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On Super Magic Algorithm for Union of Comb and Star Graph

Rongbing Huang[1], Muhammad Farhan Hanif[2], Muhammad Kamran Siddiqui[2*] and Muhammad Faisal Nadeem[2]

[1]*School of Computer Science Chengdu University, Chengdu, China*
[2]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan*
[*]*Corresponding author*

Abstract

In [7], Enomoto *et al.* identified the concept of super edge magic total labeling of graphs by getting motivation from the idea of edge-magic labeling of graphs that was brought into light by Kotzig and Rosa [20]. An edge magic total labeling of a graph G is a one to one map ϕ from $V(G) \cup E(G)$ onto the set $\{1,2,\ldots,|V(G)|+|E(G)|\}$ with the property that, there is an integer constant α such that $\phi(u)+\phi(uv)+\phi(v)=\alpha$ for any $(u,v) \in E(G)$. Moreover if $\phi(V(G))=\{1,2,\ldots,|V(G)|\}$, then edge magic total labeling is called super edge magic total labeling. In this paper, we study the super edge magic total labeling of generalized comb graph.

Keywords: *magic graph; SEM total labeling; comb graph; star*
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1. Introduction

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set by $V(G)$, edge set by $E(G)$ of a graph G and their cardinalities by p and q respectively, i.e. $|V(G)|=p$ and $|E(G)|=q$. A *labeling* of a graph is a map that carries the graph elements to numbers (usually positive integers). In this paper the domain will usually be the set of all vertices and edges, such labelings are called *total labeling*. Some labelings use the vertex-set only, or the edge-set only, this type of labeling is called *vertex-labeling* and *edge-labeling* respectively. Other domains are also possible like the set of faces of the graph. There are many types of labelings namely, graceful labeling, alpha labeling, antimagic labeling etc [1,2].
In this paper, we focus on one type of labeling called *edge-magic total labeling*. An *edge magic total labeling* of a graph G is a bijection $\phi: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that $\phi(x)+\phi(xy)+\phi(y)$ is constant, for every edge $xy \in E(G)$. A graph with an edge magic total labeling is called *edge magic graph*. An edge magic total labeling ϕ is called *super edge magic total* if $\phi(V(G))=\{1,2,\ldots,p\}$. A graph with super edge magic total labeling is called a *super edge magic graph*[14,15,16,17] .
Graph theory can be applied to a wide range of fields and problems. In chemistry and physics graph theory can be used to study molecules or chemical reactions. A graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. Graph theory has applications in cheminformatics, medicinal chemistry, biology, and biochemistry[6]. This approach is especially used in computer processing of molecular structures, ranging from chemical editors, database searching, find similarity functions, sub-structure searching, all of which are important in drug design algorithms [7,8]. Zhang et. al [9, 10, 11] discuss the topological indices of generalized bridge molecular graphs, Carbon Nanotubes and product of chemical graphs. Zhang et. al [12, 13, 14] provided the physical analysis of heat for formation and entropy of Ceria Oxide
The subject of edge-magic total labeling of graph has its origin in the work of Kotzig and Rosa [20], on what they called magic valuations of graphs. The notation of super edge-magic labeling was introduced by Enomotp *et al.* in [7] as super edge magic total labeling. A number of classification studies on super edge-magic total graphs has been intensively investigated. More detail, the results on edge magic and super edge magic labeling of some graphs can be seen in [9,10,11,12,13] and a complete survey [15] and for more details see [3, 4, 5]. In this paper, we mainly focussing on the comb and star graphs.

2. The generalized comb graph

A generalized comb graph is obtained from a path P_{n+1} having vertices $u_{0,1},u_{1,1},u_{2,1},u_{3,1},\ldots,u_{n,1}$ by joining n new paths P_i ; $1 \le i \le n$ of order t_i with vertices $u_{1,1},u_{2,1},u_{3,1},\ldots,u_{n,1}$ respectively and is denoted by $Cb_n(t_1,t_2,\ldots,t_{n-1},t_n)$. So the vertex set and the edge set of

Email addresses: **huangrb2006@126.com (Rongbing Huang), farhanlums@gmail.com (Muhammad Farhan Hanif), kamransiddiqui75@gmail.com (Muhammad Kamran Siddiqui), mfaisalnadeem@ymail.com (Muhammad Faisal Nadeem)**

generalized comb graph are defined as follows:

$$V(Cb_n) = \{u_{i,j} : 1 \le i \le n,\ 1 \le j \le t_i\} \cup \{u_{0,1}\},$$

$$E(Cb_n) = \{(u_{i-1,1}\, u_{i,1})\ : 1 \le i \le n\} \cup \{(u_{i,j}\, u_{i,j+1})\ : 1 \le i \le n,\ 1 \le j \le t_i - 1\}$$

with $|V(Cb_n)| = 1 + \sum_{i=1}^{n} t_i$ and $|E(Cb_n)| = \sum_{i=1}^{n} t_i$.

Now, we investigate the super edge magic total labeling of generalized comb graph. In order to prove our main results, we will frequently use the following lemma:

Lemma 2.1. [8] *A (p,q) graph G is super edge magictotal if and only if there exists a bijective function $\phi : V(G) \to \{1,2,\cdots,p\}$ such that the set $S = \{\phi(x)+\phi(y) : xy \in E(G)\}$ consists of q consecutive integers. In such a case, ϕ extends to a super edge magictotal labeling of G with magicconstant $\alpha = p+q+s$, where $s = min(S)$.*

Theorem 2.2. *For $n \ge 3$, $t_1 \ge 3$, the generalized comb $Cb_n(t_1, t_1+1, t_1+2, \ldots, t_1+n-2, t_1+n-1)$ admits super edge magictotal labeling.*

Proof: By the construction of generalized comb, we find that
$t_i = t_1 - 1 + i,\ \ for\ \ 2 \le i \le n$, with $|V(Cb_n)| = \frac{n(2t_1+n-1)+2}{2}$ and $|E(Cb_n)| = \frac{n(2t_1+n-1)}{2}$.
Now Its come to show that Cb_n is super edge magic we define the labeling $\phi : V(Cb_n) \to \{1,2,3,\ldots,\frac{n(2t_1+n-1)+2}{2}\}$ for $1 \le i \le n$ and $1 \le j \le t_i$ as follows:

***Case 1*:** when t_1 is odd

(a) *If* i, j have same parity,

$$\phi(u_{i,j}) = \begin{cases} \frac{j+1}{2}, & if\ i=1 \\ \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-2} t_r + \frac{i+j}{2}, & if\ 3 \le i \le n, odd \\ \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-1} t_r + 1 + \frac{i-j}{2}, & if\ 2 \le i \le n, even. \end{cases}$$

(b) *If* i, j have different parity and

(i) n is odd

$$\phi(u_{i,j}) = \begin{cases} \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-2} t_r + \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-1} t_r + \frac{2n+t_1+2-j}{2}, & if\ 2 \le i \le n, even \\ \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-2} t_r + \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-2} t_r + \frac{2n+t_1+1+j}{2}, & if\ 3 \le i \le n, odd. \end{cases}$$

$$\phi(u_{0,1}) = \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-2} t_r + \frac{2n+t_1+1}{2},\ \phi(u_{1,j}) = \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-2} t_r + \frac{2n+t_1+1+j}{2},$$

It is easy to see that under the labeling ϕ the set of all edge-sums is
$S_1 = \left\{ \frac{1}{2}\left[\sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-2} 2t_r + 2n + t_1 + 1\right] + i;\ \ 1 \le i \le \frac{n(2t_1+n-1)}{2} \right\}$

(ii) n is even

$$\phi(u_{0,1}) = \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-1} t_r + \frac{n+2}{2},\ \phi(u_{1,j}) = \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-1} t_r + \frac{n+2+j}{2},$$

$$\phi(u_{i,j}) = \begin{cases} \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-1} t_r + \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-2} t_r + \frac{n+2+j}{2}, & if\ 3 \le i \le n, odd \\ \\ \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-1} t_r + \sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{i-1} t_r + \frac{n+3-j}{2}, & if\ 2 \le i \le n, even \end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums
$S_2 = \left\{ \frac{1}{2}\left[\sum_{\substack{r=1 \\ r\equiv 1(mod 2)}}^{n-1} 2t_r + n + 2\right] + i;\ \ 1 \le i \le \frac{n(2t_1+n-1)}{2} \right\}$

***Case 2*:** when t_1 is even

(a) *If* i, j have different parity,

$$\phi(u_{0,1}) = \frac{t_1+2}{2},\quad \phi(u_{1,j}) = \frac{t_1+2-j}{2},$$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{i-1} 2t_r + 4 + i + 2j\right], & if\ 2 \le i \le n, even \\ \frac{1}{4}\left[\sum_{r=1}^{i} 2t_r + 7 + i - 2j\right], & if\ 3 \le i \le n, odd \end{cases}$$

(b) *If* i, j have same parity and

(i) n is odd

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{n} 2t_r + 2t_1 + n + 5 - 2j\right], & if\ \text{i} = 1 \\ \frac{1}{4}\left[\sum_{r=1}^{n} 2t_r + \sum_{r=1}^{i} 2t_r + n + 6 - i - 2j\right], & if\ 3 \le i \le n, odd \\ \frac{1}{4}\left[\sum_{r=1}^{n} 2t_r + \sum_{r=1}^{i-1} 2t_r + n + 5 - i + 2j\right], & if\ 2 \le i \le n, even \end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$S_3=\left\{\frac{1}{4}\left[\sum_{r=1}^{n}2t_r+n+7\right]+i;\ \ 1\le i\le \frac{n(2t_1+n-1)}{2}\right\}$

(ii) n is even

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\left[\sum\limits_{r=1}^{n-1}2t_r+4t_1+3n+4-2j\right], & if\ i=1\\ \frac{1}{4}\left[\sum\limits_{r=1}^{n-1}2t_r+\sum\limits_{r=1}^{i}2t_r+2t_1+3n+5-i-2j\right], & if\ 3\le i\le n, odd\\ \frac{1}{4}\left[\sum\limits_{r=1}^{n-1}2t_r+\sum\limits_{r=1}^{i-1}2t_r+2t_1+3n+4-i+2j\right], & if\ 2\le i\le n, even\end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$S_4=\left\{\frac{1}{4}\left[\sum_{r=1}^{n-1}2t_r+2t_1+3n+6\right]+i;\ \ 1\le i\le \frac{n(2t_1+n-1)}{2}\right\}$

Clearly, $|S_1|=|S_2|=|S_3|=|S_4|=\frac{n(2t_1+n-1)}{2}$. Therefore, by using Lemma 2.1, ϕ can be extended to a super edge-magictotal labeling. Hence, the graph Cb_n admits a super edge magictotal labeling.

Theorem 2.3. *For $n\ge 5$, the generalized comb $Cb_n(2,3,4,\ldots,\lfloor\frac{n}{2}\rfloor;\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor+1;\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor-1,\ldots;2)$ admits super edge magictotal labeling.*

Proof. By the definition of generalized comb, first we notice that

$$t_i=\begin{cases} i+1, & if\ 1\le i\le\lfloor\frac{n}{2}\rfloor-1\\ \lfloor\frac{n}{2}\rfloor+1, & if\ i=\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+1\\ 2\lfloor\frac{n}{2}\rfloor+2-i, & if\ \lfloor\frac{n}{2}\rfloor+2\le i\le n\ and\ n\ even\\ 2\lfloor\frac{n}{2}\rfloor+2-i, & if\ \lfloor\frac{n}{2}\rfloor+2\le i\le n-1\ and\ n\ odd\\ 2, & if\ i=n\ odd\end{cases}$$

$|V(Cb_n)|=\frac{n^2+6n+4}{4},\ \ |E(Cb_n)|=\frac{n^2+6n}{4}$, for n even

$|V(Cb_n)|=\frac{n^2+4n+7}{4},\ \ |E(Cb_n)|=\frac{n^2+4n+3}{4}$, for n odd

Now it comes to show that Cb_n is super edge magic we define the labeling $\phi: V(Cb_n)\to\{1,2,3,\ldots,|V(Cb_n)|\}$ as follows:

$\phi(u_{0,1})=1,\ \ \phi(u_{1,2})=2,\ \ \phi(u_{2,j})=\frac{2t_1+1+j}{2},\ if\ \ 1\le j\le t_2,\ odd$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\left[\sum\limits_{r=1}^{i}2t_r+7+i-2j\right], & if\ 3\le i\le\lfloor\frac{n}{2}\rfloor, odd\\ & and\ 1\le j\le t_i, even\\ \frac{1}{4}\left[\sum\limits_{r=1}^{i-1}2t_r+4+i+2j\right], & if\ 4\le i\le\lfloor\frac{n}{2}\rfloor+1, even\\ & and\ 1\le j\le t_i, odd\end{cases}$$

***Case 1*:** when n is odd

For $3\le i\le\lfloor\frac{n}{2}\rfloor, odd\ \ and\ \ 1\le j\le t_i,\ odd$

$\phi(u_{i,j})=\frac{1}{4}\left[\sum_{r=1}^{n-2}2t_r+\sum_{r=3}^{i}2t_r+2\lfloor\frac{n}{2}\rfloor+28-n-i\right]-\frac{j+1}{2},$

For $4\le i\le\lfloor\frac{n}{2}\rfloor+1, even\ \ and\ \ 1\le j\le t_i, even$

$\phi(u_{i,j})=\frac{1}{4}\left[\sum_{r=1}^{n-2}2t_r+\sum_{r=1}^{i-1}2t_r+2\lfloor\frac{n}{2}\rfloor+25-n-i\right]+\frac{j}{2},$

For $\lfloor\frac{n}{2}\rfloor+1\le i\le n-2, odd$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\left[\sum\limits_{r=1}^{i}2t_r+2\lfloor\frac{n}{2}\rfloor+7-i-2j\right], & if\ 1\le j\le t_i, even\\ \frac{1}{4}\left[\sum\limits_{r=1}^{n-2}2t_r+\sum\limits_{r=1}^{i}2t_r+16-n+i-2j\right], & if\ 1\le j\le t_i, odd\end{cases}$$

For $\lfloor\frac{n}{2}\rfloor+2\le i\le n-1, even$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\left[\sum\limits_{r=1}^{i-1}2t_r+2\lfloor\frac{n}{2}\rfloor+6-i+2j\right], & if\ 1\le j\le t_i, odd\\ \frac{1}{4}\left[\sum\limits_{r=1}^{n-2}2t_r+\sum\limits_{r=1}^{i-1}2t_r+13-n+i+2j\right], & if\ 1\le j\le t_i, even\end{cases}$$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{n-2} 2t_r + 2\lfloor\frac{n}{2}\rfloor - n + 17\right] + i - 1, & \text{if } i = j = 1,2 \\ \sum_{r=1}^{n-2} t_r + 5, & \text{if } i = n, j = 1 \\ \frac{1}{4}\left[\sum_{r=1}^{n-2} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 13 - n\right], & \text{if } i = n, j = 2 \end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$$S_1 = \left\{ \frac{1}{4}\left[\sum_{r=1}^{n-2} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 17 - n\right] + i;\ \ 1 \le i \le \frac{n^2+4n+3}{4} \right\}$$

***Case 2*:** when n is even

For $3 \le i \le \lfloor\frac{n}{2}\rfloor, odd\ \ and\ \ 1 \le j \le t_i, odd$

$$\phi(u_{i,j}) = \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + \sum_{r=3}^{i} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 23 - n - i\right] - \frac{j+1}{2},$$

For $4 \le i \le \lfloor\frac{n}{2}\rfloor + 1, even\ \ and\ \ 1 \le j \le t_i, even$

$$\phi(u_{i,j}) = \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + \sum_{r=1}^{i-1} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 20 - n - i\right] + \frac{j}{2},$$

For $\lfloor\frac{n}{2}\rfloor + 1 \le i \le n-1, odd$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{i} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 7 - i - 2j\right], & \text{if } 1 \le j \le t_i, even \\ \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + \sum_{r=1}^{i} 2t_r + 11 - n + i - 2j\right], & \text{if } 1 \le j \le t_i, odd \end{cases}$$

For $\lfloor\frac{n}{2}\rfloor + 2 \le i \le n, even$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{i-1} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 6 - i + 2j\right], & \text{if } 1 \le j \le t_i, odd \\ \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + \sum_{r=1}^{i-1} 2t_r + 8 - n + i + 2j\right], & \text{if } 1 \le j \le t_i, even \end{cases}$$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + 2\lfloor\frac{n}{2}\rfloor - n + 12\right], & \text{if } i = j = 1 \\ \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + 2\lfloor\frac{n}{2}\rfloor - n + 16\right], & \text{if } i = j = 2 \end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$$S_2 = \left\{ \frac{1}{4}\left[\sum_{r=1}^{n-1} 2t_r + 2\lfloor\frac{n}{2}\rfloor + 12 - n\right] + i;\ \ 1 \le i \le \frac{n^2+6n}{4} \right\}$$

Clearly, $|S_1| = \frac{n^2+4n+3}{4},\ \ |S_2| = \frac{n^2+6n}{4}$. Therefore by using Lemma 2.1, ϕ can be extended to a super edge-magictotal labeling. Hence, the graph Cb_n admits a super edge magictotal labeling.
In next theorems, we formulate super-edge magictotal labeling for disjoint union of generalized combs and star.

Theorem 2.4. *For $n \ge 4$, $m \ge 3$, the graph $G \cong Cb_n(2,3,4,\ldots,n-1,n,n+1) \cup K_{1,m}$ admits super edge magictotal labeling.*

Proof. Let $G \cong Cb_n(2,3,4,\ldots,n-1,n,n+1) \cup K_{1,m}$, the vertex set and edge set of G are defined as follows.

$V(G) = \{u_{i,j} : 1 \le i \le n,\ 1 \le j \le t_i\} \cup \{u_{0,1}\} \cup \{c, v_l; 1 \le l \le m\}$, and

$E(G) = \{(u_{i-1,1}\, u_{i,1})\ : 1 \le i \le n\} \cup \{(u_{i,j}\, u_{i,j+1})\ : 1 \le i \le n,\ 1 \le j \le t_i - 1\} \cup \{(cv_l); 1 \le l \le m\}$
with $|V(G)| = \frac{n^2+3n+2m+4}{2},\ |E(G)| = \frac{n^2+3n+2m}{2}$.

Also, we observe that $t_i = 1 + i,\ \ for\ \ 1 \le i \le n$.

We define the labeling $\phi : V(G) \to \{1,2,3,\ldots,\frac{n^2+3n+2m+4}{2}\}$ for $1 \le i \le n$ and $1 \le j \le t_i$ as follows:

***Case 1*:** when i, j have different parities,

$$\phi(c) = 2,\ \phi(u_{1,2}) = 1,\ \phi(u_{2,3}) = 3,\ \phi(u_{2,1}) = 4,\ \phi(u_{0,1}) = 5,$$

$$\phi(u_{i,j}) = \begin{cases} \frac{1}{4}\left[\sum_{r=3}^{i} 2t_r + 21 + i - 2j\right], & \text{if } 3 \le i \le n,\ odd \\ \frac{1}{4}\left[\sum_{r=3}^{i-1} 2t_r + 18 + i + 2j\right], & \text{if } 4 \le i \le n,\ even \end{cases}$$

***Case 2*:** when i, j have the same parity and

(i) n is odd

$\phi(v_l)=\frac{1}{4}\Big[\sum\limits_{r=3}^{n}2t_r+17+n+4l\Big], \quad if \quad 1\leq l\leq m,$

$\phi(u_{i,j})=\frac{1}{4}\Big[\sum\limits_{r=3}^{n}2t_r+21+n+4m\Big]+i-1, \quad if \ i=j=1,2$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\Big[\sum\limits_{r=3}^{n}2t_r+\sum\limits_{r=3}^{i}2t_r+n+4m+30-i-2j\Big], if\ 3\leq i\leq n, odd\\ \frac{1}{4}\Big[\sum\limits_{r=3}^{n}2t_r+\sum\limits_{r=3}^{i-1}2t_r+n+4m+29-i+2j\Big], if\ 4\leq i\leq n, even.\end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$S_1=\Big\{\frac{1}{4}\Big[\sum\limits_{r=3}^{n}2t_r+n+25\Big]+i;\ \ 1\leq i\leq\frac{n^2+3n+2m}{2}\Big\}.$

(ii) n is even

$\phi(v_l)=\frac{1}{4}\Big[\sum\limits_{r=3}^{n-1}2t_r+3n++20+4l\Big], \quad if \quad 1\leq l\leq m,$

$\phi(u_{i,j})=\frac{1}{4}\Big[\sum\limits_{r=3}^{n-1}2t_r+24+3n+4m\Big]+i-1, \quad if \ i=j=1,2$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\Big[\sum\limits_{r=3}^{n-1}2t_r+\sum\limits_{r=3}^{i}2t_r+3n+4m+33-i-2j\Big], if\ 3\leq i\leq n, odd\\ \frac{1}{4}\Big[\sum\limits_{r=3}^{n-1}2t_r+\sum\limits_{r=3}^{i-1}2t_r+3n+4m+32-i+2j\Big], if\ 4\leq i\leq n, even\end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$S_2=\Big\{\frac{1}{4}\Big[\sum\limits_{r=3}^{n-1}2t_r+n+28\Big]+i;\ \ 1\leq i\leq\frac{n^2+3n+2m}{2}\Big\}.$

Clearly, $|S_1|=|S_2|=\frac{n^2+3n+2m}{2}$. Therefore by using Lemma 2.1, ϕ can be extended to a super edge-magictotal labeling. So, the graph G admits a super edge magictotal labeling. By now, the proof is complete.

Theorem 2.5. *For $n\geq 5$, $m\geq 3$, the graph $G\cong Cb_n(2,3;2\lceil\frac{i}{2}\rceil+1)\cup K_{1,m}$ for $3\leq i\leq n$, admits super edge magictotal labeling.*

Proof. Let $G\cong Cb_n(2,3;2\lceil\frac{i}{2}\rceil+1)\cup K_{1,m}$, the vertex set and the edge set of G are defined as follows:

$V(G)=\{u_{i,j}:1\leq i\leq n,\ 1\leq j\leq t_i\}\cup\{u_{0,1}\}\cup\{c,v_l;1\leq l\leq m\}$

$E(G)=\{(u_{i-1,1}\,u_{i,1})\ :1\leq i\leq n\}\cup\{(u_{i,j}\,u_{i,j+1})\,1\leq i\leq n,\ 1\leq j\leq t_i-1\}\cup\{(cv_l);1\leq l\leq m\}$

with $|V(G)|=\frac{n^2+4n+2m+3}{2}$, $|E(G)|=\frac{n^2+4n+2m-1}{2}$, for n *odd*

$|V(G)|=\frac{n^2+4n+2m+2}{2}$, $|E(G)|=\frac{n^2+4n+2m-2}{2}$, for n *even*

Also, we observe that

$$t_i=\begin{cases}1+i, & if\ i=1,2\\ 2\lceil\frac{i}{2}\rceil+1 & if\ 3\leq i\leq n\end{cases}$$

We define the labeling $\phi:V(G)\rightarrow\{1,2,3,\ldots,|V(G)|\}$ for $1\leq i\leq n$ and $1\leq j\leq t_i$ as follows:

***Case 1*:** when i,j have different parity

$\phi(c)=2,\phi(u_{2,3})=1,\phi(u_{1,2})=4,\ \phi(u_{2,1})=5,\phi(u_{0,1})=3,$
$\phi(u_{3,2})=6,\ \phi(u_{3,4})=7,$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{2}\Big[\sum\limits_{r=3}^{i-1}t_r+10+j\Big], & if\ 5\leq i\leq n, odd\\ \frac{1}{2}\Big[\sum\limits_{r=3}^{i}t_r+11-j\Big], & if\ 4\leq i\leq n, even\end{cases}$$

***Case 2*:** when i,j have same parity and

(i) n is odd

$\phi(v_l)=\frac{1}{2}\Big[\sum\limits_{r=3}^{n-1}t_r+2\lceil\frac{n}{2}\rceil+10+2l\Big], \quad if \quad 1\leq l\leq m$

$\phi(u_{i,j})=\frac{1}{2}\Big[\sum\limits_{r=3}^{n-1}t_r+2\lceil\frac{n}{2}\rceil+2m+12\Big]+i-1, \quad if \ i=j=1,2$

$\phi(u_{3,j})=\frac{1}{2}\Big[\sum\limits_{r=3}^{n-1}t_r+2\lceil\frac{n}{2}\rceil+2m+15+j\Big], \quad if \ 1\leq j\leq t_3\,odd$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{2}\left[\sum_{r=3}^{n-1}t_r+\sum_{r=3}^{i-1}t_r+2\lceil\frac{n}{2}\rceil+2m+15+j\right], if\ 5\le i\le n,\ odd\\ \frac{1}{2}\left[\sum_{r=3}^{n-1}t_r+\sum_{r=3}^{i}t_r+2\lceil\frac{n}{2}\rceil+2m+16-j\right], if\ 4\le i\le n,\ even\end{cases}$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$$S_1=\left\{\frac{1}{2}\left[\sum_{r=3}^{n-1}t_r+2\lceil\frac{n}{2}\rceil+14\right]+i;\quad 1\le i\le\frac{n^2+4n+2m-1}{2}\right\}.$$

(ii) n is even

$$\phi(u_{i,j})=\begin{cases}\frac{1}{2}\left[\sum_{r=3}^{n}t_r+\sum_{r=3}^{i-1}t_r+2m+15+j\right], if\ 5\le i\le n, odd\\ \frac{1}{2}\left[\sum_{r=3}^{n}t_r+\sum_{r=3}^{i}t_r+2m+16-j\right], if\ 4\le i\le n, even\end{cases}$$

$$\phi(v_l)=\frac{1}{2}\left[\sum_{r=3}^{n}t_r+10+2l\right],\quad if\ \ 1\le l\le m$$

$$\phi(u_{i,j})=\frac{1}{2}\left[\sum_{r=3}^{n}t_r+2m+12\right]+i-1,\quad if\ i=j=1,2$$

$$\phi(u_{3,j})=\frac{1}{2}\left[\sum_{r=3}^{n}t_r+2m+15+j\right],\quad if\ \ 1\le j\le t_3, odd$$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$$S_2=\left\{\frac{1}{2}\left[\sum_{r=3}^{n}t_r+14\right]+i;\quad 1\le i\le\frac{n^2+4n+2m-2}{2}\right\}$$

Clearly, $|S_1|=\frac{n^2+4n+2m-1}{2}$, $|S_2|=\frac{n^2+4n+2m-2}{2}$. Therefore by using Lemma 2.1, ϕ can be extended to a super edge-magictotal labeling. So, the graph G admits a super edge magictotal labeling.

Theorem 2.6. *For $n\ge 6$, even and*

$$G\cong Cb_n(2,3,4,\dots,n,n+1)\cup C'b_n(3,4,5;6,6,8,8,\dots,n-2,n-2,n,n;n+2)$$

then the graph G admits super edge-magictotal labeling.

Proof. The vertex set and edge set of G are defined as follows:

$V(G)=\{u_{i,j}:1\le i\le n,\ 1\le j\le t_i\}\cup\{u_{0,1}\}\cup\{u'_{i,j}:1\le i\le n,\ 1\le j\le t'_i\}\cup\{u'_{0,1}\}$

$E(G)=\{(u_{i-1,1}u_{i,1})\ :1\le i\le n\}\cup\{(u_{i,j}u_{i,j+1})\ :1\le i\le n,\ 1\le j\le t_i-1\}\cup\{(u'_{i-1,1}u'_{i,1})\ :1\le i\le n\}\cup\{(u'_{i,j}u'_{i,j+1})\ :1\le i\le n,\ 1\le j\le t'_i-1\}$

with $|V(G)|=\frac{2n^2+7n+8}{2}$, $|E(G)|=\frac{2n^2+7n+4}{2}$

Also we find that $t_i=1+i, for\ \ 1\le i\le n$

$$t'_i=\begin{cases}2+i, & if\ i=1,2,3\\ 2\lfloor\frac{i}{2}\rfloor+2 & if\ 4\le i\le n-1\\ n+2, & if\ i=n\end{cases}$$

We define the labeling $\phi:V(G)\to\{1,2,3,\dots,\frac{2n^2+7n+8}{2}\}$ as follows:

$\phi(u'_{0,1})=1,\ \phi(u_{0,1})=2,\ \phi(u_{1,2})=3,\ \phi(u_{2,1})=4,\ \phi(u_{2,3})=5,$

$$\phi(u_{i,j})=\begin{cases}\frac{1}{4}\left[\sum_{r=2}^{i}2t_r+15+i-2j\right], & if\ 3\le i\le n, odd\\ & and\ 1\le j\le t_i, even\\ \frac{1}{4}\left[\sum_{r=2}^{i-1}2t_r+12+i+2j\right], & if\ 4\le i\le n, even\\ & and\ 1\le j\le t_i, odd\end{cases}$$

$$\phi(u'_{i,j})=\begin{cases}\frac{1}{4}\left[\sum_{r=2}^{n-1}2t_r+\sum_{r=i+1}^{n}2t'_r+14+3n+2j\right], & if\ 3\le i\le n-1, odd\\ & and\ 1\le j\le t'_i, even\\ \frac{1}{4}\left[\sum_{r=2}^{n-1}2t_r+\sum_{r=i}^{n}2t'_r+16+3n-2j\right], & if\ 4\le i\le n, even\\ & and\ 1\le j\le t'_i, odd\end{cases}$$

For $3\le i\le n-1, odd$ and $1\le j\le t_i, odd$

$$\phi(u_{i,j})=\frac{1}{4}\left[\sum_{r=2}^{n-1}2t_r+\sum_{r=3}^{n}2t'_r+\sum_{r=3}^{i}2t_r+37+3n-i-2j\right]$$

For $4\le i\le n, even$ and $1\le j\le t_i, even$

$\phi(u_{i,j}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + \sum_{r=3}^{i-1} 2t_r + 36 + 3n - i + 2j\right]$

For $3 \leq i \leq n-1, odd$ and $1 \leq j \leq t'_i, odd$

$\phi(u'_{i,j}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 4t_r + \sum_{r=3}^{n} 2t'_r + \sum_{r=i+1}^{n} 2t'_r + 32 + 4n + 2j\right]$

For $4 \leq i \leq n, even$ and $1 \leq j \leq t'_i, even$

$\phi(u'_{i,j}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 4t_r + \sum_{r=3}^{n} 2t'_r + \sum_{r=i}^{n} 2t'_r + 34 + 4n - 2j\right]$

$\phi(u'_{2,j}) = \sum_{r=2}^{n-1} t_r + \sum_{r=3}^{n} t'_r + n + 8 + \frac{j}{2}, \text{ for } 1 \leq j \leq 4, \text{ } even$

$\phi(u'_{1,j}) = \sum_{r=2}^{n-1} t_r + \sum_{r=3}^{n} t'_r + n + 13 - \frac{j+1}{2}, \text{ for } 1 \leq j \leq 3, \text{ } odd$

$\phi(u_{i,j}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + 3n + 28\right] + i - 1, \text{ } i = j = 1,2$

$\phi(u'_{2,3}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + 3n + 16\right]$

$\phi(u'_{1,2}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + 3n + 20\right]$

$\phi(u'_{2,1}) = \frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + 3n + 24\right]$

It is easy to see that under the labeling ϕ the set of all edge-sums is

$S = \left\{\frac{1}{4}\left[\sum_{r=2}^{n-1} 2t_r + \sum_{r=3}^{n} 2t'_r + 3n + 32\right] + i; \text{ } 1 \leq i \leq \frac{2n^2+7n+4}{2}\right\}$

Clearly $|S| = \frac{2n^2+7n+4}{2}$. Therefore by using Lemma 2.1, ϕ can be extended to a super edge-magic total labeling. So, the graph G admits a super edge magictotal labeling.

Conclusion:

In this paper, it has been shown the super edge-magicness of certain types of generalized comb as well as disjoint union of generalized combs and star. Additionally, we prove the super edge-magicness of
$G \cong Cb_n(2,3,4,\ldots,n,n+1) \cup Cb_n(3,4,5;6,6,8,8,\ldots,n-2,n-2,n,n;n+2)$
for n, even only. However, much more effort is to be done in order to get a comprehensive understanding the super edge-magicness of generalized comb. We encourage researchers to try to determine the super edge magic total labeling of other graphs for further research. Therefor, we raise an open question.

Open problem: For $n \geq 5$, odd and $G \cong Cb_n(2,3,4,\ldots,n,n+1) \cup Cb_n(3,4,5;6,6,8,8,\ldots,n-2,n-2,n,n;n+2)$, Find the super edge magictotal labeling of G.

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