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Research Article

Characterization of plane to plane map germs by invariants

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Abstract

We characterize the map germs of corank at most 1 with $\mathcal{A}$-codimension ≤ 6 in terms of certain invariants such as multiplicity and the number of cusps of map germs. On the basis of this characterization we present an algorithm to classify the map germs of corank at most 1 from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ with $\mathcal{A}$-codimension ≤ 6 and also give its implementation in the computer algebra system Singular.

Mathematics Subject Classification (2020). 30C45, 30C50

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1. Introduction

Singularities of map germs have been studied for a long time. There exist extensive classification and recognition problems from the plane into the plane under $\mathcal{A}$-equivalence and $\mathcal{K}$-equivalence. Gaffney [4] differentiated the term "classification" and "recognition" explicitly. According to him classification for map germs with respect to some equivalence relation means to discover a list of germs and showing that all germs satisfying certain conditions are equivalent to a germ in the list. Classification is a well understood term and has been a subject of large number of investigation in the literature (cf. [4], [6], [7]). Recognition means finding a criteria by using some useful approaches which describe that a given map germ is equivalent to which map germ in the list. There are many classification results for singularities of map germs, however there are not many studies on recognition problem.

In the present paper we deal with recognition problem by studying $\mathcal{A}$-classification of map germs with codimension at most 6 given by Rieger [6]. Saji [8] gave a criteria for these map germs of $\mathcal{A}$-codimension ≤ 3. Since Saji used only the Taylor coefficients of the germs, so it can be applied directly for the recognition of lips, beaks, and swallowtails on explicitly parameterized maps. Using this criteria he studied the singularities of conservation law about a time variable. Kabata [5] discussed the recognition problem for map germs obtained in [6]. He gave two phased criteria; one is about geometric conditions on specified jets for topological $\mathcal{A}$-type in terms of intrinsic derivatives and the second is about algebraic conditions on the Taylor coefficients of germs with each specified jet. A characterization for the classifications given in [6] and [7] can be found in [1] and [2].

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The objective of this paper is to establish sequence of substantial propositions in an algorithmic way, which enables us to determine the type of map germs given in Table 1 of [6]. We use explicit numerical invariants which characterize all the types except type 8 and type 9. We can distinguish them by codimension of the modular stratum†. Also we give an implementation of this characterization in the computer algebra system SINGULAR.

The rest of the article is organized as follows. In Section 2 we recall some essential definitions and numerical invariants that are used to characterize the map germs. In Section 3 we give the characterization of the map germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ with $\mathcal{A}$-codimension ≤ 6, in terms of certain invariants. We also give the algorithm for the characterization of the map germs. In Section 4, examples are obtained by using the algorithms.

2. Preliminaries and some invariants

We shall denote by $\mathbb{C}$ the set of complex numbers and $A(2,2) =< x, y > \mathbb{C}[[x,y]]^2$ be the set of map germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$. Let $\mathcal{A} = \text{Aut}_{\mathbb{C}}(\mathbb{C}^2, 0) \times \text{Aut}_{\mathbb{C}}(\mathbb{C}^2, 0)$. Then we have a canonical action of $\mathcal{A}$ on $A(2,2)$ defined by

$$\mathcal{A} \times A(2,2) \to A(2,2)$$

such that

$$((\varphi, \psi), f) \mapsto \psi \circ f \circ \varphi^{-1}.$$

Definition 2.1. Let $f_1, f_2 \in A(2,2)$. Then f_1 is said to be $\mathcal{A}$-equivalent to f_2 if there exist two diffeomorphisms, $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and $\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\psi \circ f_1 = f_2 \circ \varphi$, denoted by $f_1 \sim_{\mathcal{A}} f_2$.

Definition 2.2. Let $f \in A(2,2)$, we define the orbit map $\theta_f : \mathcal{A} \to A(2,2)$ as $\theta_f(\phi, \psi) = \phi^{-1} \circ f \circ \psi$, $\phi : \mathbb{C}[[x,y]] \to \mathbb{C}[[x,y]]$ and $\psi : \mathbb{C}[[x,y]] \to \mathbb{C}[[x,y]]$. Exceptionally we have $\theta_f(id) = f$. The orbit of f under the action of $\mathcal{A}$ is the image of θ_f, assume $\mathcal{A}_f := Im(\theta_f)$. The tangent space to the orbit at f, $T_{\mathcal{A}_f, f}$ is defined as

$$T_{\mathcal{A}_f, f} =< x, y >_{\mathbb{C}[[x,y]]} < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >_{\mathbb{C}[[x,y]]} + < f_1, f_2 >_{\mathbb{C}[[f_1,f_2]]} \mathbb{C}[[f_1, f_2]]^2.$$

Definition 2.3. Let $f \in A(2,2)$ and the tangent space $T_{\mathcal{A}_f, f}$. The $\mathcal{A}$-codimension at f is defined as

$$\text{cod}_{\mathcal{A}}(f) := \dim_{\mathbb{C}} \frac{\mathcal{A}(2,2)}{T_{\mathcal{A}_f, f}}.$$

$\text{cod}_{\mathcal{A}}(f)$ exists with the restriction that f should be $\mathcal{A}$-finite and has been implemented in the computer algebra system SINGULAR (see library `"classifyMapGerms.lib"`).

Definition 2.4. Let $f = (x, g(x,y))$ be a map germ. Then multiplicity is denoted by $m(f)$ and defined as

$$m(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{< x, g >}.$$

Definition 2.5. Let $f = (x, g(x,y))$ be a map germ. Then the number of cusps is denoted by $c(f)$ and defined as

$$c(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{< g_y, g_{yy} >}.$$

†Rieger used this term in [6] and [7]. It is simply a codimension of germ minus number of moduli if exists. Codimension of a map germ can be computed by using library.classifyMapGerms.lib in SINGULAR

3. Characterization of map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$

In this section we give a characterization of map germs of corank ≤ 1 and $\mathcal{A}$-codimension ≤ 6 in terms of easy computable invariants.

Proposition 3.1. *The map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of corank ≤ 1 and $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) ≤ 6 are given in Table 1.*

Table 1

Type	*Normal form*	*Conditions*
1	(x, y)	–
2	(x, y^2)	–
3	$(x, xy + y^3)$	–
4_k	$(x, y^3 + x^k y)$	$2 \leq k \leq 5$
5	$(x, xy + y^4)$	–
6	$(x, xy + y^5 + y^7)$	–
7	$(x, xy + y^5)$	
8	$(x, xy + y^6 + y^8 + \alpha y^9)$	
9	$(x, xy + y^6 + y^9)$	
10	$(x, xy + y^7 + y^9 + \alpha y^{10} + \beta y^{11})$	
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1})$	$2 \leq k \leq 4$
12	$(x, xy^2 + y^5 + y^6)$	–
13	$(x, xy^2 + y^5 + +y^9)$	–
15	$(x, xy^2 + y^6 + y^7 + \alpha y^9)$	–
16	$(x, x^2y + y^4 + y^5)$	–
17	$(x, x^2y + y^4)$	–
18	$(x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7)$	–
19	$(x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2)$	–

Proof. For a proof see [6]. □

Lemma 3.2. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank ≤ 1. Then $f \sim_{\mathcal{A}} (x, g(x, y))$ for suitable g with $g(x, 0) = 0$.*

Proof. This is obvious. □

Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of corank ≤ 1 then one can use Algorithm 1 to find a map germ of the form $(x, g(x, y))$ such that $f \sim_{\mathcal{A}} (x, g(x, y))$.

Algorithm 1 Normal form of a map (`normalMap`)

Input: A map germ $f(x, y) = (f_1, f_2)$ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of corank ≤ 1.
Output: A map germ of the form $(x, g(x, y))$ such that $f \sim_{\mathcal{A}} (x, g(x, y))$.

1: Permute the generators of f such that $\mathrm{ord}(f_1) = 1$.
2: Compute $p = \mathrm{determinacy}(f)$.
3: Compute a map ψ such that $\psi(f_1) = x \mod < x, y >^{p+1}$.
4: Define $g(x, y) = (g_1, g_2) = \mathrm{jet}((x, \psi(f_2)), p)$.
5: **return** (g).

In the following example we have as input the map $f(x, y) = (f_1, f_2)$ of corank ≤ 1, where
$f_1 = x + 2y + x^2y + 3xy^2 + x^2y^2$,
$f_2 = x^4 - 3x^3y + x^5 - 15x^4y + 90x^3y^2 - 270x^2y^3 + 405xy^4 - 243y^5$.

```
ring R=0,(x,y),ds;
> poly f1=x+2y+x2y+3xy2+x2y2;
> poly f2=x4-3x3y+x5-15x4y+90x3y2-270x2y3+405xy4-243y5;
> ideal f=f1,f2;
```

If one wants to find the 6-jet of a map germ of the form $(x, g(x,y))$ $\mathcal{A}$-equivalent to f, one can use the following procedure:

```
normalMap(f,6);
[1]=x
[2]=27/8000x3y-27/800x2y2+y4-59049/1310720x5-32805/32768x4y-
18225/2048x3y2-10125/256x2y3-5625/64xy4-625/8y5+3375027/20480000000x6
-1336257/256000000x5y+91989/2560000x4y2+3609/16000x3y3-2799/1600x2y4
-617/100xy5-17/5y6
```

Thus,

$$\begin{aligned} g(x,y) =& \frac{27}{8000}x^3y - \frac{27}{800}x^2y^2 + y^4 - \frac{59049}{1310720}x^5 - \frac{32805}{32768}x^4y - \frac{18225}{2048}x^3y^2 - \frac{10125}{256}x^2y^3 \\ &- \frac{5625}{64}xy^4 - \frac{625}{8}y^5 + \frac{3375027}{20480000000}x^6 - \frac{1336257}{256000000}x^5y + \frac{91989}{2560000}x^4y^2 \\ &+ \frac{3609}{16000}x^3y^3 - \frac{2799}{1600}x^2y^4 - \frac{617}{100}xy^5 - \frac{17}{5}y^6, \end{aligned}$$

i.e.,

$$f \sim_{\mathcal{A}} (x, g(x,y)).$$

Proposition 3.3. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ of corank at most 1. If $m(f) > 7$, then $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) > 6.*

Proof. It is an immediate consequence of [6]. □

Now in the following sequence of propositions, we give a characterization of map germs of corank ≤ 1 and $\mathcal{A}$-codimension ≤ 6.

Proposition 3.4. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. Then corank of f is zero if and only if $m(f) = 1$. In this case $f \sim_{\mathcal{A}} (x,y)$.*

Proof. The first part of the statement is obvious. To prove the second part, we assume that $f \sim_{\mathcal{A}} (x, \sum_{i+j\geq 1} a_{i,j}x^iy^j)$. By using left coordinate changes $\bar{Y} = Y - a_{i,0}X^i$ where $i > 0$, we can transform $(x, \sum_{i+j\geq 1} a_{i,j}x^iy^j)$ into $(x, a_{0,1}y + \sum_{\substack{i+j\geq 2\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$. If $m(f) = 1$ then $a_{01} \neq 0$. We can take $a_{01} = 1$ and therefore

$$f \sim_{\mathcal{A}} (x, y + \sum_{i+j\geq 2} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{k,0}x^k + a_{k-1,1}x^{k-1}y + \ldots + a_{0,k}y^k),\ k > 1$ gives

$$f \sim_{\mathcal{A}} (x,y).$$

□

Proposition 3.5. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0)$ to $(\mathbb{C}^2,0)$ of corank 1 and $\mathcal{A}$-codimension ≤ 6. If $m(f) = 2$, then f is $\mathcal{A}$-equivalent to (x,y^2).*

Proof. We may assume that $f \sim_{\mathcal{A}} (x, \sum_{i+j\geq 1} a_{i,j}x^iy^j)$. The left coordinate changes $\bar{Y} = Y - a_{i,0}X^i$ give $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$.

If $m(f) = 2$, then $a_{0,1} = 0$ and $a_{0,2} \neq 0$. We can take $a_{0,2} = 1$ and therefore

$$f \sim_{\mathcal{A}} (x, y^2 + a_{1,1}xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

Now if $a_{1,1} = 0$, then we have $f \sim_{\mathcal{A}} (x, y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$ and if $a_{1,1} \neq 0$, then we have

$$f \sim_{\mathcal{A}} (x, (y + \frac{a_{1,1}}{2}x)^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - \frac{a_{1,1}}{2}x$ gives

$$f \sim_{\mathcal{A}} (x, y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Rieger's classification implies that $f \sim_{\mathcal{A}} (x, y^2)$.

□

Proposition 3.6. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank 1 and $\mathcal{A}$-codimension ≤ 6. If $m(f) = 3$, then f is $\mathcal{A}$-equivalent to one of the germs given in Table 2.*

Table 2

Type	*Normal form*	$c(f)$
3	$(x, xy + y^3)$	1
4_k	$(x, y^3 + x^ky),\ 2 \leq k \leq 5$	k

Proof. Since corank of f is 1 therefore we can assume $f \sim_{\mathcal{A}} (x, \sum_{i+j\geq 1} a_{i,j}x^iy^j)$. The left coordinate changes $\bar{Y} = Y - a_{i,0}X^i$ give $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$. If $m(f) = 3$, then coefficients $a_{0,1}$ and $a_{0,2}$ are zero and $a_{0,3}$ must be non-zero. Then

$$f \sim_{\mathcal{A}} (x, a_{1,1}xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

(1) Now if $c(f) = 1$ then $a_{1,1} \neq 0$. We have

$$f \sim_{\mathcal{A}} (x, xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{2,1}xy + a_{1,2}y^2)$, gives

$$f \sim_{\mathcal{A}} (x, xy + \alpha y^3 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Rieger's classification implies that $f \sim_{\mathcal{A}} (x, xy + y^3)$.

(2) If $c(f) \geq 2$, then $a_{1,1} = 0$ and therefore

$$f \sim_{\mathcal{A}} (x, a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

Now if $c(f) = 2$, then $3a_{2,1}a_{0,3} - a_{1,2}^2 \neq 0$. The transformation $y \to y - \frac{a_{12}}{3a_{03}}x$ gives

$$f \sim_{\mathcal{A}} (x, y^3 + \alpha x^2y + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Then from Rieger's classification it follows that $f \sim_{\mathcal{A}} (x, y^3 + x^2y)$.
If $3 \leq k \leq 5$ then $3a_{2,1}a_{0,3} - a_{1,2}^2 = 0$ and the transformation $y \to y - \frac{a_{12}}{3a_{03}}x$ gives

$$f \sim_{\mathcal{A}} (x, y^3 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

By using the transformation

$$y \to y - \frac{1}{3}(b_{k-1,2}x^{k-1} + b_{k-2,3}x^{k-2}y + \ldots + b_{2,k-1}x^2y^{k-3} + b_{1,k}xy^{k-2} + b_{0,k+1}y^{k-1}),$$

we get

$$f \sim_{\mathcal{A}} (x, y^3 + \beta x^ky + \sum_{\substack{i+j\geq k+2\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

It follows from the Rieger's classification that $f \sim_{\mathcal{A}} (x, y^3 + x^ky)$. □

Proposition 3.7. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank 1 and $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) ≤ 6. If $m(f) = 4$ then f is $\mathcal{A}$-equivalent to one of the germs given in Table 3.*

Table 3

Type	*Normal form*	$c(f)$	$\mathrm{cod}_{\mathcal{A}}(f)$
5	$(x, xy + y^4)$	2	3
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1})$, $2 \leq k \leq 4$	3	$k + 2$
16	$(x, x^2y + y^4 + y^5)$	4	5
17	$(x, x^2y + y^4)$	4	6
19	$(x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2)$	6	7(6†)

Proof. We may assume that $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$. If $m(f) = 4$, then coefficients $a_{0,1}, a_{0,2}, a_{0,3} = 0$, and $a_{0,4} \neq 0$. Then

$$f \sim_{\mathcal{A}} (x, a_{1,1}xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

(1) If $c(f) = 2$, then $a_{1,1} \neq 0$. We can take $a_{1,1} = 1$, then

$$f \sim_{\mathcal{A}} (x, xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{2,1}xy + a_{1,2}y^2)$ gives

$$f \sim_{\mathcal{A}} (x, xy + b_{3,1}x^3y + b_{2,2}x^2y^2 + b_{1,3}xy^3 + b_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Now by using the transformation $y \to y - (b_{3,1}x^2y + b_{2,2}xy^2 + b_{1,3}y^3)$, we get

$$f \sim_{\mathcal{A}} (x, xy + \alpha y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

Rieger's classification gives $f \sim_{\mathcal{A}} (x, xy + y^4)$.

(2) If $c(f) = 3$, then $a_{1,1} = 0$ and $a_{1,2} \neq 0$. We can take $a_{1,2} = 1$, then

$$f \sim_{\mathcal{A}} (x, xy^2 + a_{2,1}x^2y + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - \frac{a_{2,1}}{2a_{1,2}}x$ gives

$$f \sim_{\mathcal{A}} (x, xy^2 + \sum_{\substack{i+j\geq 4\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

By using the transformation

$$y \to y - \frac{1}{2}(b_{k-1,1}x^{k-2} + b_{k-2,2}x^{k-3}y + \ldots + b_{2,k-2}xy^{k-3} + b_{1,k-1}y^{k-2}),\ k \geq 4,$$

we get $f \sim_{\mathcal{A}} (x, xy^2 + c_{0,4}y^4 + \sum_{i=5}^{k} c_{0,i}y^i)$. Since $\mathcal{A}$-codimension of f is ≤ 6 therefore $f \sim_{\mathcal{A}} (x, xy^2 + c_{04}y^4 + \sum_{i=5}^{9} c_{0,i}y^i)$. Moreover, $c_{0,i} \neq 0$ for some odd i, otherwise f will not be $\mathcal{A}$-finite.
It is easy to see that $\mathrm{cod}_{\mathcal{A}}(f) = 4 \Leftrightarrow c_{0,5} \neq 0$, $\mathrm{cod}_{\mathcal{A}}(f) = 5 \Leftrightarrow c_{0,5} = 0$ and $c_{0,7} \neq 0$ and $\mathrm{cod}_{\mathcal{A}}(f) = 6 \Leftrightarrow c_{0,5} = 0, c_{0,7} = 0$ and $c_{0,9} \neq 0$. Consequently Rieger's classification implies that $f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^{2k+1}), 2 \leq k \leq 4$.

(3) If $c(f) = 4$, then $a_{1,1} = a_{1,2} = 0$ and $a_{2,1} \neq 0$. We can take $a_{2,1} = 1$, then

$$f \sim_{\mathcal{A}} (x, x^2y + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{3,1}xy + a_{2,2}y^2)$ gives

$$f \sim_{\mathcal{A}} (x, x^2y + \alpha xy^3 + \beta y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

If $\alpha = 0$, then $f \sim_{\mathcal{A}} (x, x^2y + \beta y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j)$ and if $\alpha \neq 0$, then the transformation $y \to y - \frac{\alpha}{4\beta}x$ gives

$$f \sim_{\mathcal{A}} (x, x^2y + \beta y^4 + \alpha_1x^3y + \alpha_2x^2y^2 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

Again by using the transformation $y \to y - \alpha_1xy$ we have

$$f \sim_{\mathcal{A}} (x, x^2y + \beta_1x^2y^2 + \beta y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j).$$

Now the transformation $y \to y - \beta_1y^2$ gives, $f \sim_{\mathcal{A}} (x, x^2y + \beta y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} e_{i,j}x^iy^j)$. Furthermore, it follows from the Rieger's classification if $\mathrm{cod}_{\mathcal{A}}(f) = 6$ then $f \sim_{\mathcal{A}} (x, x^2y + y^4)$

and if $\mathrm{cod}_{\mathcal{A}}(f) = 5$ then $f \sim_{\mathcal{A}} (x, x^2y + y^4 + y^5)$.

(4) Now if $c(f) = 6$, then $a_{1,1} = a_{2,1} = a_{1,2} = 0$, and $a_{3,1}$, $a_{2,2}$ are not zero simultaneously, then

$$f \sim_{\mathcal{A}} (x, x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

Then the transformation $y \to y - \frac{a_{1,3}}{4}x$ gives

$$f \sim_{\mathcal{A}} (x, x^3y + b_{2,2}x^2y^2 + b_{0,4}y^4 + \sum_{\substack{i+j\geq 5\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Moreover, $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) ≤ 6 so,

$$\begin{aligned} f \sim_{\mathcal{A}} (x, x^3y &+ b_{2,2}x^2y^2 + b_{0,4}y^4 + b_{4,1}x^4y + b_{3,2}x^3y^2 + b_{2,3}x^2y^3 + b_{1,4}xy^4 + b_{0,5}y^5 \\ &+ \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j) \end{aligned}$$

with one of the coefficients of term of degree 5 must be non-zero.

Again by using the transformation $y \to y - b_{41}xy$ we get

$$\begin{aligned} f \sim_{\mathcal{A}} (x, x^3y &+ c_{2,2}x^2y^2 + c_{0,4}y^4 + c_{4,1}x^4y + c_{3,2}x^3y^2 + c_{2,3}x^2y^3 + c_{1,4}xy^4 + c_{0,5}y^5 \\ &+ \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j). \end{aligned}$$

Now if $\Delta = -12c_{3,2} - 4{c_{2,2}}^2c_{2,3} + 4c_{2,2}c_{1,4} + (3 + 2{c_{2,2}}^3)c_{0,5} \neq 0$, then it is an immediate consequence of Proposition 3.2.3 : 1 [6] that

$$f \sim_{\mathcal{A}} (x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2).$$

□

Proposition 3.8. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank 1 and $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) ≤ 6. If $m(f) = 5$, then f is $\mathcal{A}$-equivalent to one of the germs given in Table 4.*

Table 4

Type	*Normal form*	$c(f)$	$\mathrm{cod}_{\mathcal{A}}(f)$
6	$(x, xy + y^5 + y^7)$	3	4
7	$(x, xy + y^5)$	3	5
12	$(x, xy^2 + y^5 + y^6)$	4	5
13	$(x, xy^2 + y^5 + +y^9)$	4	6
18	$(x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7)$	6	$8^*(6\dagger)$

The notations †and ∗ are used in [6] for the codimension of stratum and excluding exception values of the moduli, respectively.

Proof. We may assume that $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$. If $m(f) = 5$, then coefficients $a_{0,1}, a_{0,2}, a_{0,3}, a_{0,4} = 0$, and $a_{0,5} \neq 0$. Then

$$f \sim_{\mathcal{A}}(x, a_{1,1}xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

(1) If $c(f) = 3$, then $a_{1,1} \neq 0$. We can take $a_{1,1} = 1$, then

$$f \sim_{\mathcal{A}}(x, xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{2,k}xy^k + a_{1,k+1}y^{k+1})$, $k = 1, 2, 3$ gives

$$f \sim_{\mathcal{A}} (x, xy + b_{3,2}x^3y^2 + b_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Again by using the transformation $y \to y - b_{3,2}x^2y^2$ we get

$$f \sim_{\mathcal{A}} (x, xy + c_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

Now if $\mathcal{A}$-codimension is 5, then f is of type $(x, xy+y^5)$ and if $\mathcal{A}$-codimension is 4, then f is of type $(x, xy + y^5 + y^7)$.

(2) If $c(f) = 4$, then $a_{1,1} = 0$ and $a_{1,2} \neq 0$ so. We can take $a_{1,2} = 1$, then

$$f \sim_{\mathcal{A}}(x, xy^2 + a_{2,1}x^2y + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - \frac{a_{2,1}}{2}x$ gives

$$f \sim_{\mathcal{A}}(x, xy^2 + b_{3,1}x^3y + b_{2,2}x^2y^2 + b_{1,3}xy^3 + b_{4,1}x^4y + b_{3,2}x^3y^2 + b_{2,3}x^2y + b_{1,4}xy^4 + b_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Now the transformation for $k \geq 3$

$$y \to y - \frac{1}{2}(b_{k-1,1}x^{k-2} + b_{k-2,2}x^{k-3}y + \ldots + b_{2,k-2}xy^{k-3} + b_{1,k-1}y^{k-2})$$

transforms f into $(x, xy^2 + c_{0,5}y^5 + \sum_{i\geq 6} c_{0,i}y^i)$. Now if $\mathcal{A}$-codimension is 5, then f is of type $(x, xy^2 + y^5 + y^6)$ and if $\mathcal{A}$-codimension is 6, then f is of type $(x, xy^2 + y^5 + y^9)$.

(3) If $c(f) = 6$, then $a_{1,1} = a_{1,2} = 0$, and $a_{2,1} \neq 0$. We can take $a_{2,1} = 1$, then

$$f \sim_{\mathcal{A}}(x, x^2y + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - (a_{k-1,1}x^{k-3}y + a_{k-2,2}y^{k-4} + \ldots + a_{3,k-3}xy^{k-3} + a_{2,k-2}y^{k-2})$ gives

$$f \sim_{\mathcal{A}} (x, x^2y + b_{1,3}xy^3 + b_{1,4}xy^4 + b_{0,5}y^5 + \sum_{\substack{i+j\geq 6\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Let $j^6f = (x, x^2y + xy^3 + b_{1,4}xy^4 + b_{0,5}y^5 + b_{1,5}xy^5 + b_{0,6}y^6)$. If $b_{1,4} = 0$, then $j^6f = (x, x^2y + xy^3 + b_{0,5}y^5 + b_{1,5}xy^5 + b_{0,6}y^6)$ and if $b_{1,4} \neq 0$, then it follows from Proposition 3.6(6) in [5], that

$$j^6f \sim_{\mathcal{A}} (x, x^2y + xy^3 + y^5 + (b_{0,6} - \frac{15b_{0,5}b_{1,4}}{5b_{0,5} - 9})y^6).$$

One can observe that $\mathrm{cod}(j^6f) = 8 \Leftrightarrow (b_{0,6} - \frac{15b_{0,5}b_{1,4}}{5b_{0,5}-9}) \neq 0$.

Now it is an immediate consequence of Proposition 3.6(6) in [5] that

$$f \sim_{\mathcal{A}} (x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7),$$

and $\alpha \neq 0$ otherwise $\mathrm{cod}_{\mathcal{A}}(f) = 10$. □

Proposition 3.9. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank 1 and $\mathcal{A}$-codimension (or codimension of the stratum in the presence of moduli) ≤ 6. If $m(f) = 6$ then f is $\mathcal{A}$-equivalent to one of the germs given in Table 5.*

Table 5

Type	*Normal form*	$c(f)$	$\mathrm{cod}_{\mathcal{A}}(f)$
8	$(x, xy + y^6 + y^8 + \alpha y^9)$	4	6(5†)
9	$(x, xy + y^6 + y^9)$	4	6
15	$(x, xy^2 + y^6 + y^7 + \alpha y^9)$	5	7(6†)

Proof. We may assume that $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$. If $m(f) = 6$, then coefficients $a_{0,1}, a_{0,2}, a_{0,3}, a_{0,4} = 0, a_{0,5}$, and $a_{0,6} \neq 0$. Then

$$\begin{aligned} f \sim_{\mathcal{A}} (x, &a_{1,1}xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y \\ &+ a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{5,1}x^5y + a_{4,2}x^4y^2 + a_{3,3}x^3y^3 \\ &+ a_{2,4}x^2y^4 + a_{1,5}xy^5 + a_{0,6}y^6 + \sum_{\substack{i+j\geq 7\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j). \end{aligned}$$

(1) If $c(f) = 4$, then $a_{1,1} \neq 0$. We can take $a_{1,1} = 1$, then

$$\begin{aligned} f \sim_{\mathcal{A}} (x, &xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 \\ &+ a_{2,3}x^2y + a_{1,4}xy^4 + a_{5,1}x^5y + a_{4,2}x^4y^2 + a_{3,3}x^3y^3 + a_{2,4}x^2y^4 + a_{1,5}xy^5 \\ &+ a_{0,6}y^6 + \sum_{\substack{i+j\geq 7\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j). \end{aligned}$$

The transformation

$$y \to y - (a_{(k-2)2}x^{k-3}y^2 + a_{(k-3)3}x^{k-4}y^3 + \ldots + a_{2(k-2)}xy^{k-2} + a_{1(k-1)}y^{k-1})$$

gives $f \sim_{\mathcal{A}} (x, xy + b_{0,6}y^6 + \sum_{\substack{i+j\geq 7\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j)$.

Now if $\mathcal{A}$-codimension is 6 and codimension of the stratum in the presence of moduli is 5 then f is of type $(x, xy + y^6 + y^8 + \alpha y^9)$ and if $\mathcal{A}$-codimension is 6, then f is of type

$(x, xy + y^6 + y^9)$.

(2) If $c(f) = 5$, then $a_{1,1} = 0$ and $a_{1,2} \neq 0$. We can take $a_{1,2} = 1$, then

$$f \sim_{\mathcal{A}} (x, xy^2 + a_{2,1}x^2y + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{4,1}x^4y + a_{3,2}x^3y^2 + a_{2,3}x^2y + a_{1,4}xy^4 + a_{5,1}x^5y + a_{4,2}x^4y^2 + a_{3,3}x^3y^3 + a_{2,4}x^2y^4 + a_{1,5}xy^5 + a_{0,6}y^6 + \sum_{\substack{i+j\geq 7\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation $y \to y - \frac{a_{21}}{2}x$ gives

$$f \sim_{\mathcal{A}} (x, xy^2 + b_{3,1}x^3y + b_{2,2}x^2y^2 + b_{1,3}xy^3 + b_{4,1}x^4y + b_{3,2}x^3y^2 + b_{2,3}x^2y + b_{1,4}xy^4 + b_{5,1}x^5y + b_{4,2}x^4y^2 + b_{3,3}x^3y^3 + b_{2,4}x^2y^4 + b_{1,5}xy^5 + b_{0,6}y^6 + \sum_{\substack{i+j\geq 7\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j).$$

Now the transformation for $k \geq 4$

$$y \to y - \frac{1}{2}(b_{k-1,1}x^{k-2} + b_{k-2,2}x^{k-3}y + \ldots + b_{2,k-2}xy^{k-3} + b_{1,k-1}y^{k-2})$$

transforms f into $(x, xy^2 + \sum_{i\geq 6} c_{0,i}y^i)$ since $\mathcal{A}$-codimension of f is ≤ 6 so,

$$f \sim_{\mathcal{A}} (x, xy^2 + c_{0,6}y^6 + \sum_{i\geq 7} c_{0,i}y^i).$$

Moreover, $c_{0,i} \neq 0$ for some odd i, otherwise f will not be $\mathcal{A}$-finite. Now if $\mathcal{A}$-codimension of the stratum in the presence of moduli is 6, then f is of type $(x, xy^2 + y^6 + y^7 + \alpha y^9)$. □

Proposition 3.10. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ of corank 1 and $\mathcal{A}$-codimension ≤ 6. If $m(f) = 7$ and $c(f) = 5$, then f is $\mathcal{A}$-equivalent to $(x, xy + y^7 + \gamma y^9 + \alpha y^{10} + \beta y^{11})$.*

Proof. We may assume that $f \sim_{\mathcal{A}} (x, \sum_{i+j\geq 1} a_{i,j}x^iy^j)$. The left coordinate changes $\bar{Y} = Y - a_{i,0}X^i$ give $f \sim_{\mathcal{A}} (x, \sum_{\substack{i+j\geq 1\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j)$.

If $m(f) = 7$ then

$$f \sim_{\mathcal{A}} (x, a_{1,1}xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j),$$

with $a_{0,i} = 0$ for $1 \leq i \leq 6$ and $a_{0,7} \neq 0$. Now if $c(f) = 5$, then $a_{1,1} \neq 0$. We can take $a_{1,1} = 1$, then

$$f \sim_{\mathcal{A}} (x, xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} a_{i,j}x^iy^j).$$

The transformation

$$y \to y - (a_{k-1,1}x^{k-2}y + a_{k-2,2}x^{k-3}y^2 + \ldots + a_{2,k-2}xy^{k-2} + a_{1,k-1}y^{k-1})$$

remove all the terms of degree k which are divisible by x at the level of k-jet, $k > 2$, and it transforms f into $(x, xy + y^7 + \sum_{i\geq 8} b_{0,i}y^i)$.

Now if $b_{0,8} = 0$, then $f \sim_{\mathcal{A}} (x, xy + y^7 + \gamma y^9 + \alpha y^{10} + \beta y^{11})$ with $\gamma = b_{0,9}, \alpha = b_{0,10}$ and $\beta = b_{0,11}$ and if $b_{0,8} \neq 0$, then we can easily get

$$f \sim_{\mathcal{A}} (x, xy + y^7 + y^8 + \frac{b_{0,9}}{{b_{0,8}}^2}y^9 + \frac{b_{0,10}}{{b_{0,8}}^3}y^{10} + \frac{b_{0,11}}{{b_{0,8}}^4}y^{11} + \ldots) = (x, g).$$

Using the morphisms φ^{-1} respectively ψ defined by $\varphi^{-1}(x) = x - \frac{1}{6}g$ and $\varphi^{-1}(y) = 6\sum_{v=1}^{10}(\frac{1}{6}y)^v$ respectively $\psi(x) = x+\frac{1}{6}y$ and $\psi(y) = y$, then we obtain, for $\gamma \neq 0$, $f \sim_{\mathcal{A}} (x, xy+y^7+\gamma y^9+\alpha y^{10}+\beta y^{11})$, $\gamma = d - \frac{7}{12}, \alpha = c - \frac{12}{9}d + \frac{14}{27}$ and $\beta = e - \frac{3}{2}c + d - \frac{7}{24}$ and $d = \frac{b_{0,9}}{{b_{0,8}}^2}, c = \frac{b_{0,10}}{{b_{0,8}}^3}$ and $e = \frac{b_{0,11}}{{b_{0,8}}^4}$, since $\varphi^{-1}(xy+y^7+\gamma y^9+\alpha y^{10}+\beta y^{11}) = g$. This can be checked with SINGULAR as follows.

```
ring R=(0,c,d,e),(x,y),ds;
poly g=xy+y7+y8+d*y9+c*y10+e*y11;
poly h=xy+y7+(d-7/12)*y9+(c-12*d/9+14/27)*y10+(e-3*c/2+d-7/24)*y11;
map phi_invers=R, x-1/6*g,y+1/6*y2+1/36*y3+1/216*y4+1/1296*y5+
        1/7776*y6+1/46656*y7+1/279936*y8+1/1679616*y9+1/10077696*y10;
jet(phi_invers(h),11);
xy+y7+y8+(d)*y9+(c)*y10+(e)*y11
```

□

4. Singular examples

We have implemented Algorithm 2 in the computer algebra system Singular [3]. The code can be downloaded from https://www.mathcity.org/files/ahsan/Proc-classifyPlane Maps1.txt. We give some examples.

```
ring r=0,(x,y),(c,ds);
```

In the first example we have as an input the map $f(x,y) = (f_1, f_2)$, where

$$\begin{aligned} f_1 =& 3x - 2y - xy, \\ f_2 =& -9x^3 + 21x^2y - 16xy^2 + 4y^3 - 3x^4 + 17x^3y - 25x^2y^2 + 13xy^3 - 2y^4 - x^5 + 6x^4y \\ & - 14x^3y^2 + 14x^2y^3 - 6xy^4 + y^5 + x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 \\ & + y^6 - x^7 + 7x^6y - 21x^5y^2 + 35x^4y - 35x^3y^4 + 21x^2y^5 - 7xy^6 + y^7. \end{aligned}$$

In SINGULAR this can be written as

```
ideal I=3x-2y-xy,-9x3+21x2y-16xy2+4y3-3x4+17x3y-25x2y2+13xy3-2y4-x5
        +6x4y-14x3y2+14x2y3-6xy4+y5+x6-6x5y+15x4y2-20x3y3+15x2y4-6xy5
        +y6-x7+7x6y-21x5y2+35x4y-35x3y4+21x2y5-7xy6+y7;
```

To compute the required type of map germs we use the procedure:

```
classifyPlaneMaps1(I);
```

and we obtain the output:

```
[1]=x
[2]=x2y+xy3+y5+y6+y7
```

i.e.,

$$f \sim_{\mathcal{A}} (x, x^2y + xy^3 + y^5 + y^6 + y^7).$$

Algorithm 2 Plane to plane maps (classifyPlaneMaps1)

Input: A germ $f(x, y)$ from the plane to the plane of corank ≤ 1.
Output: $(x, g(x, y))$, the type or 0 if $\mathcal{A}$-codimension of f is greater than 6.

Transform f into a germ $\mathcal{A}$-equivalent to $(x, g(x, y))$ and $g(x, 0) = 0$.
Compute $\mathrm{cod}_{\mathcal{A}}(f)$, the codimension of f,$m(f)$, the multiplicity of f and $c(f)$, the number of cusps of f.
if $m(f) = 1$ **then**
  return (x, y);
if $m(f) = 2$ **then**
  return (x, y^2);
if $m(f) = 3$ **then**
  if $c(f) = 1$ **then**
    return $(x, xy + y^3)$;
    if $c(f) = k, 2 \leq k \leq 5$ **then**
      return $(x, y^3 + x^k y)$;
if $m(f) = 4$ **then**
  if $c(f) = 2$ **then**
    return $(x, xy + y^4)$;
  if $c(f) = 3$ and $\mathrm{cod}_{\mathcal{A}}(f) = k + 2, 2 \leq k \leq 4$ **then**
    return $(x, xy^2 + y^4 + y^{2k+1})$;
  if $c(f) = 4$ **then**
    if $\mathrm{cod}_{\mathcal{A}}(f) = 5$ **then**
      return $(x, x^2y + y^4 + y^5)$;
      if $\mathrm{cod}_{\mathcal{A}}(f) = 6$ **then**
        return $(x, x^2y + y^4)$;
  if $c(f) = 6$ **then**
    return $(x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2)$;
if $m(f) = 5$ **then**
  if $c(f) = 3$ **then**
    if $\mathrm{cod}_{\mathcal{A}}(f) = 4$ **then**
      return $(x, xy + y^5 + y^7)$;
      if $\mathrm{cod}_{\mathcal{A}}(f) = 5$ **then**
        return $(x, xy + y^5)$;
    if $c(f) = 4$ **then**
      if $\mathrm{cod}_{\mathcal{A}}(f) = 5$ **then**
        return $(x, xy^2 + y^5 + y^6)$;
        if $\mathrm{cod}_{\mathcal{A}}(f) = 6$ **then**
          return $(x, xy^2 + y^5 + y^9)$;
  if $c(f) = 6$ **then**
    return $(x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7)$;
if $m(f) = 6$ **then**
  if $c(f) = 4$ and $\mathrm{cod}_{\mathcal{A}}(f) = 6$ **then**
    Compute $g = jet(g, 9)$;
    return $(x, xy + y^6 + y^8 + \alpha y^9)$ or $(x, xy + y^6 + y^9)$;
    if $c(f) = 5$ **then**
      return $(x, xy^2 + y^6 + y^7 + \alpha y^9)$;
if $m(f) = 7$ and $c(f) = 5$ **then**
  return $(x, xy + y^7 + \gamma y^9 + \alpha y^{10} + \beta y^{11})$;
else
return 0.

In the second example we have as an input the map $f(x,y) = (f_1, f_2)$, where

$$f_1 = x - xy,$$
$$f_2 = -x^2 + xy + x^2y - xy^2 - x^7 + 7x^6y - 21x^5y^2 + 35x^4y^3 - 35x^3y^4 + 21x^2y^5 - 7xy^6 + y^7 - x^9 + 9x^8y - 36x^7y^2 + 84x^6y^3 - 126x^5y^4 + 126x^4y^5 - 84x^3y^6 + 36x^2y^7 - 9xy^8 + y^9 + x^{10} - 10x^9y + 45x^8y^2 - 120x^7y^3 + 210x^6y^4 - 252x^5y^5 + 210x^4y^6 - 120x^3y^7 + 45x^2y^8 - 10xy^9 + y^{10} - x^{11} + 11x^{10}y - 55x^9y^2 + 165x^8y^3 - 330x^7y^4 + 462x^6y^5 - 462x^5y^6 + 330x^4y^7 - 165x^3y^8 + 55x^2y^9 - 11xy^10 + y^{11}.$$

In SINGULAR this can be written as

```
ideal I=x-xy,-x2+xy+x2y-xy2-x7+7x6y-21x5y2+35x4y3-35x3y4+21x2y5-7xy6+
        y7-x9+9x8y-36x7y2+84x6y3-126x5y4+126x4y5-84x3y6+36x2y7-9xy8+y9
        +x10-10x9y+45x8y2-120x7y3+210x6y4-252x5y5+210x4y6-120x3y7+45x2y8
        -10xy9+y10-x11+11x10y-55x9y2+165x8y3-330x7y4+462x6y5-462x5y6+
         330x4y7-165x3y8+55x2y9-11xy10+y11;
```

To compute the required type of map germs we use the procedure:

```
classifyPlaneMaps1(I);
```

and we obtain the output:

```
[1]=x
[2]=xy+y7+y9+y10+y11
```

i.e.,

$$f \sim_{\mathcal{A}} (x, xy + y^7 + y^9 + y^{10} + y^{11}).$$

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References

[1] S. Aslam, M.A. Binyamin, and G. Pfister, *Recognition of unimodal map germs from the plane to the plane by invariants*, Int. J. Algebra Comput. **28** (7), 1199–1208, 2018.

[2] M.A. Binyamin, H. Mahmood, and S. Kanwal, *On the classification of simple maps from the plane to the plane*, J. Algebra Appl. **16** (10), 2017, 1750199.

[3] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 4-1-0 — A computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2017.

[4] T. Gaffney, *The structure of $T\mathcal{A}_f$ classification and an application to differential geometry, In singularities, Part I*, Proc. Sympos. in Pure Math., Amer. Math. Soc. **40**, 409–427, 1983.

[5] Y. Kabata, *Recognition of plane-to-plane map-germs*, Topol. Appl. **202**, 216–238, 2016.

[6] J.H. Rieger, *Families of maps from the plane to the plane*, J. London Math. Soc. **26** (2), 351–369, 1987.

[7] J.H. Rieger, *A-unimodal map-germs into the plane*, Hokkaido Math. J. **33**, 47–64, 2004.

[8] K. Saji, *Criteria for singularities of smooth maps from the plane into the plane and their applications*, Hiroshima Math. J. **40**, 229–239, 2010.