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THE COMPLEMENTARY NABLA BENNETT-LEINDLER TYPE INEQUALITIES

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Abstract. We aim to find the complements of the Bennett-Leindler type inequalities in nabla time scale calculus by changing the exponent from $0 < \zeta < 1$ to $\zeta > 1$. Different from the literature, the directions of the new inequalities, where $\zeta > 1$, are the same as that of the previous nabla Bennett-Leindler type inequalities obtained for $0 < \zeta < 1$. By these settings, we not only complement existing nabla Bennett-Leindler type inequalities but also generalize them by involving more exponents. The dual results for the delta approach and the special cases for the discrete and continuous ones are obtained as well. Some of our results are novel even in the special cases.

1. Introduction

The theory of inequalities containing series or integrals has been shown to be of great importance due to their effective usage in differential equations and in their applications after the celebrated discrete and continuous inequalities of Hardy have been obtained. In 1920, when Hardy [24] tried to find a simple and elementary proof of Hilbert's inequality [32]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m c_n}{m+n} \leq \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty} c_n^2\right)^{1/2},$$

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where $a_m, c_n \geq 0$ and $\sum_{m=1}^{\infty} a_m^2$ and $\sum_{n=1}^{\infty} c_n^2$ are convergent, he showed the following pioneering discrete inequality

$$\sum_{j=1}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{m} c(i)\right)^{\zeta} \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\sum_{j=1}^{\infty} c^{\zeta}(j), \quad c(j)\geq 0,\ \zeta>1 \tag{1}$$

and pioneering continuous inequality for a nonnegative function Γ and for a real constant $\zeta > 1$, as

$$\int_0^{\infty}\left(\frac{1}{t}\int_0^t \Gamma(s)ds\right)^{\zeta} dt \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\int_0^{\infty}\Gamma^{\zeta}(t)dt, \tag{2}$$

where $\int_0^{\infty}\Gamma^{\zeta}(t)dt < \infty$. In fact, Hardy only stated inequality (2) in [24] but did not prove it. Later in 1925, the proof of inequality (2), which depends on the calculus of variations, was shown by Hardy in [25].

The constant $\left(\frac{\zeta}{\zeta-1}\right)^{\zeta}$ that appears in the above inequalities also has been found as the best possible one, since if it is replaced by a smaller constant then inequalities (1) and (2) are not fulfilled anymore for the involved sequences and functions, respectively.

Hardy et al. [26, Theroem 330] developed inequality (2) and derived the following integral inequality for a nonnegative function Γ as

$$\int_0^{\infty}\frac{\Psi^{\zeta}(t)}{t^{\theta}}dt \leq \left|\frac{\zeta}{\theta-1}\right|^{\zeta}\int_0^{\infty}\frac{\Gamma^{\zeta}(t)}{t^{\theta-\zeta}}dt, \quad \zeta>1, \tag{3}$$

where $\Psi(t) = \begin{cases} \int_0^t \Gamma(s)ds, & \text{if } \theta > 1, \\ \int_t^{\infty}\Gamma(s)ds, & \text{if } \theta < 1. \end{cases}$

The exhibition of the results containing the improvements, generalizations and applications of the discrete and continuous Hardy inequalities can be found in the books [7, 26, 32, 33, 38] and references therein.

Since various generalizations and numerous variants of the discrete Hardy inequality (1) exist in the literature, all of which can not be covered here, we only focus on the extensions which have been established by Copson [15, Theorem 1.1, Theorem 2.1]. We refer these inequalities as Hardy-Copson type inequalities. The discrete Hardy inequality (1) or Copson's discrete inequalities were generalized in [9, 14, 34–37] and references therein.

The investigation of the reverse Hardy-Copson inequalities, which are called Bennett-Leindler inequalities, were started almost at the same time with the original inequalities.

The first reverse discrete Hardy-Copson inequalities were obtained by Hardy and Littlewood [23] in 1927 for $0 < \zeta < 1$ without finding the best possible constants. Then Copson [15], Bennett [10] and Leindler [35] established discrete Bennett-Leindler inequalities by means of the following: Assume that the sequences z and h are nonnegative. If $0 < \zeta < 1$, then

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=m}^{\infty} h(j)z(j)\right)^{\zeta} \geq \zeta^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad 0 \leq \theta < 1, \tag{4}$$

where $\overline{G}(m) = \sum_{j=1}^{m} z(j)$ and

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=m}^{\infty} h(j)z(j)\right)^{\zeta} \geq \left(\frac{\zeta}{1-\theta}\right)^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad \theta < 0 \tag{5}$$

and for $0 < L \leq \frac{z(m)}{z(m+1)}$,

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=1}^{m} h(j)z(j)\right)^{\zeta} \geq \left(\frac{L\zeta}{\theta-1}\right)^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad \theta > 1. \tag{6}$$

There are some results in [36] about the reverse discrete Hardy-Copson inequalities different than the above ones and in [19] about finding conditions on the sequence $z(m)$ for $0 < \zeta < 1$ to obtain best possible constant.

The following results are interesting due to the fact that in contrast to the literature, discrete Bennett-Leindler inequalities were obtained for $\zeta > 1$, which is the same interval as for the Hardy-Copson inequalities. In 1986, Renaud [45] established the following discrete Bennett-Leindler inequality for the nonnegative and nonincreasing sequence $h(m)$ whenever $\zeta > 1$ as

$$\sum_{m=1}^{\infty} \frac{1}{m^{\zeta}} \left(\sum_{j=1}^{m} h(j)\right)^{\zeta} \geq Z(\zeta) \sum_{m=1}^{\infty} h^{\zeta}(m), \tag{7}$$

where $Z(\zeta)$ is Riemann-Zeta function.

Similar to the discrete Hardy inequality (1), the continuous versions (2) or (3) have attracted many mathematicians' interests and expansions of these continuous inequalities have appeared in the literature. The first continuous refinements were obtained by Copson [16, Theorem 1, Theorem 3] and after these results many papers were devoted to continuous analogues and continuous improvements of the discrete Hardy-Copson inequalities, see [8, 27, 39, 41, 42].

The first continuous Bennett-Leindler inequality, which is the reverse version of the continuous Hardy-Copson inequality (3), when $\theta = \zeta$, was established in [26, Theorem 337] for $0 < \zeta < 1$ and for $\overline{H}(t) = \int_t^\infty h(s)ds$ as

$$\int_0^\infty \frac{\overline{H}^\zeta(t)}{t^\zeta} dt \geq \left(\frac{\zeta}{1-\zeta}\right)^\zeta \int_0^\infty h^\zeta(t)dt, \quad h(t) \geq 0. \tag{8}$$

Then Copson derived continuous analogues of the discrete Bennett-Leindler inequalities (5) and (6), which are called continous Bennett-Leindler inequalities, in [16, Theorem 4, Theorem 2], respectively, for $z(t) \geq 0$ and $h(t) \geq 0$ and $\overline{G}(t) = \int_0^t z(s)ds$, $H(t) = \int_0^t z(s)h(s)ds$, $\overline{H}(t) = \int_t^\infty z(s)h(s)ds$ in the following manners: If $0 < \zeta \leq 1$, $\theta < 1$ then

$$\int_0^b \frac{z(t)}{[\overline{G}(t)]^\theta} [\overline{H}(t)]^\zeta dt \geq \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_0^b z(t)[\overline{G}(t)]^{\zeta-\theta} h^\zeta(t)dt, \quad 0 < b \leq \infty. \tag{9}$$

If $0 < \zeta \leq 1 < \theta$, $a > 0$, then

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta} [H(t)]^\zeta dt \geq \left(\frac{\zeta}{\theta-1}\right)^\zeta \int_a^\infty z(t)[\overline{G}(t)]^{\zeta-\theta} h^\zeta(t)dt. \tag{10}$$

Unlike the above classical results, for $\zeta > 1$, the continuous counterpart of the discrete Bennett-Leindler inequality (7) was obtained in [45] as follows: Let $\zeta > 1$ and for nonnegative and decreasing function h, we have

$$\int_0^\infty \frac{1}{t^\zeta} \left[\int_0^t h(s)ds\right]^\zeta dt \geq \frac{\zeta}{\zeta-1} \int_0^\infty h^\zeta(t)dt. \tag{11}$$

Following the development of the time scale concept [6, 12, 13, 20, 21], the analysis of dynamic inequalities have become a popular research area and most classical inequalities have been extended to an arbitrary time scale. The surveys [1, 46] and the monograph [3] can be used to see these extended dynamic inequalities for delta approach. Although the nabla dynamic inequalities are less attractive compared to the delta ones, some of the nabla dynamic inequalities can be found in [5, 11, 22, 40, 43].

The growing interest to Hardy-Copson type inequalities take place in the time scale calculus as well and delta unifications of these inequalities are established in the book [4] and in the articles [2, 18, 44, 47, 48, 50–54] whereas their nabla counterparts and extensions can be seen in [29–31] for $\zeta > 1$.

In the delta time scale calculus, the reverse Hardy-Copson type inequalities, which are called delta Bennett-Leindler inequalities, can be found in [17,47,49,54,55] for $0 < \zeta < 1$. These results are unifications of discrete and continuous Bennett-Leindler inequalities mentioned above except the ones in [45]. In addition to delta calculus, the above discrete and continuous Bennett-Leindler inequalities can be unified by nabla calculus and the previous reverse Hardy-Copson type inequalities can be obtained for the nabla case, see [28] for $0 < \zeta < 1$. Then these inequalities are called nabla Bennett-Leindler inequalities.

For our further purposes, we will show the nabla Bennett-Leindler inequalities established for $0 < \zeta < 1$ in [28] and use them in the sequel. As is customary, ρ denotes the backward jump operator and $f^\rho(t) = (f \circ \rho)(t) = f(\rho(t))$.

The following theorem presented in [28, Theorem 3.1] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for $0 < \zeta < 1$.

Theorem 1. *[28] For nonnegative functions z and h, let us define the functions $G(t) = \int_t^\infty z(s)\nabla s$ and $H(t) = \int_a^t z(s)h(s)\nabla s$. If $\theta \le 0 < \zeta < 1$, then*

$$\int_a^\infty \frac{z(t)}{[G^\rho(t)]^\theta}[H(t)]^\zeta \nabla t \ge \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[G^\rho(t)]^{\zeta-\theta}\nabla t. \tag{12}$$

The following theorem presented in [28, Theorem 3.9] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for $0 < \zeta < 1$.

Theorem 2. *[28] For nonnegative functions z and h, let us define the functions $\overline{G}(t) = \int_a^t z(s)\nabla s$ and $\overline{H}(t) = \int_t^\infty z(s)h(s)\nabla s$. If $\theta \le 0 < \zeta < 1$, then*

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta}[\overline{H}^\rho(t)]^\zeta \nabla t \ge \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[\overline{G}(t)]^{\zeta-\theta}\nabla t. \tag{13}$$

The following theorem presented in [28, Theorem 3.12] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.4] for $0 < \zeta < 1$.

Theorem 3. *[28] For nonnegative functions z and h, let us define the functions $\overline{G}(t) = \int_a^t z(s)\nabla s$ and $H(t) = \int_a^t z(s)h(s)\nabla s$. For $L = \inf_{t\in\mathbb{T}} \frac{\overline{G}^\rho(t)}{\overline{G}(t)} > 0$, if $0 < \zeta < 1 < \theta$, then*

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta}[H(t)]^\zeta \nabla t \ge \left(\frac{\zeta L^\theta}{\theta-1}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[\overline{G}(t)]^{\zeta-\theta}\nabla t. \tag{14}$$

The following theorem presented in [28, Theorem 3.4] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.2] for $0 < \zeta < 1$.

Theorem 4. *[28] For nonnegative functions z and h, let us define the functions $G(t) = \int_t^\infty z(s)\nabla s$ and $\overline{H}(t) = \int_t^\infty z(s)h(s)\nabla s$. If $0 < \zeta < 1 < \theta$, then*

$$\int_a^\infty \frac{z(t)}{[G^\rho(t)]^\theta}[\overline{H}^\rho(t)]^\zeta \nabla t \ge \left(\frac{\zeta}{\theta-1}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[G^\rho(t)]^{\zeta-\theta}\nabla t. \tag{15}$$

Although delta and nabla Bennett-Leindler type inequalities for the case $0 < \zeta < 1$ have been deeply analyzed, the case $\zeta > 1$ has been investigated neither via nabla and delta approaches nor for continuous and discrete cases. Hence the main aim of this article is to complement aforementioned Bennett-Leindler type inequalities obtained for $0 < \zeta < 1$ to the case $\zeta > 1$ by using nabla and delta time scale calculi without changing the directions of the inequalities derived for $0 < \zeta < 1$. We preserve the directions of the known inequalities since otherwise we obtain the reverse Bennett-Leindler type inequalities, which are called Hardy-Copson type inequalities and have already been established for the case $\zeta > 1$ in delta [53] and nabla settings [29]. Our results are inspired by the papers [28] and [55] which contain nabla and delta Bennett-Leindler type inequalities for the case $0 < \zeta < 1$. We notice that the cases $\theta \leq 0$ and $\theta > 1$ were considered in [28] and [55] while the case $0 \leq \theta < 1$ was not investigated therein. By taking account of another constant $\eta \geq 0$, we not only generalize the nabla and delta Bennett-Leindler type inequalities presented in [28] and [55] for $\eta \geq 0$, but also complement them from the case $0 < \zeta < 1$ to the case $\zeta > 1$. Furthermore novel discrete and continuous Bennett-Leindler type inequalites, which are complementary and generalized inequalities of inequalities (4)-(11), are established for $\zeta > 1$ and $\eta \geq 0$.

The organization of this paper can be seen as follows. The nabla time scale calculus and its main properties are introduced in Section 2. The delta version can be obtained similarly. The contribution of Section 3, which includes the main result, is to extend the recently developed nabla and delta results, which were established for $0 < \zeta < 1$ and presented in [28, 55], to the case $\zeta > 1$ by using the properties of nabla and delta derivatives and integrals. Then the special cases of the nabla and delta $\zeta > 1$ type inequalities, which are continuous and discrete inequalities, are stated.

2. Preliminaries

This section is devoted to present the main definitions and theorems of the nabla time scale calculus. The fundamental theories of the delta and nabla calculi can be found in [6, 12].

If $\mathbb{T} \neq \emptyset$ is a closed subset of $\mathbb{R}$, then $\mathbb{T}$ is called a time scale. If $t > \inf \mathbb{T}$, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := sup\{\tau < t : \tau \in \mathbb{T}\}$. The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\nu(t) := t - \rho(t)$, for $t \in \mathbb{T}$.

The ∇-derivative of $\Gamma : \mathbb{T} \to \mathbb{R}$ at the point $t \in \mathbb{T}_\kappa = \mathbb{T}/[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$ denoted by $\Gamma^\nabla(t)$ is the number enjoying the property that for all $\epsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that

$$|\Gamma(s) - \Gamma(\rho(t)) - \Gamma^\nabla(t)(s - \rho(t))| \leq \epsilon|s - \rho(t)|$$

for all $s \in V$.

The nabla derivative satisfies the following.

Lemma 1. *[6, 12] Let $\Lambda : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$.*

(1) *If Λ is continuous at a left scattered point t, then Λ is nabla differentiable at t with* $\Lambda^{\nabla}(t) = \dfrac{\Lambda(t) - \Lambda(\rho(t))}{\nu(t)}$.

(2) Λ *is nabla differentiable at a left dense point t if and only if the limit* $\Lambda^{\nabla}(t) = \lim\limits_{s \to t} \dfrac{\Lambda(t) - \Lambda(s)}{t - s}$ *exists as a finite number.*

(3) *If Λ is nabla differentiable at t, then* $\Lambda^{\rho}(t) = \Lambda(t) - \nu(t)\Lambda^{\nabla}(t)$.

A function $\Gamma : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim\limits_{s \to t^+} \Gamma(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T}, \mathbb{R})$ denotes the class of real, ld-continuous functions defined on a time scale $\mathbb{T}$.

If $\Gamma \in C_{ld}(\mathbb{T}, \mathbb{R})$, then there exists a function $\overline{\Gamma}(t)$ such that $\overline{\Gamma}^{\nabla}(t) = \Gamma(t)$ and the nabla integral of Γ is defined by $\displaystyle\int_a^b \Gamma(s)\nabla s = \overline{\Gamma}(b) - \overline{\Gamma}(a)$.

Some of the properties of the nabla integral are gathered next.

Lemma 2. *[6, 12] Let $t_1, t_2, t_3 \in \mathbb{T}$ with $t_1 < t_3 < t_2$ and $a, b \in \mathbb{R}$. If $\Lambda, \Gamma : \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

1) $\displaystyle\int_{t_1}^{t_2} [a\Lambda(s) + b\Gamma(s)]\nabla s = a\int_{t_1}^{t_2} \Lambda(s)\nabla(s) + b\int_{t_1}^{t_2} \Gamma(s)\nabla s.$

2) $\displaystyle\int_{t_1}^{t_1} \Lambda(s)\nabla(s) = 0.$

3) $\displaystyle\int_{t_1}^{t_3} \Lambda(s)\nabla s + \int_{t_3}^{t_2} \Lambda(s)\nabla s = \int_{t_1}^{t_2} \Lambda(s)\nabla s = -\int_{t_2}^{t_1} \Lambda(s)\nabla s.$

4) *integration by parts formula holds:*

$$\int_{t_1}^{t_2} \Lambda(s)\Gamma^{\nabla}(s)\nabla s = \Lambda(t_2)\Gamma(t_2) - \Lambda(t_1)\Gamma(t_1) - \int_{t_1}^{t_2} \Lambda^{\nabla}(s)\Gamma(\rho(s))\nabla s.$$

Lemma 3 (Hölder's inequality)**.** *[40] Let $t_1, t_2 \in \mathbb{T}$. For $\Lambda, \Gamma \in C_{ld}([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ and for constants $\kappa, \varpi > 1$ with $\frac{1}{\kappa} + \frac{1}{\varpi} = 1$, Hölder's inequality*

$$\int_{t_1}^{t_2} |\Lambda(s)\Gamma(s)|\nabla s \le \left[\int_{t_1}^{t_2} |\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi} \text{ holds true.}$$

If $0 < \kappa < 1$ or $\kappa < 0$ with $\frac{1}{\kappa} + \frac{1}{\varpi} = 1$, then the reversed Hölder's inequality

$$\int_{t_1}^{t_2} |\Lambda(s)\Gamma(s)|\nabla s \ge \left[\int_{t_1}^{t_2} |\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi} \tag{16}$$

is satisfied.

Lemma 4 (Chain rule for the nabla derivative)**.** *[22] If $\Lambda:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\Gamma:\mathbb{T}\to\mathbb{R}$ is nabla differentiable, then $\Lambda\circ\Gamma$ is nabla differentiable and*

$$(\Lambda\circ\Gamma)^{\nabla}(s)=\Gamma^{\nabla}(s)\left[\int_0^1 \Lambda'(\Gamma(\rho(s))+h\nu(s)\Gamma^{\nabla}(s))dh\right].$$

3. Bennett-Leindler type inequalities

In the sequel, we will obtain several Bennett-Leindler type inequalities for nonnegative, ld-continuous, ∇-differentiable and locally nabla integrable functions z and h and for the functions $G, H, \overline{G}$ and $\overline{H}$ defined in Theorem 1-Theorem 4.

The next theorem, which is proven for $\zeta>1,\ \eta\geq 0$ and $\eta+\theta\leq 0$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0<\zeta<1,\ \eta=0$ and $\theta\leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality obtained by Saker et al. [55, Remark 2] or Kayar et al. [28, Remark 3.3].
(b) The continuous inequality obtained by Saker et al. [55, Remark 1] or Kayar et al. [28, Remark 3.2].
(c) The delta counterpart of the nabla inequality (12) in Theorem 1 obtained by Saker et al. [55, Theorem 2.1].
(d) The nabla inequality (12) in Theorem 1 obtained by Kayar et al. [28, Theorem 3.1].

Theorem 5. *Let the functions z, h, G and H be defined as in Theorem 1. For a constant $L_1>0$, assume that $\dfrac{G^{\rho}(t)}{G(t)}\leq L_1$ for $t\in(a,\infty)_{\mathbb{T}}$. Let $\zeta>1,\ \eta\geq 0$ be real constants. If $\eta+\theta\leq 0$, then we have*

(1)
$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t\geq\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^{\infty}\frac{z(t)h(t)[H^{\rho}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t,\tag{17}$$

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t\geq\left[\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[H^{\rho}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t.\tag{18}$$

(2)
$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}}\nabla t\geq\frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty}\frac{z(t)h(t)[H^{\rho}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t,\tag{19}$$

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}}\nabla t\geq\left[\frac{L_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[H^{\rho}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t.\tag{20}$$

Proof. The same methodology used in the proof of [28, Theorem 3.1] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.2) in the proof of [28, Theorem 3.1] as

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t = \int_a^\infty -u(t)\left[H^{\eta+\zeta}(t)\right]^\nabla \nabla t, \tag{21}$$

where $u(t) = -\int_t^\infty \frac{z(s)}{[G(s)]^{\eta+\theta}}\nabla s$. Observe that since $\eta+\zeta > 1$,

$$[H^{\eta+\zeta}(t)]^\nabla \geq (\eta+\zeta)z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}, \tag{22}$$

which is different than (3.3) in the proof of [28, Theorem 3.1]. In our case, when $\eta+\theta \leq 0$, since

$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \geq -(1-\eta-\theta)\frac{z(t)}{[G^\rho(t)]^{\eta+\theta}} \geq -(1-\eta-\theta)\frac{z(t)}{L_1^{\eta+\theta}[G(t)]^{\eta+\theta}},$$

using (22) and

$$-u(t) = \int_t^\infty \frac{z(s)\nabla s}{[G(s)]^{\eta+\theta}} \geq \int_t^\infty \frac{-L_1^{\eta+\theta}\left[G^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{1-\eta-\theta} = \frac{L_1^{\eta+\theta}[G(t)]^{1-\eta-\theta}}{1-\eta-\theta}$$

in (21) implies the desired result (17). In order to obtain inequality (18), we apply reversed Hölder inequality (16) to inequality (17) with the constants $\frac{1}{\zeta} < 1$ and $\frac{1}{1-\zeta} < 0$.

(2) When the above process is repeated for the left hand side of inequality (19) with $u(t) = -\int_t^\infty \frac{z(s)}{[G^\rho(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 1. *The nabla Bennett-Leindler type inequalities* (17)-(20) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.1] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 1. *From inequalities* (17)-(20) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^ρ, G, H^ρ, H *presented in Theorem 1 by* G, G^σ, H, H^σ*, repectively, where*

$$G(t) = \int_t^\infty z(s)\Delta s \quad \textit{and} \quad H(t) = \int_a^t z(s)h(s)\Delta s \tag{23}$$

and $\sigma : \mathbb{T} \to \mathbb{T}$ *denotes the forward jump operator defined by* $\sigma(t) := inf\,\{\tau > t : \tau \in \mathbb{T}\}$ *with* $f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t))$.

Let z and h be nonnegative functions and G and H be defined as in (23). *For a constant $M_1 > 0$, assume that $\dfrac{G(t)}{G^\sigma(t)} \le M_1$ for $t \in (a,\infty)_{\mathbb{T}}$. In this case for $\zeta > 1$, $\eta \ge 0$ and $\eta+\theta \le 0$, nabla Bennett-Leindler type inequalities* (17)-(20) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (18) *and* (20) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \ge \left[\frac{M_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \ge \left[\frac{M_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (17)-(20) *obtained for $\zeta > 1$, $\eta \ge 0$ and $\eta+\theta \le 0$ are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for $0 < \zeta < 1$, $\eta = 0$ and $\theta \le 0$.*

Remark 2. *If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_1 = 1$ in* (17)-(20). *Hence inequalities* (17) *and* (19) *as well as inequalities* (18) *and* (20) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (17) *and* (18) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \ge \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \ge \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1$, $\eta \ge 0$ and $\eta+\theta \le 0$ and the functions G and H are defined as

$$G(t) = \int_t^\infty z(s)ds \quad \text{and} \quad H(t) = \int_a^t z(s)h(s)ds. \tag{24}$$

For the continuous case, when $0 < \zeta < 1$, $\eta = 0$ and $\theta \le 0$, the first Bennett-Leindler type inequalities were established in [55, Remark 1] and [28, Remark 3.2] for the given aforementioned functions G and H. These inequalities are extended to the cases $\zeta > 1$, $\eta \ge 0$ and $\eta+\theta \le 0$ by the above novel continuous Bennett-Leindler type inequalities.

Remark 3. *If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t-1$ in* (17)-(20).

Using $\int_t^\infty z(s)\nabla s = \sum_{k=t+1}^{\infty} z(k)$, *we have* $G^\rho(t) = G(t-1) = \sum_{k=t}^{\infty} z(k)$, *where* $G(t) = \sum_{k=t+1}^{\infty} z(k)$. *Moreover* $H(t) = \sum_{k=a+1}^{t} z(k)h(k)$. *For a constant $L_1 > 0$, let us assume that* $\frac{G(t-1)}{G(t)} \le L_1$. *For $a = 0,\ \zeta > 1,\ \eta \ge 0$ and $\eta + \theta \le 0$, in the set of natural numbers, inequalities* (17)-(20) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (18) *and* (20) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \ge \left[\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \ge \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

For the discrete case, when $0 < \zeta < 1,\ \eta = 0$ and $\theta \le 0$, the first Bennett-Leindler type inequalities were established in [55, Remark 2] and [28, Remark 3.3] for the given aforementioned series G and H. These inequalities are extended to the cases $\zeta > 1,\ \eta \ge 0$ and $\eta + \theta \le 0$ by the above novel discrete Bennett-Leindler type inequalities.

The next theorem, which is proven for $\zeta > 1,\ \eta \ge 0$ and $0 \le \eta + \theta < 1$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0$ and $\theta \le 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality obtained by Saker et al. [55, Remark 2] or Kayar et al. [28, Remark 3.3].
(b) The continuous inequality obtained by Saker et al. [55, Remark 1] or Kayar et al. [28, Remark 3.2].
(c) The delta counterpart of the nabla inequality (12) in Theorem 1 obtained by Saker et al. [55, Theorem 2.1].
(d) The nabla inequality (12) in Theorem 1 obtained by Kayar et al. [28, Theorem 3.1].

Theorem 6. *Let the fuctions z, h, G and H be defined as in Theorem 1. For a constant $L_2 > 0$, let us assume that* $1 \le \frac{G^\rho(t)}{G(t)} \le \frac{1}{L_2}$ *for $t \in (a,\infty)_{\mathbb{T}}$. Let $\zeta > 1,\ \eta \ge 0$ be real constants. If $0 \le \eta + \theta < 1$, then we have*

(1)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \frac{L_2^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t, \tag{25}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{26}$$

(2)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t, \tag{27}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{28}$$

Proof. The same methodology used in the proofs of [28, Theorem 3.1] and Theorem 6 works for the proof of this theorem except that for $0 \leq \eta+\theta < 1$, we have

$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \geq -(1-\eta-\theta)\frac{z(t)}{[G(t)]^{\eta+\theta}}.$$

□

Remark 4. *The nabla Bennett-Leindler type inequalities* (25)-(28) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.1] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 2. *From inequalities* (25)-(28) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^ρ, G, H^ρ, H *presented in Theorem 1 by* G, G^σ, H, H^σ *defined in* (23)*, repectively.*

Let z *and* h *be nonnegative functions and* G *and* H *be defined as in* (23)*. For a constant* $M_2 > 0$*, let us assume that* $1 \leq \dfrac{G(t)}{G^\sigma(t)} \leq \dfrac{1}{M_2}$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$*, the nabla Bennett-Leindler type inequalities* (25)-(28) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (26) *and* (28) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \geq \left[M_2\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (25)-(28) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Remark 5. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_2 = 1$ *in* (25)-(28). *Hence inequalities* (25) *and* (27) *as well as inequalities* (26) *and* (28) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (25) *and* (26) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *and the functions* G *and* H *are defined as in* (24).

For the continuous case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$, *the first Bennett-Leindler type inequalities were established in [55, Remark 1] and [28, Remark 3.2] for the given aforementioned functions* G *and* H. *These inequalities are extended to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *by the above novel continuous Bennett-Leindler type inequalities.*

Remark 6. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t-1$ *in* (25)-(28). *Suppose that the series* G *and* H *are defined as in Remark 3. For a constant* $L_2 > 0$, *let us assume that* $1 \leq \dfrac{G(t-1)}{G(t)} \leq \dfrac{1}{L_2}$. *For* $a = 0,\ \zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$, *in the set of natural numbers, inequalities* (25)-(28) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (26) *and* (28) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \geq \left[L_2\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

For the discrete case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$, *the first Bennett-Leindler type inequalities were established in [55, Remark 2] and [28, Remark 3.3] for the given aforementioned series* G *and* H. *These inequalities are extended to the* *cases $\zeta > 1,\ \eta \geq 0$ and $0 \leq \eta + \theta < 1$ by the above novel discrete Bennett-Leindler type inequalities.*

The next theorem, which is proven for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta \leq 0$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0$ and $\eta + \theta \leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (5) obtained by Copson [15, Theorem 2.3] and by Bennett [10, Corollary 1] or Leindler [35, Proposition 6].
(b) The continuous inequality (8) obtained by Hardy et al. [26, Theorem 337] and the continuous inequality (9) obtained by Copson [16, Theorem 4].
(c) The delta counterpart of the nabla inequality (13) in Theorem 2 obtained by Saker et al. [55, Theorem 2.3].
(d) The nabla inequality (13) in Theorem 2 obtained by Kayar et al. [28, Theorem 3.9].

Theorem 7. *Let the fuctions $z, h, \overline{G}$ and $\overline{H}$ be defined as in Theorem 2. For a constant $L_3 > 0$, assume that $\dfrac{\overline{G}(t)}{\overline{G}^{\rho}(t)} \leq L_3$ for $t \in (a,\infty)_{\mathbb{T}}$. Let $\zeta > 1,\ \eta \geq 0$ be real constants. If $\eta + \theta \leq 0$, then we have*

(1)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \geq \frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta} \int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}} \nabla t, \tag{29}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{30}$$

(2)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta} \int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}} \nabla t, \tag{31}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{32}$$

Proof. The same methodology used in the proof of [28, Theorem 3.9] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.11) in the proof of [28, Theorem 3.9] as

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t = \int_a^{\infty} u^{\rho}(t) \left\{ -\left[\overline{H}^{\eta+\zeta}(t)\right]^{\nabla} \right\} \nabla t, \tag{33}$$

where $u(t) = \int_a^t \frac{z(s)}{[\overline{G}^\rho(s)]^{\eta+\theta}} \nabla s$. Observe that since $\eta + \zeta > 1$,

$$-[\overline{H}^{\eta+\zeta}(t)]^\nabla \geq (\eta+\zeta) z(t) h(t) [\overline{H}(t)]^{\eta+\zeta-1}, \tag{34}$$

which is different than (3.12) in the proof of [28, Theorem 3.9]. In our case, when $\eta + \theta \leq 0$, since

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^\nabla \leq (1-\eta-\theta) \frac{z(t)}{[\overline{G}(t)]^{\eta+\theta}} \leq (1-\eta-\theta) \frac{z(t)}{L_3^{\eta+\theta} [\overline{G}^\rho(t)]^{\eta+\theta}},$$

using (34) and

$$u^\rho(t) = \int_a^{\rho(t)} \frac{z(s)\nabla s}{[\overline{G}^\rho(s)]^{\eta+\theta}} \geq \int_a^{\rho(t)} \frac{L_3^{\eta+\theta} \left[\overline{G}^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{1-\eta-\theta} = \frac{L_3^{\eta+\theta} [\overline{G}^\rho(t)]^{1-\eta-\theta}}{1-\eta-\theta}$$

in (33) implies the desired result (29). In order to obtain inequality (30), we apply reversed Hölder inequality (16) to inequality (29) with the constants $\frac{1}{\zeta} < 1$ and $\frac{1}{1-\zeta} < 0$.

(2) When the above process is repeated for the left hand side of inequality (31) with $u(t) = \int_a^t \frac{z(s)}{[\overline{G}(s)]^{\eta+\theta}} \nabla s$, the desired results can be obtained.

□

Remark 7. *The nabla Bennett-Leindler type inequalities* (29)-(32) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta \leq 0$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.9] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 3. *From inequalities* (29)-(32) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^\rho, \overline{G}, \overline{H}^\rho, \overline{H}$ *presented in Theorem 2 by* $\overline{G}, \overline{G}^\sigma, \overline{H}, \overline{H}^\sigma$ *defined as*

$$\overline{G}(t) = \int_a^t z(s)\Delta s \quad \text{and} \quad \overline{H}(t) = \int_t^\infty z(s)h(s)\Delta s, \tag{35}$$

respectively.

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* $\overline{H}$ *be defined as in* (35)*. For a constant* $M_3 > 0$*, let us assume that* $\frac{\overline{G}^\sigma(t)}{\overline{G}(t)} \leq M_3$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta \leq 0$*, the nabla Bennett-Leindler type inequalities* (29)-(32) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (30) *and* (32) *can be written as follows*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \Delta t \geq \left[\frac{M_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t) h^{1/\zeta}(t) [\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \Delta t$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (29)-(32) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for* $0<\zeta<1,\ \eta=0$ *and* $\theta \leq 0$.

Remark 8. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t)=t$ *and* $L_3=1$ *in* (29)-(32). *Hence inequalities* (29) *and* (31) *as well as inequalities* (30) *and* (32) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (29) *and* (30) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta\leq 0$ *and the functions* $\overline{G}$ *and* $\overline{H}$ *are defined as*

$$\overline{G}(t)=\int_a^t z(s)ds \quad \textit{and} \quad \overline{H}(t)=\int_t^\infty z(s)h(s)ds. \tag{36}$$

These novel inequalities complement and generalize the continuous inequality (8) *obtained by Hardy et al. [26, Theorem 337] for* $0<\zeta<1,\ \eta=0$ *and* $\theta=\zeta$ *and the continuous inequality* (9) *obtained by Copson [16, Theorem 4] for* $0<\zeta<1,\ \eta=0$ *and* $\theta<1$ *to the cases* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta\leq 0$.

Remark 9. *If the time scale is the set of natural numbers, then for all* $t\in\mathbb{N}$, *the backward jump operator results in* $\rho(t)=t-1$ *in* (29)-(32).

Using $\overline{G}(t)=\displaystyle\int_a^t z(s)\nabla s=\sum_{k=a+1}^{t} z(k)$, *we have* $\overline{G}^\rho(t)=\overline{G}(t-1)=\displaystyle\sum_{k=a+1}^{t-1} z(k)$.
Moreover $\overline{H}(t)=\displaystyle\sum_{k=t+1}^{\infty} z(k)f(k)$. *For a constant* $L_3>0$, *let us assume that* $\dfrac{\overline{G}(t)}{\overline{G}(t-1)} \leq L_3$. *For* $a=0,\ \zeta>1,\ \eta\geq 0$, *and* $\eta+\theta\leq 0$, *in the set of natural numbers, inequalities* (29)-(32) *become novel discrete Bennett-Leindler type*

inequalities, two of which obtained from (30) *and* (32) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \geq \left[\frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \geq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

The discrete Bennett-Leindler type inequality (5) *obtained by Copson [15, Theorem 2.3] and by Bennett [10, Corollary 1] or Leindler [35, Proposition 6] for* $0<\zeta<1,\ \eta=0,\ \theta<0$ *is complemented and generalized to the cases* $\zeta>1,\ \eta\geq 0,\ \eta+\theta\leq 0$ *by Theorem 7 and particularly by this remark.*

The next theorem, which is proven for $\zeta>1,\ \eta\geq 0$ and $0\leq\eta+\theta<1$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0<\zeta<1,\ \eta\geq 0$ and $\eta+\theta\leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (4) obtained by Copson [15, Theorem 2.3].
(b) The continuous inequality (8) obtained by Hardy et al. [26, Theorem 337] and the continuous inequality (9) obtained by Copson [16, Theorem 4].
(c) The delta analogue of the inequality (13) in Theorem 2 obtained by Saker et al. [55, Theorem 2.3].
(d) The nabla inequality (13) in Theorem 2 obtained by Kayar et al. [28, Theorem 3.9].

Theorem 8. *Let the functions* $z, h, \overline{G}$ *and* $\overline{H}$ *be defined as in Theorem 2. For a constant* $L_4>0$*, let us assume that* $1\leq \dfrac{\overline{G}(t)}{\overline{G}^{\rho}(t)}\leq \dfrac{1}{L_4}$ *for* $t\in(a,\infty)_{\mathbb{T}}$*. Let* $0<\zeta<1,\ \eta\geq 0$ *be real constants. If* $0\leq\eta+\theta<1$*, then we have*

(1)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \frac{L_4^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}}\nabla t, \tag{37}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{38}$$

(2)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}}\nabla t, \tag{39}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{40}$$

Proof. The same methodology used in the proofs of [28, Theorem 3.9] and Theorem 7 works for the proof of this theorem except that for $0 \leq \eta+\theta < 1$, we have

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^{\nabla} \leq (1-\eta-\theta)\frac{z(t)}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}.$$

□

Remark 10. *The nabla Bennett-Leindler type inequalities* (37)-(40) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.9] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 4. *From inequalities* (37)-(40) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^{\rho}, \overline{G}, \overline{H}^{\rho}, \overline{H}$ *presented in Theorem 2 by* $\overline{G}, \overline{G}^{\sigma}, \overline{H}, \overline{H}^{\sigma}$ *defined in* (35), *repectively.*

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* $\overline{H}$ *be defined as in* (35). *For a constant* $M_4 > 0$, *let us assume that* $1 \leq \dfrac{\overline{G}^{\sigma}(t)}{\overline{G}(t)} \leq \dfrac{1}{M_4}$ *for* $t \in (a,\infty)_{\mathbb{T}}$. *In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$, *the nabla Bennett-Leindler type inequalities* (37)-(40) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (38) *and* (40) *can be written as follows*

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}}\Delta t \geq \left[M_4\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (37)-(40) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Remark 11. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_4 = 1$ *in* (37)-(40). *Hence inequalities* (37) *and* (39) *as well as inequalities* (38) *and* (40) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (37) *and* (38) *reduce to the following inequalities as*

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty}\frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *and the functions* $\overline{G}$ *and* $\overline{H}$ *are defined as in* (36).

These novel inequalities complement and generalize the continuous inequality (8) *obtained by Hardy et al. [26, Theorem 337] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta = \zeta$ *and the continuous inequality* (9) *obtained by Copson [16, Theorem 4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta < 1$ *to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$.

Remark 12. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t-1$ *in* (37)-(40). *Suppose that the series* $\overline{G}$ *and* $\overline{H}$ *are defined as in Remark 9. For a constant* $L_4 > 0$, *let us assume that* $\dfrac{\overline{G}(t)}{\overline{G}(t-1)} \leq \dfrac{1}{L_4}$. *For* $a = 0,\ \zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$, *in the set of natural numbers, inequalities* (37)-(40) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (38) *and* (40) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \geq \left[L_4 \frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

The discrete Bennett-Leindler type inequality (4) *obtained by Copson [15, Theorem 2.3] for* $0 < \zeta < 1,\ \eta = 0,\ 0 \leq \theta < 1$ *is complemented and generalized to the case* $\zeta > 1,\ \eta \geq 0,\ 0 \leq \eta+\theta < 1$ *by Theorem 8 and particularly by this remark.*

The next theorem, which is proven for $\zeta > 1,\ \eta \geq 0$ and $\eta+\theta > 1$, provides complements and generalizations of some of the previous Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0,\ \theta > 1$ or $\zeta > 1,\ \eta = 0,\ \theta = \zeta$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (6) obtained by Copson [15, Theorem 1.3] and Bennett [10, Corollary 3] or Leindler [35, Proposition 7] as well as the discrete inequality (7) obtained by Renaud [45, Theorem 1].
(b) The continuous inequality (10) obtained by Copson [16, Theorem 2] and the continuous inequality (11) obtained by Renaud in [45, Theorem 3].
(c) The delta counterpart of the nabla inequality (14) in Theorem 3 obtained by Saker et al. [55, Theorem 2.4].
(d) The nabla inequality (14) in Theorem 3 obtained by Kayar et al. [28, Theorem 3.12].

Theorem 9. *Suppose that the functions $z, h, \overline{G}$ and H are defined as in Theorem 3 and the constant L_4 is defined as in Theorem 8. Let $\zeta > 1$, $\eta \geq 0$ be real numbers. If $\eta + \theta > 1$, then we have*

(1)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1} \int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}} \nabla t, \tag{41}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{42}$$

(2)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}} \nabla t \geq \frac{\eta+\zeta}{\eta+\theta-1} \int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}} \nabla t, \tag{43}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{44}$$

Proof. The same methodology used in the proof of [28, Theorem 3.12] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.16) in the proof of [28, Theorem 3.12] as

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t = \int_a^\infty -u(t) \left[H^{\eta+\zeta}(t)\right]^\nabla \nabla t, \tag{45}$$

where $u(t) = \displaystyle\int_t^\infty \frac{z(s)}{[\overline{G}(s)]^{\eta+\theta}} \nabla s$. In our case, when $\eta + \theta > 1$, since

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^\nabla \geq -(\eta+\theta-1)\frac{z(t)}{[\overline{G}^\rho(t)]^{\eta+\theta}} \geq -(\eta+\theta-1)\frac{z(t)}{L_4^{\eta+\theta}[\overline{G}(t)]^{\eta+\theta}},$$

using (22) and

$$-u(t) = \int_t^\infty \frac{z(s)\nabla s}{[\overline{G}(s)]^{\eta+\theta}} \geq \int_t^\infty \frac{-L_4^{\eta+\theta}\left[\overline{G}^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = \frac{L_4^{\eta+\theta}[\overline{G}(t)]^{1-\eta-\theta}}{\eta+\theta-1}$$

in (45) implies the desired result (41). In order to obtain inequality (42), we apply reversed Hölder inequality (16) to inequality (41) with the constants $\dfrac{1}{\zeta} < 1$ and $\dfrac{1}{1-\zeta} < 0$.

(2) When the above process is repeated for the left hand side of inequality (43) with $u(t) = \displaystyle\int_t^\infty \frac{z(s)}{[\overline{G}^\rho(s)]^{\eta+\theta}} \nabla s$, the desired results can be obtained.

□

Remark 13. *The nabla Bennett-Leindler type inequalities* (41)-(44) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.12] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Corollary 5. *From inequalities* (41)-(44) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^{\rho}, \overline{G}, H^{\rho}, H$ *presented in Theorem 3 by* $\overline{G}, \overline{G}^{\sigma}, H, H^{\sigma}$ *defined in* (35) *and* (23)*, repectively.*

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* H *be defined as in* (35) *and* (23)*, repectively, and the constant* M_4 *be defined as in Corollary 4. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$*, the nabla Bennett-Leindler type inequalities* (41)-(44) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (42) *and* (44) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}} \Delta t \geq \left[\frac{M_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \Delta t \geq \left[\frac{M_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (41)-(44) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Remark 14. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$*, the backward jump operator results in* $\rho(t) = t$ *and* $L_4 = 1$ *in* (41)-(44)*. Hence inequalities* (41) *and* (43) *as well as inequalities* (42) *and* (44) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (41) *and* (42) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} dt \geq \frac{\eta+\zeta}{\eta+\theta-1} \int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}} dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} dt \geq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} dt.$$

respectively, where $\zeta < 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *and the functions* $\overline{G}$ *and* H *are defined as in* (36) *and* (24)*, respectively.*

These novel inequalities complement and generalize the continuous inequality (10) *obtained by Copson [16, Theorem 2] for* $0<\zeta<1,\ \eta=0$ *and* $\theta>1$ *and the continuous inequality* (11) *obtained by Renaud in [45, Theorem 3] for* $\zeta>1,\ \eta=0$ *and* $\theta=\zeta$ *to the cases* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta>1$.

Remark 15. *If the time scale is the set of natural numbers, then for all* $t\in\mathbb{N}$, *the backward jump operator results in* $\rho(t)=t-1$ *in* (41)-(44). *Let the constant* L_4 *be defined as in Remark 12. For* $a=0,\ \zeta>1,\ \eta\geq 0$ *and* $\eta+\theta>1$, *in the set of natural numbers, inequalities* (41)-(44) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (42) *and* (44) *can be written as follows*

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\geq\left[\frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}}\geq\left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively, where the series $\overline{G}$ *and* H *are defined as in Remark 9 and Remark 3, respectively.*

The discrete Bennett-Leindler type inequality (6) *obtained by Copson [15, Theorem 1.3] and Bennett [10, Corollary 3] or Leindler [35, Proposition 7] for* $0<\zeta<1,\ \eta=0,\ \theta>1$ *as well as the discrete inequality* (7) *obtained by Renaud [45, Theorem 1] for* $\zeta>1,\ \eta=0,\ \theta=\zeta$ *are complemented and generalized to the cases* $\zeta>1,\ \eta\geq 0,\ \eta+\theta>1$ *by Theorem 9 and particularly by this remark.*

The next theorem, which is proven for $\zeta>1,\ \eta\geq 0$ and $\eta+\theta>1$, provides complements and generalizations of some of the previous Bennett-Leindler type inequalities given for $0<\zeta<1,\ \eta=0$ and $\theta>1$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequalities obtained by Saker et al. [55, Remark 4] and by Kayar et al. [28, Remark 3.8].
(b) The continuous inequalities obtained by Saker et al. [55, Remark 3] and by Kayar et al. [28, Remark 3.7].
(c) The delta counterpart of the nabla inequality (15) in Theorem 4 obtained by Saker et al. [55, Theorem 2.2].
(d) The nabla inequality (15) in Theorem 4 obtained by Kayar et al. [28, Theorem 3.4].

Theorem 10. *Suppose that the functions* z,h,G *and* $\overline{H}$ *are defined as in Theorem 4 and the constant* L_2 *is defined as in Theorem 6. Let* $\zeta>1,\ \eta\geq 0$ *be real numbers. If* $\eta+\theta>1$, *then we have*

(1)

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G^\rho(t)]^{\eta+\theta-1}}\nabla t, \tag{46}$$

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{47}$$

(2)

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G^\rho(t)]^{\eta+\theta-1}}\nabla t, \tag{48}$$

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{49}$$

Proof. The same methodology used in the proof of [28, Theorem 3.4] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.7) in the proof of [28, Theorem 3.4] as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t = \int_a^\infty u^\rho(t)\left\{-\left[\overline{H}^{\eta+\zeta}(t)\right]^\nabla\right\}\nabla t, \tag{50}$$

where $u(t) = \int_a^t \frac{z(s)}{[G^\rho(s)]^{\eta+\theta}}\nabla s$. In our case, when $\eta+\theta>1$, since

$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \leq (\eta+\theta-1)\frac{z(t)}{[G(t)]^{\eta+\theta}} \leq (\eta+\theta-1)\frac{z(t)}{L_2^{\eta+\theta}[G^\rho(t)]^{\eta+\theta}},$$

using (34) and

$$u^\rho(t) = \int_a^{\rho(t)} \frac{z(s)\nabla s}{[G^\rho(s)]^{\eta+\theta}} \geq \int_a^{\rho(t)} \frac{L_2^{\eta+\theta}\left[G^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = \frac{L_2^{\eta+\theta}[G^\rho(t)]^{1-\eta-\theta}}{\eta+\theta-1}$$

in (50) implies the desired result (46). In order to obtain inequality (47), we apply reversed Hölder inequality (16) to inequality (46) with the constants $\frac{1}{\zeta}<1$ and $\frac{1}{1-\zeta}<0$.

(2) When the above process is repeated for the left hand side of inequality (48) with $u(t) = \int_a^t \frac{z(s)}{[G(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 16. *The nabla Bennett-Leindler type inequalities* (46)-(49) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Corollary 6. *From inequalities* (46)-(49) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $G^\rho, G, \overline{H}^\rho, \overline{H}$ *presented in Theorem 4 by* $G, G^\sigma, \overline{H}, \overline{H}^\sigma$ *defined in* (23) *and* (35), *repectively.*

Let z *and* h *be nonnegative functions and* $\overline{H}$ *be defined as in* (23) *and* (35), *repectively, and the constant* M_2 *be defined as in Corollary 2. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$, *the nabla Bennett-Leindler type inequalities* (46)-(49) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (47) *and* (49) *can be written as follows*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively. The delta variants of the nabla Bennett-Leindler type inequalities (46)-(49) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.2] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Remark 17. *If the time scale is set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_2 = 1$ *in* (46)-(49). *Hence inequalities* (46) *and* (48) *as well as inequalities* (47) *and* (49) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (46) *and* (47) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *and the functions* G *and* $\overline{H}$ *are defined as in* (24) *and* (36), *repectively.*

For the continuous case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$, *the first Bennett-Leindler type inequalities were established in [55, Remark 3] and [28, Remark 3.7] for the given aforementioned functions* G *and* $\overline{H}$. *By this remark, these inequalities* *are extended to the cases $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ by the above novel continuous Bennett-Leindler type inequalities.*

Remark 18. *If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (46)-(49). Let the constant L_2 be defined as in Remark 5. For $a = 0$, $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$, in the set of natural numbers, inequalities (46)-(49) become novel discrete Bennett-Leindler type inequalities, two of which obtained from (47) and (49) can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \geq \left[\frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \geq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively, where the series $\overline{H}$ and G are defined as in Remark 9 and Remark 3, respectively.

For the discrete case, when $0 < \zeta < 1$, $\eta = 0$ and $\theta > 1$, the first Bennett-Leindler type inequalities were established in [55, Remark 4] and [28, Remark 3.8] for the given aforementioned series G and $\overline{H}$. By this remark, these inequalities are extended to the cases $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ by the above novel discrete Bennett-Leindler type inequalities.

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