t (t-s)^{n-\vartheta-1} \varphi^{(n)}(s)\, ds \\ &= J^{n-\vartheta}\varphi^{(n)}(t), t > 0, \ n-1 < \vartheta < n, \ n \in N^{*}, \end{aligned}$$

and the Riemann-Liouville fractional integral of order $\vartheta > 0$, defined by

$$J^{\vartheta}\varphi(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \varphi(s)\, ds, t > 0,$$

where $\Gamma(\vartheta) = \int_0^{\infty} e^{-u} x^{\vartheta-1} dx$.

Now, we give the following lemmas [19, 21]:

Lemma 1.1. *Let $m, s > 0$, $\varphi \in L_1([a, b])$. Then $J^{m}J^{s}\varphi(t) = J^{m+s}\varphi(t)$, $D^{s}J^{s}\varphi(t) = \varphi(t)$, $t \in [a, b]$.*

Lemma 1.2. *Let $s > m > 0$, $\varphi \in L_1([a, b])$. Then $D^{m}J^{s}\varphi(t) = J^{s-m}\varphi(t)$, $t \in [a, b]$.*

Also we give the following lemmas [19]:

Lemma 1.3. *For $\vartheta > 0$, the general solution of the fractional differential equation $D^{\vartheta}u(t) = 0$ is given by*

$$u(t) = \sum_{i=0}^{n-1} c_i t^i,$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$, $n = [\vartheta] + 1$.

Lemma 1.4. *Let $\vartheta > 0$. Then*

$$J^{\vartheta}D^{\vartheta}u(t) = u(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, $n = [\vartheta] + 1$.

Let us now define the space $W = \{u : u \in C([0,T], \mathbb{R}), D^{\beta}u \in C([0,T], \mathbb{R})\}$ equipped, with the norm $\|u\|_W = 2\|u\| + \|D^{\beta}u\|$, where

$$\|u\| = \sup_{t\in[0,T]} |u(t)| \text{ and } \left\|D^{\beta}u\right\| = \sup_{t\in[0,T]} \left|D^{\beta}u(t)\right|.$$

It is clear that $(W, \|u\|_W)$ is a Banach space.

In that follows, we present the Ulam stability for the sequential fractional pantograph differential equations (1).

Definition 1.5. *The sequential fractional pantograph differential equations (1) is Ulam-Hyers stable if there exists a real number $\lambda_{\phi} > 0$ such that for each $\mu > 0$ and for each solution $v \in W$ of the inequality*

$$\left|\left[D^{\alpha} + kD^{\beta}\right]v(t) - \phi\left(t, v(t), v(\eta t), D^{\beta}v(\eta t)\right)\right| \leq \mu, \ t \in [0,T], \tag{3}$$

there exists a solution $u \in W$ of the sequential fractional pantograph differential equations (1) with

$$|v(t) - u(t)| \leq \lambda_{\phi}\mu, \ t \in [0,T].$$

Definition 1.6. *The sequential fractional pantograph differential equations (1) is generalized Ulam-Hyers stable if there exists $g_{\phi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g_{\phi}(0) = 0$, such that for each solution $v \in W$ of the inequality (3), there exists a solution $u \in W$ of the sequential fractional pantograph differential equations (1) with*

$$|v(t) - u(t)| \leq g_{\phi}(\mu), \ t \in [0,T].$$

Definition 1.7. *The sequential fractional pantograph differential equations (1) is Ulam-Hyers-Rassias stable with respect to $h \in W$ if there exists a real number $\lambda_{\varphi} > 0$ such that for each $\mu > 0$ and for each solution $v \in W$ of the inequality*

$$\left|\left[D^{\alpha} + kD^{\beta}\right]v(t) - \phi\left(t, v(t), v(\eta t), D^{\beta}v(\eta t)\right)\right| \leq \mu h(t), \ t \in [0,T], \tag{4}$$

there exists a solution $u \in W$ of sequential fractional pantograph differential equations (1) with

$$|v(t) - u(t)| \leq \lambda_{\varphi}\mu h(t), \ t \in [0,T].$$

Definition 1.8. *The sequential fractional pantograph differential equations (1) is generalized Ulam-Hyers-Rassias stable with respect to $h \in C(J, \mathbb{R}_+)$ if there exists a real number $\lambda_{\phi,h} > 0$ such that for each solution $v \in W$ of the inequality*

$$\left|\left[D^{\alpha} + kD^{\beta}\right]v(t) - \phi\left(t, v(t), v(\eta t), D^{\beta}v(\eta t)\right)\right| \leq h(t), \ t \in [0,T], \tag{5}$$

there exists a solution $u \in W$ of the sequential fractional pantograph differential equations (1) with

$$|v(t) - u(t)| \leq \lambda_{\phi,h}h(t), \ t \in [0,T].$$

Remark 1.9. *A function $v \in W$ is a solution of the inequality (3) if and only if there exists a function $\psi : [0,T] \to \mathbb{R}$ (which depend on v) such that*

(1) : $|\psi(t)| \leq \mu, t \in [0,T]$.

(2) : $\left[D^{\alpha} + kD^{\beta}\right]v(t) = \phi\left(t, v(t), v(\eta t), D^{\beta}v(\eta t)\right) + \psi(t), t \in [0,T]$.

2. Main Results

We prove the following auxiliary lemma which is pivotal to define the solution for the problem (1)-(2).

Lemma 2.1. *Suppose that* $h \in C([0,T], \mathbb{R})$ *and consider the problem*

$$\begin{cases} \left[D^{\alpha} + kD^{\beta}\right] u(t) = h(t), \ t \in [0,T], \\ u(0) = f(u), \quad u(T) = \theta, \theta \in \mathbb{R}, \\ k > 0, \ 1 < \alpha \leq 2, \ 0 < \beta \leq 1. \end{cases} \tag{6}$$

Then, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-x)^{\alpha-\beta-1} \left(\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\, ds \right) dx \\ &- \frac{k}{\Gamma(\alpha-\beta)} \int_0^t (t-x)^{\alpha-\beta-1} u(x)\, dx \\ &+ \frac{t^{\alpha-\beta}}{T^{\alpha-\beta}} \left[\theta - \frac{1}{\Gamma(\alpha-\beta)} \int_0^T (T-x)^{\alpha-\beta-1} \left(\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\, ds \right) dx \right. \\ &\left. + \frac{k}{\Gamma(\alpha-\beta)} \int_0^T (T-x)^{\alpha-\beta-1} u(x)\, dx \right] - \left[\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}} - 1 \right] f(u). \end{aligned} \tag{7}$$

Proof. We have

$$\left[D^{\alpha} + kD^{\beta}\right] u(t) = h(t). \tag{8}$$

Now, writing the linear sequential fractional pantograph differential equation in (8) as

$$D^{\beta} \left[D^{\alpha-\beta} + k\right] u(t) = h(t). \tag{9}$$

By taking the Riemann-Liouville fractional integral of order β for (9), we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-x)^{\alpha-\beta-1} \left(\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\, ds \right) dx \\ &- \frac{k}{\Gamma(\alpha-\beta)} \int_0^t (t-x)^{\alpha-\beta-1} u(x)\, dx + \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} c_0 + d_0, \end{aligned} \tag{10}$$

where c_0 and d_0 are arbitrary constants. By the boundary condition $u(0) = f(u)$, we conclude that $d_0 = f(u)$.

Using the boundary condition $u(T) = \theta$, we obtain that

$$\begin{aligned} c_0 &= \frac{\Gamma(\alpha-\beta+1)}{T^{\alpha-\beta}} \left[\theta - \frac{1}{\Gamma(\alpha-\beta)} \int_0^T (T-x)^{\alpha-\beta-1} \left(\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\, ds \right) dx \right. \\ &\left. + \frac{k}{\Gamma(\alpha-\beta)} \int_0^T (T-x)^{\alpha-\beta-1} u(x)\, dx - f(u) \right]. \end{aligned} \tag{11}$$

Substituting the values of c_0 and c_1 in (10), we obtain the solution (7). This completes the proof.

□

In view of Lemma 2.1, we define the operator $O : W \to W$ by:

$$\begin{aligned} Ou(t) &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1} \\ &\times\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\phi\left(t,u(s),u(\eta s),D^\beta u(\eta s)\right)ds\right)dx \\ &-\frac{k}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}u(x)\,dx \\ &+\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}}\left[\theta-\frac{1}{\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}\right. \\ &\times\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)ds\right)dx \\ &\left.+\frac{k}{\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}u(x)\,dx\right]-\left[\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}}-1\right]f(u). \end{aligned} \tag{12}$$

In the sequel, we need the following hypotheses:

(H_1) : $\phi : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exists nonnegative constant ω such that for all $t \in [0,1]$ and $y_i,\ z_i \in \mathbb{R}\ (i = 1,2,3)$,

$$|\phi(t,y_1,y_2,y_3)-\phi(t,z_1,z_2,z_3)| \le \omega\left(|t_1-z_1|+|y_2-z_2|+|y_3-z_3|\right).$$

(H_2) : $f : C([0,T],\mathbb{R}) \to \mathbb{R}$ is a continuous function with $f(0) = 0$ and there exists constant $\varpi > 0$ such that

$$|f(u)-f(v)| \le \varpi\|u-v\|,\ \forall u,v \in C([0,T],\mathbb{R}).$$

For convenience, we define:

$$\begin{aligned} \nabla_0 &:= \frac{\omega T^\alpha}{\Gamma(\alpha+1)}+\frac{kT^\alpha}{\Gamma(\alpha-\beta+1)}+\varpi, \\ \nabla_1 &:= \frac{\omega T^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\alpha-\beta)\,\omega T^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{kT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{(\alpha-\beta)\,kT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}+\frac{\varpi(\alpha-\beta)}{T^\alpha}, \\ \Pi_0 &:= \frac{2T^\alpha L}{\Gamma(\alpha+1)}+|\theta|,\ \Pi_1 = \frac{(\alpha-\beta)}{T^\alpha}|\theta|. \end{aligned} \tag{13}$$

2.1. Existence and uniqueness of solution

The result is concerned with the existence and uniqueness of the solution for the problem (1)-(2 and is based on Banach's fixed point theorem.

Theorem 2.2. *Assume that* (H_1) *and* (H_2) *hold. If the inequality*

$$\nabla_1 T^{1-\beta} < \Gamma(2-\beta)(1-4\nabla_0), \tag{14}$$

is valid, then the problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Let us fix $\sup_{t\in[0,T]}\phi(t,0,0,0) = L < \infty$ and define

$$r \ge \frac{2\Pi_0+\frac{2T^{1-\beta}}{\Gamma(2-\beta)}\Pi_1}{1-4\nabla_0-\frac{T^{1-\beta}}{\Gamma(2-\beta)}\nabla_1}.$$

We show that $OB_r \subset B_r$, where O defined by (10) and $B_r = \{u \in W : \|u\|_W \le r\}$.

For $u \in B_r$, we have

$$\begin{aligned}
& \left|\phi\left(t, u(t), u(\eta t), D^{\beta} u(\eta t)\right)\right| \\
\leq\ & \left(\left|\phi\left(t, u(t), u(\eta t), D^{\beta} u(\eta t)\right)-\phi(t, 0,0,0)\right|+|\varphi(t, 0,0,0)|\right) \\
& \left(\left|\phi\left(t, u(t), u(\eta t), D^{\beta} u(\eta t)\right)-\phi(t, 0,0,0)\right|\right)+|\varphi(t, 0,0,0)| \\
\leq\ & \omega\left(|u(t)|+|u(\eta t)|+\left|D^{\beta} u(t)\right|\right)+L \leq \omega\left(2\|u\|+\left\|D^{\beta} u\right\|\right)+L \\
\leq\ & \omega\|u\|_{W}+N \leq \omega r+L,
\end{aligned} \tag{15}$$

and

$$|f(u)| \leq \varpi\|u\| \leq \varpi\|u\|_{W} \leq \varpi r. \tag{16}$$

It follows from (15) and (16), that

$$\begin{aligned}
|O u(t)| \leq\ & \frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{t}(t-x)^{\alpha-\beta-1} \\
& \times\left(\frac{1}{\Gamma(\beta)} \int_{0}^{x}(x-s)^{\beta-1}\left|\phi\left(t, u(s), u(\eta s), D^{\beta} u(\eta s)\right)\right| d s\right) d x \\
& +\frac{k}{\Gamma(\alpha-\beta)} \int_{0}^{t}(t-x)^{\alpha-\beta-1}|u(x)| d x \\
& +\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}}\left[|\theta|+\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{T}(T-x)^{\alpha-\beta-1}\right. \\
& \times\left(\frac{1}{\Gamma(\beta)} \int_{0}^{x}(x-s)^{\beta-1}\left|\phi\left(s, u(s), u(\eta s), D^{\beta} u(\eta s)\right)\right| d s\right) d x \\
& \left.+\frac{k}{\Gamma(\alpha-\beta)} \int_{0}^{T}(T-x)^{\alpha-\beta-1}|u(x)| d x\right]+\left|\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}}-1\right||f(u)| \\
\leq\ & 2\left(\frac{\omega T^{\alpha}}{\Gamma(\alpha+1)}+\frac{k T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\varpi\right) r+2\left(\frac{T^{\alpha} L}{\Gamma(\alpha+1)}\right)+|\theta| . \\
=\ & 2 \nabla_{0} r+\Pi_{0} .
\end{aligned} \tag{17}$$

On the other hand, we have

$$\begin{aligned}
&\left|O(u)'(t)\right| \leq \frac{1}{\Gamma(\alpha-\beta-1)}\int_0^t (t-x)^{\alpha-\beta-2} \\
&\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left|\phi\left(t,u(s),u(\eta s),D^\beta u(\eta s)\right)\right| ds\right) dx \\
&+\frac{k}{\Gamma(\alpha-\beta-1)}\int_0^t (t-x)^{\alpha-\beta-2}|u(x)|\,dx \\
&+\frac{(\alpha-\beta)t^{\alpha-\beta-1}}{T^{\alpha-\beta}}\left[|\theta|+\frac{1}{\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}\right. \\
&\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left|\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)\right| ds\right) dx \\
&\left.+\frac{k}{\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}|u(x)|\,dx\right]+\frac{(\alpha-\beta)t^{\alpha-\beta-1}}{T^{\alpha-\beta}}|f(u)| \\
\leq\ &\left(\frac{\omega T^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\alpha-\beta)\omega T^{\alpha}}{\Gamma(\alpha+1)}+\frac{kT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{(\alpha-\beta)kT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\right. \\
&\left.+\frac{\varpi(\alpha-\beta)}{T^\alpha}\right)r+\frac{(\alpha-\beta)}{T^\alpha}|\theta| \\
=\ &\nabla_1 r+\Pi_1,
\end{aligned} \tag{18}$$

which implies that

$$\left|D^\beta Ou(t)\right| \leq \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\left|O(u)'(t)\right| ds \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}\left(\nabla_1 r+\Pi_1\right). \tag{19}$$

Thus

$$\begin{aligned}
\|Ou\|_W &= 2\|Ou\|+\left\|D^\beta Ou\right\| \\
&\leq \left(4\nabla_0+\frac{T^{1-\beta}}{\Gamma(2-\beta)}\nabla_1\right)r+2\Pi_0+\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Pi_1 \leq r,
\end{aligned} \tag{20}$$

which implies that $OB_r \subset B_r$. Now for $u, v \in B_r$ and for all $t \in [0,T]$, we obtain

$$\begin{aligned}
&|Ou(t)-Ov(t)| \\
\leq\ &\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\right. \\
&\left.\left|\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)-\phi\left(s,v(s),v(\eta s),D^\beta v(\eta s)\right)\right| ds\right) dx \\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}|u(x)-v(x)|\,dx \\
&+\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\right. \\
&\left.\left|\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)-\phi\left(s,v(s),v(\eta s),D^\beta v(\eta s)\right)\right| ds\right) dx \\
&+\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}|u(x)-v(x)|\,dx+\left|\frac{t^{\alpha-\beta}}{T^{\alpha-\beta}}-1\right||f(u)-f(v)| \\
\leq\ &2\left[\frac{T^\alpha\omega}{\Gamma(\alpha+1)}+\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\varpi\right]\|u-v\|_W \\
=\ &2\nabla_0\|u-v\|_W.
\end{aligned} \tag{21}$$

We also have

$$\begin{aligned}
&\left|O(u)'(t)-O(v)'(t)\right| \qquad (22)\\
\leq\ & \frac{1}{\Gamma(\alpha-\beta-1)}\int_0^t (t-x)^{\alpha-\beta-2}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\right.\\
&\left.\left|\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)-\phi\left(s,v(s),v(\eta s),D^\beta v(\eta s)\right)\right|ds\right)dx\\
&+\frac{1}{\Gamma(\alpha-\beta-1)}\int_0^t (t-x)^{\alpha-\beta-2}\left|u(x)-v(x)\right|dx\\
&+\frac{(\alpha-\beta)t^{\alpha-\beta-1}}{T^{\alpha-\beta}\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\right.\\
&\left.\left|\phi\left(s,u(s),u(\eta s),D^\beta u(\eta s)\right)-\phi\left(s,v(s),v(\eta s),D^\beta v(\eta s)\right)\right|ds\right)dx\\
&\frac{(\alpha-\beta)t^{\alpha-\beta-1}}{T^{\alpha-\beta}\Gamma(\alpha-\beta)}\int_0^T (T-x)^{\alpha-\beta-1}\left|u(x)-v(x)\right|dx+\frac{(\alpha-\beta)t^{\alpha-\beta}}{T^{\alpha-\beta}}\left|f(u)-f(v)\right|\\
\leq\ & \left(\frac{\omega T^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\alpha-\beta)\omega T^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{(\alpha-\beta)T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\right.\\
&\left.+\frac{\varpi(\alpha-\beta)}{T^\alpha}\right)\|u-v\|_W\\
=\ & \nabla_1\|u-v\|_W.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\left|D^\beta Ou(t)-D^\beta Ov(t)\right| &\leq \int_0^t\frac{(t-s)^{1-(\alpha+\beta)}}{\Gamma(2-\alpha-\beta)}\left|O(u)'(t)-O(v)'(t)\right|ds\\
&\leq \frac{\nabla_1 T^{1-\beta}}{\Gamma(2-\beta)}\|u-v\|_W. \qquad (23)
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
\|Ou-Ov\|_W &= 2\|Ou-Ov\|+\left\|D^\beta Ou-D^\beta Ov\right\|\\
&\leq \left(4\nabla_0+\frac{\nabla_1 T^{1-\beta}}{\Gamma(2-\beta)}\right)\|u-v\|_W. \qquad (24)
\end{aligned}$$

By (14), we see that O is a contractive operator. Consequently, by the Banach fixed point theorem, O has a fixed point which is a solution of system (1)-(2). This completes the proof. □

2.2. Ulam-Hyers-Rassias stability

In the following section, we will study Ulam's type stability of the sequential fractional pantograph differential equations (1).

Theorem 2.3. *Assume that $\phi:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying (H_1). If*

$$\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}<1-\frac{\omega T^\alpha}{\Gamma(\alpha+1)}. \qquad (25)$$

Then the sequential fractional pantograph differential equations (1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.

Proof. Let $v\in W$ be a solution of the inequality (3), i.e.

$$\left|\left[D^\alpha+kD^\beta\right]v(t)-\phi\left(t,v(t),v(\eta t),D^\beta v(\eta t)\right)\right|\leq\mu, t\in[0,T],$$

and let us denote by $u \in W$ the unique solution of the problem

$$\begin{aligned} \left[D^{\alpha}+kD^{\beta}\right]u(t) &= \phi\left(t,u(t),u(\eta t),D^{\beta}u(\eta t)\right), k>0, t\in[0,T], 0<\eta<1, \\ u(0) &= v(0),\ u(T)=v(T). \end{aligned}$$

Thanks to Lemma 2.1, we can write

$$u(t)=I^{\alpha-\beta}\left[I^{\beta}h(t)\right]-kI^{\alpha-\beta}u(t)+c_0\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+d_0.$$

By integration of the inequality (3), we obtain

$$\begin{aligned} &\left|v(t)-I^{\alpha-\beta}\left[I^{\beta}h_v(t)\right]-kI^{\alpha-\beta}v(t)-c_1\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-d_1\right| \\ \leq\ & \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)}\leq\frac{\mu T^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{26}$$

where

$$h_v(t)=\phi\left(t,v(t),v(\eta t),D^{\beta}v(\eta t)\right). \tag{27}$$

On the other hand, if $u(0)=v(0)$ and $u(T)=v(T)$, then

$$c_0=c_1 \text{ and } d_0=d_1.$$

For any $t\in[0,T]$, we have

$$\begin{aligned} v(t)-u(t) &= v(t)-I^{\alpha-\beta}\left[I^{\beta}h_u(t)\right]-kI^{\alpha-\beta}v(t)-c_1\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-d_1 \\ &\quad +I^{\alpha-\beta}\left[I^{\beta}\left(h_v(t)-h_u(t)\right)\right]-kI^{\alpha-\beta}\left(v(t)-u(t)\right), \end{aligned} \tag{28}$$

where

$$h_u(t)=\phi\left(t,u(t),u(\eta t),D^{\beta}u(\eta t)\right), \tag{29}$$

then

$$\begin{aligned} &I^{\alpha-\beta}\left[I^{\beta}\left(h_v(t)-h_u(t)\right)\right] \\ &=I^{\alpha-\beta}\left[I^{\beta}\left(\phi\left(t,u(t),u(\eta t),D^{\beta}u(\eta t)\right)-\phi\left(t,v(t),v(\eta t),D^{\beta}v(\eta t)\right)\right)\right] \\ &=\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x(x-s)^{\beta-1}\phi\left(t,u(t),u(\eta t),D^{\beta}u(\eta t)\right)\right. \\ &\left.-\phi\left(t,v(t),v(\eta t),D^{\beta}v(\eta t)\right)ds\right)dx. \end{aligned} \tag{30}$$

By (H_1), we can write

$$\begin{aligned} &I^{\alpha-\beta}\left[I^{\beta}\left(h_v(t)-h_u(t)\right)\right] \\ \leq\ &\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x(x-s)^{\beta-1}\omega\left\|v(s)-u(s)\right\|_W ds\right)dx. \end{aligned} \tag{31}$$

Using (31), we get

$$\begin{aligned} |v(t)-u(t)| &\leq \left|v(t)-I^{\alpha-\beta}\left[I^{\beta}h_u(t)\right]-kI^{\alpha-\beta}v(t)-c_1\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-d_1\right| \\ &+\frac{\omega}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\|v(s)-u(s)\|_W\,ds\right)dx \\ &+\frac{k}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\|v(s)-u(s)\|_W\,ds. \end{aligned} \tag{32}$$

Hence,

$$|v(t)-u(t)| \leq \frac{\mu T^{\alpha}}{\Gamma(\alpha+1)}+\left(\frac{\omega T^{\alpha}}{\Gamma(\alpha+1)}+\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|v(s)-u(s)\|_W\,. \tag{33}$$

Then

$$\|v(s)-u(s)\|_W\ \left[1-\left(\frac{\omega T^{\alpha}}{\Gamma(\alpha+1)}+\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\right] \leq \frac{\mu T^{\alpha}}{\Gamma(\alpha+1)}. \tag{34}$$

For each $t\in[0,T]$

$$\|u(t)-v(t)\| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)\left[1-\left(\frac{\omega T^{\alpha}}{\Gamma(\alpha+1)}+\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\right]}\mu := \lambda_{\phi}\mu. \tag{35}$$

Therefore, the sequential fractional pantograph differential equations (1) is Ulam-Hyers stable. By taking $g_{\phi}(\sigma)=\lambda_{\phi}\mu, g_{\phi}(0)=0$ yields that the sequential fractional pantograph differential equations (1) is generalized Ulam-Hyers stable. □

Now, we will study Ulam-Hyers-Rassias stability of the sequential fractional pantograph differential equations (1).

Theorem 2.4. *Let $\varphi:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function and suppose that (H_1) and (25) hold. In addition, the following hypothesis holds*

(H_3) : There exists an function $h\in C([0,T],\mathbb{R}_+)$ and there exists $\pi_h>0$ such that for any $t\in[0,T]$

$$\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}h(s)\,ds\right)dx \leq \pi_h h(t).$$

Then the sequential fractional pantograph differential equations (1) is Ulam-Hyers-Rassias stable.

Proof. Let us denote by $v\in W$ the solution of the inequality (4), i.e.

$$\left|\left[D^{\alpha}+kD^{\beta}\right]v(t)-\phi\left(t,v(t),v(\eta t),D^{\beta}v(\eta t)\right)\right| \leq \mu h(t)\,,\ t\in[0,T]\,.$$

Let $u\in W$ be a the unique solution of the problem

$$\begin{aligned} \left[D^{\alpha}+kD^{\beta}\right]u(t) &= \phi\left(t,u(t),u(\eta t),D^{\beta}u(\eta t)\right),k>0,t\in[0,T],0<\eta<1, \\ u(0) &= v(0)\,,\ u(T)=v(T)\,. \end{aligned}$$

Applying Lemma 2.1, we get

$$u(t)=I^{\alpha-\beta}\left[I^{\beta}h(t)\right]-kI^{\alpha-\beta}u(t)+c_0\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+d_0.$$

Integrating the inequality (4), we can state that

$$\begin{aligned} &\left| v\left(t\right) - I^{\alpha-\beta}\left[I^{\beta}h_v\left(t\right)\right] - kI^{\alpha-\beta}v\left(t\right) - c_1\frac{t^{\alpha-\beta}}{\Gamma\left(\alpha-\beta+1\right)} - d_1\right| \\ \leq\ & \frac{\mu}{\Gamma\left(\alpha-\beta\right)}\int_0^t \left(t-x\right)^{\alpha-\beta-1}\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^x \left(x-s\right)^{\beta-1} h\left(s\right)ds\right)dx, \end{aligned} \tag{36}$$

where $h_v\left(t\right)$ is given by (27).

So, by (H_1), we obtain

$$\begin{aligned} \left|v\left(t\right)-u\left(t\right)\right| \ \leq\ & \frac{\mu}{\Gamma\left(\alpha-\beta\right)}\int_0^t \left(t-x\right)^{\alpha-\beta-1}\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^x \left(x-s\right)^{\beta-1} h\left(s\right)ds\right)dx \\ &+\frac{1}{\Gamma\left(\alpha-\beta\right)}\int_0^t \left(t-x\right)^{\alpha-\beta-1}\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^x \left(x-s\right)^{\beta-1} \|v(s)-u(s)\|_W\,ds\right)dx \\ &+\frac{k}{\Gamma\left(\alpha-\beta\right)}\int_0^t \left(t-s\right)^{\alpha-\beta-1}\|v(s)-u(s)\|_W\,ds. \end{aligned} \tag{37}$$

Now, using (H_3), we have

$$\left|v\left(t\right)-u\left(t\right)\right| \leq \mu\pi_h h(t) + \left(\frac{\omega T^{\alpha}}{\Gamma\left(\alpha+1\right)} + \frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|v(s)-u(s)\|_W\,, \tag{38}$$

which implies that

$$\|v(s)-u(s)\|\left[1-\frac{\omega T^{\alpha}}{\Gamma\left(\alpha+1\right)}-\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] \leq \mu\pi_h h(t). \tag{39}$$

For any $t\in[0,T]$, we have

$$\|u\left(t\right)-v\left(t\right)\| \leq \left[\frac{\pi_h}{1-\frac{\omega T^{\alpha}}{\Gamma(\alpha+1)}-\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}}\right]\mu h(t) := \lambda_{\phi}\mu h(t). \tag{40}$$

Then, the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable. □

3. An example

To illustrate our main results, we treat the following example.

Example 3.1. *Let us consider the following fractional boundary value problem*

$$\begin{cases} \left(D^{\frac{5}{3}}+\frac{1}{10}D^{\frac{3}{4}}\right)u\left(t\right) = \frac{1}{2.20^2\pi}\sin\left(2\pi u\left(t\right)\right)+\frac{1}{20^2}\sin\left(t\right)u\left(\frac{1}{2}t\right) \\ \qquad +\frac{1}{20^2}D^{\frac{3}{4}}u\left(\frac{1}{2}t\right)+\frac{5}{23},\ t\in[0,1]\,, \\ x\left(0\right)=\frac{1}{60}\tan^{-1}u\left(t\right),\ u\left(1\right)=\frac{7}{5}. \end{cases} \tag{41}$$

For this example, we have

$$\begin{aligned} \varphi\left(t,u,v,w\right) &= \frac{1}{2.20^2\pi}\sin\left(2\pi u\right)+\frac{1}{20^2}\sin\left(t\right)u \\ +\frac{1}{20^2}w+\frac{5}{23},\ t &\in [0,1]\,, \end{aligned}$$

and

$$f\left(u\right)=\frac{1}{60}\tan^{-1}u.$$

So, for any $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathbb{R}^3$ *and* $t \in [0,1]$, *we can write*

$$|\varphi(t, u_1, v_1, w_1) - \varphi(t, u_2, v_2, w_2)| \leq \frac{1}{20^2}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|).$$

Hence the condition (H_1) *holds with* $\omega = \frac{1}{20^2}$. *Also for all* $u, v \in C([0,1], \mathbb{R})$, *we have*

$$|f(u) - f(v)| \leq \frac{1}{60}|u - v|.$$

So, (H_2) *is satisfied with* $\varpi = \frac{1}{60}$.

For $\alpha = \frac{5}{3}, \beta = \frac{3}{4}, k = \frac{1}{10}, T = 1$ *and* $\eta = \frac{1}{2}$, *we have*

$$\nabla_0 = 0.121\,68, \quad \nabla_1 = 0.209\,05.$$

Using the given data, we find that

$$\frac{\nabla_1 T^{1-\beta}}{\Gamma(2-\beta)} = 0.230\,64 < 1 - 4\nabla_0 = 0.513\,2.$$

Hence by Theorem 11, the problem (41) has a unique solution.

Also, we have

$$\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} = 0.103\,35 < 1 - \frac{\omega T^{\alpha}}{\Gamma(\alpha+1)} = 0.998\,34.$$

Then, all the hypotheses of Theorem 12 are satisfied. Thus, by the conclusion of Theorem 12, problem (41) is Ulam-Hyers stable.

Let $h(t) = \gamma t^2, \gamma \in \mathbb{R}$. *We have*

$$\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-x)^{\alpha-\beta-1}\left(\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} h(s)\, ds\right) dx \leq \frac{2\gamma}{\Gamma\left(\frac{14}{3}\right)} t^2 = \pi_h h(t).$$

Thus condition (H_3) *is satisfied with* $h(t) = \gamma t^2$ *and* $\pi_h = \frac{2\gamma}{\Gamma\left(\frac{14}{3}\right)}$. *It follows from Theorem 13 problem (41) is Ulam-Hyers-Rassias stable.*

4. Conclusion

In this work, we have discussed the uniqueness and different types of Ulam-stability of solutions for sequential fractional pantograph differential equations with nonlocal boundary conditions. We have establish the uniqueness results applying the Banach contraction principle. Also, we have proved different types of Ulam stability results including Ulam-Hyers stability, generalized Ulam-Hyers stability and Ulam-Hyers-Rassias. For justification, a numerical example has been given to illustrate our main results.

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