Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (5) (2022), 1403 – 1418
DOI : 10.15672/hujms.930601

Research Article

The modified objective-constraint scalarization approach for multiobjective optimization problems

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Abstract

In this article, a novel scalarization methodology, called the modified objective-constraint technique, is proposed for determining efficient solutions a given multiobjective programming problem. The suggested scalarized problem extends some existing problems. It is shown that how adding slack variables to the constraints, can help us to find easily checked conditions concerning (weak, proper) Pareto optimality. By applying the suggested problem, we generate an almost even approximation of the efficient front. The performance and capability of the developed approach are demonstrated in test problems containing disconnected or nonconvex fronts and feasible points. In particular, we apply the suggested approach in an engineering design problem with two objective functions.

Mathematics Subject Classification (2020). 90C29, 90C30, 49M37

Keywords. multiobjective optimization, objective-constraint scalarization, Pareto optimality, proper efficiency

1. Introduction

In the majority of real-world optimization problems, several competing criterion functions are minimized, simultaneously. In many cases, it is impractical to earn a single point minimizing all the criteria together, and hence other optimality concepts are discussed. Many optimality notions in the literature were introduced. Among them, the so-called (weakly and properly) Pareto minimality concepts are the most popular.The multiobjective optimization problems (MOPs) have important applications in many real-world problems including medical treatment, game theory, Internet, economics, management, machine learning, engineering design, control, and transportation. We refer to [1, 4, 11, 13, 14, 17, 19, 25, 27, 29, 34].

Researchers have discussed MOPs from various perspectives, and therefore there exist several solution philosophies for solving them. The main goal in these procedures is to construct a set of Pareto minimal points (see [3, 5, 9, 10, 23, 24, 31, 33]). A common technique to gain a collection of Pareto minimal solutions of a given MOP is reformulating it to an optimization problem with one objective function, including some added parameters, and solving it for different parameters. This technique is called scalarization. Therefore,

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Received: 01.05.2021; Accepted: 06.05.2022

by this approach, the MOP is converted into a single-objective programming problem, named a scalarization model. The optimal solutions of the scalarized single-objective programming problem may then be attained by applying suitable existing approaches and appropriate softwares.

Several scalarization techniques that are widely studied by various other researchers are the weighted sum approach [20], the ϵ-constraint technique [8] and the Pascoletti-Serafini procedure and its generalizations [2, 16, 18, 22, 30]. Some other scalarization methods can be found in [12, 15, 17, 19, 26]. The most interesting scalarization techniques are those that by changing their parameters or weights, an acceptable approximation of the real Pareto front can be constructed. As typical cases, the weighted-constraint approach [6], the feasible-value-constraint technique [7] and the objective-constraint approach [32] are in this class. In the objective-constraint scalarizing technique [32], we devote a weight to each of the objectives and then one of the weighted objectives (kth) is optimized, while all the rest weighted objectives are added to the constraints.

It is worth mentioning that in references [7] and [6] necessary and/or sufficient results for (weakly) Pareto solutions were obtained, that are valid under some conditions. However, these scalarization techniques have no result on proper efficiency. As we know, sometimes the trade-offs among the objectives is unbounded. Pareto optimal solutions that have bounded trade-offs are named properly Pareto optimal [12, 16, 21, 28]. Rizvi [32], provided a condition for Karush-Kuhn-Tucker properly Pareto optimal points [28] of an MOP, utilizing the optimal solutions of the objective-constraint approach [32]. However, that result is valid under some conditions like differentiability of the functions and constraints and regularity conditions.

In this paper, we modify the introduced method in [32] by adding slack variables in the constraints. Here, we will investigate how adding slack variables in the constraints can help to identify conditions for (weak, proper) efficiency. We establish an MOP by means of creating a scalar-valued optimization problem. Furthermore, we find several necessary and sufficient conditions for different types of Pareto optimal solutions of an MOP. On the basis of the suggested scalarization technique and by varying the parameters, a set of optimal points is generated. We show the performance of the suggested technique on numerical test problems and an application in design engineering. The test problems display that the introduced method produces Pareto points located in the convex and nonconvex parts of the front. We show that even if the feasible regions and/or Pareto front are disconnected, our proposed technique can generate the Pareto points.

The rest of this article unfolds as follows. Some preliminaries on multiobjective programming are reviewed in Section 2. The new scalarization approach and its useful proved properties are contained in Section 3. In Section 4, the fruitfulness of the proposed approach on some examples is illustrated. In Section 5, the proposed approach is applied on an application problem and finally Section 6 is devoted to the conclusions.

2. Preliminary definitions

This section contains some basic preliminaries and fundamental notions which are utilized in the later sections of the paper. Let $y^1, y^2 \in \mathbb{R}^n$. In this text, the following conventions and notations are used for the special orders in $\mathbb{R}^n$.

$$y^1 \leqq y^2 \iff y^1_j \leqslant y^2_j, \forall j \in \{1, \dots, n\},$$

$$y^1 \leq y^2 \iff y^1 \leqq y^2 \text{ and } y^1 \neq y^2,$$

$$y^1 < y^2 \iff y^1_j < y^2_j, \forall j \in \{1, \dots, n\}.$$

With the above-mentioned orders, we describe the nonnegative orthant as $\mathbb{R}^n_{\geqq} = \{y \in \mathbb{R}^n : y \geqq 0\}$. Let $X \subseteq \mathbb{R}^m$ be a nonempty set and $f_j : X \longrightarrow \mathbb{R}, j = 1, 2, \ldots, n$ be real-valued functions defined over X. We investigate the following multiobjective optimization problem (MOP):

$$MOP \quad \min_{x \in X} \quad F(x) = (f_1(x), f_2(x), \cdots, f_n(x)), \tag{2.1}$$

The image set of the feasible set X, is illustrated by $Y = f(X) = \{f(x) : x \in X\}$. The boundedness of all the objective functions is assumed over the X.

Definition 2.1. Investigate MOP (2.1) and a feasible solution $x^* \in X$.

(i) $x^* \in X$ is named Pareto optimal (or efficient), if there is no $x \in X$ with $F(x) \leq F(x^*)$.

(ii) x^* is named weak Pareto optimal (or weak efficient), if there is no $x \in X$ with $f_j(x) < f_j(x^*), \ \ \forall j \in \{1, 2, \ldots, n\}$.

Definition 2.2. A feasible solution x^* is named properly Pareto optimal solution of MOP (2.1) if it is Pareto optimal and if there is some number $M > 0$ such that, for any $x \in X$ if $f_l(x) < f_l(x^*)$ for some $l \in \{1, 2, \ldots, n\}$, we have $(f_l(x^*) - f_l(x))/(f_t(x) - f_t(x^*)) \leqslant M$, for some $t \in \{1, 2, \ldots, n\}$ such that $f_t(x^*) < f_t(x)$.

Notation: We denote by $WP(MOP)$, $P(MOP)$ and $PP(MOP)$ the set of weakly Pareto, Pareto and properly Pareto solutions of MOP (2.1), respectively.

In general, MOPs have a set of optimal points instead of only one. Therefore, it is an interesting issue to develop efficient approaches for finding these optimal points. Some approaches developed for determining efficient solutions are stochastic techniques [33], interactive approaches [9, 10] or evolutionary algorithms [5, 10]. Scalarization is the most popular approach for solving MOPs. Scalarization means transforming the MOP into a suitable real-valued optimization problem, perhaps including some additional constraints and/or parameters. By solving the single objective optimization problems for different parameters, a variety of efficient solutions can be obtained. There exist many scalarization methods in the literature. Now, we recall two of the recent scalarization approaches that are related to our proposed method.

The weighted-constraint approach. This approach developed by Burachik et al. [6] for transforming the MOP into a scalarized problem by optimizing the kth weighted objective function and incorporating the other weighted functions as constraints. The associated scalarized problem is stated as:

$$\begin{array}{ll} \min & w_k f_k(x) \\ s.t. & w_i f_i(x) \leqslant w_k f_k(x), \ \ \forall i \in \{1, 2, \ldots, n\}, \ \ i \neq k \\ & x \in X, \end{array} \tag{2.2}$$

where k, is fixed, $w_i > 0 \ \forall i$ and $\displaystyle\sum_{i=1}^{n} w_i = 1$.

If $\hat{x} \in X$ solves the scalarized problem (2.2), $\forall k$ with $w \in \mathbb{R}^n_{>}$, then $\hat{x} \in WP(MOP)$. However, if a point is optimal of this problem for all k, then it is not necessarily efficient (for more details refer to [6]).

The objective-constraint approach. This scalarized optimization problem described in [32] and for some fixed $\bar{x} \in X$ and $k \in \{1, 2, ..., n\}$ is stated as

$$\begin{array}{ll} \min & w_k f_k(x) \\ s.t. & w_i f_i(x) \leqslant w_k f_k(\bar{x}), \ \ \forall i \in \{1, 2, \ldots, n\}, \ \ i \neq k \\ & x \in X, \end{array} \tag{2.3}$$

where

$$w_i = \frac{1/f_i(\bar{x})}{\sum_{t=1}^{n} 1/f_t(\bar{x})}. \tag{2.4}$$

If $\hat{x} \in X$ solves the scalarized problem (2.3), for all k and with w defined in (2.4), then $\hat{x} \in P(MOP)$ (refer to [32]).

3. The modified objective-constraint approach

In this section, we suggest a new scalarization technique, named *the modified objective-constraint approach*. Let $\bar{x}$ be an arbitrarily selected feasible solution for MOP (2.1). For fixed $k \in \{1, 2, \ldots, n\}$, we introduce the associated scalar problem as follows:

$$\begin{aligned} (SOP_{\bar{x}}^k): \quad \min \quad & w_k f_k(x) - \sum_{i \neq k} \lambda_i t_i \\ s.t. \quad & w_i f_i(x) + t_i \leqslant w_k f_k(\bar{x}), \quad i \neq k, \\ & t_i \geqslant 0, \quad i \neq k, \\ & x \in X, \end{aligned} \tag{3.1}$$

where $w_i,\ i \in \{1, 2, \ldots, n\}$ and $\lambda_i,\ i \neq k$ are nonnegative weights.
This approach modifies the objective-constraint scalarized problem by adding nonnegative slack variables to the constraints. The kth-objective function equipped with the weight w_k and the negative summation of the weighted slacks, form the objective function of this scalarized problem. It is obvious that, by choosing appropriate values for the available parameters in (3.1), one can get scalarized problems (2.2) and (2.3). Let $\bar{X}^k \subseteq X$ be the feasible set of $(SOP_{\bar{x}}^k)$. We define $Sol_{\bar{x}}^k = \{(\tilde{x}, \tilde{t}) \in \bar{X}^k | (\tilde{x}, \tilde{t}) \text{ solves } (SOP_{\bar{x}}^k)\}$. Before we present our investigations, we state the next lemma.

Lemma 3.1. *Let $\lambda \geqq 0$ and $Sol_{\bar{x}}^k \neq \emptyset$. Then, there exists an optimal solution $(\tilde{x}, \tilde{t})$ for $(SOP_{\bar{x}}^k)$ such that $w_i f_i(\tilde{x}) + \tilde{t}_i = w_k f_k(\bar{x})$ for each $i \neq k$. If $\lambda > 0$, then $w_i f_i(\tilde{x}) + \tilde{t}_i = w_k f_k(\bar{x})$ for all $i \neq k$ is satisfied for every optimal solution of $(SOP_{\bar{x}}^k)$.*

Proof. Let $(\tilde{x}, \tilde{t}) \in Sol_{\bar{x}}^k$. Assume that there is an index $j \in \{1, 2, \ldots, n\} \backslash \{k\}$ with $w_j f_j(\tilde{x}) + \tilde{t}_j < w_k f_k(\bar{x})$. Therefore, there exists $\bar{t}_j > 0$ such that $w_j f_j(\tilde{x}) + \tilde{t}_j + \bar{t}_j = w_k f_k(\bar{x})$. We consider

$$t_i = \begin{cases} \tilde{t}_i, & i \in \{1, 2, \ldots, n\} \backslash \{j, k\}, \\ \tilde{t}_i + \bar{t}_i, & i = j. \end{cases}$$

The point $(\tilde{x}, t)$ is feasible for $(SOP_{\bar{x}}^k)$. Since $t_j > \tilde{t}_j$, it yields

$$w_k f_k(\tilde{x}) - \sum_{i \neq k} \lambda_i t_i \leqslant w_k f_k(\tilde{x}) - \sum_{i \neq k} \lambda_i \tilde{t}_i.$$

This result means that if $\lambda_j > 0$ then $(\tilde{x}, t)$ implies a better objective value for $(SOP_{\bar{x}}^k)$ than $(\tilde{x}, \tilde{t})$ and it is the same as $(\tilde{x}, \tilde{t})$, if $\lambda_j = 0$. □

Before providing theoretical results, we present a geometric interpretation of the presented approach. Investigate the particular case of $n = 2$, Lemma 3.1 helps us to visualize the implementations in the objective space. Consider the feasible set of $(SOP_{\bar{x}}^2)$, namely $\bar{X}^2$. First we set $w_2 f_2(\bar{x}) = b$. Consider a feasible solution (x^*, t_1^*) and assume that the objective function value of $(SOP_{\bar{x}}^2)$ is $d^* = w_2 f_2(x^*) - \lambda_1 t_1^*$. We can interpret the level set of $v = d/w_2 = f_2(x) - (\lambda_1/w_2) t_1$ as a line in the $t_1 - f_2-$ space that passes through $(t_1^*, f_2(x^*))$ and has the slope $-\lambda_1/w_2$.
Suppose that $(\hat{x}, \hat{t}) \in \mathbb{R}^2 \times \mathbb{R}$ is an optimal solution of $(SOP_{\bar{x}}^2)$. Suppose that $\hat{d} = w_2 f_2(\hat{x}) - \lambda_1 \hat{t}_1$ shows the optimal value of $(SOP_{\bar{x}}^2)$. Note that due to Lemma 3.1, the added constraints are active at optimality, therefore $\hat{t}_1/w_1 = b/w_1 - f_1(\hat{x})$. If $\hat{t}_1 = 0$, we

have $w_1 f_1(\hat{x}) = b$ and $w_2 f_2(\hat{x}) = \hat{d}$. Suppose that $\hat{t}_1 \neq 0$. If we substitute $\hat{t}_1$ into the objective function of $(SOP^2_{\tilde{x}})$, we attain

$$\frac{-\lambda_1 w_1}{w_2} = \frac{\hat{v} - f_2(\hat{x})}{(b/w_1) - f_1(\hat{x})}.$$

The negative slope of the line passing through $(f_1(\hat{x}), f_2(\hat{x}))$ and $(b/w_1, \hat{v})$ equals The scalar $(\lambda_1 w_1)/w_2$. To visualize this observation see Figure 1. The additional constraint $f_1(x) \leqslant \frac{b}{w_1}$, reduces the feasible set of the main MOP. Thus, a line with slope $m = -(\lambda_1 w_1)/w_2$ is transformed in parallel toward the origin till it supports the restricted nondominated set. Hence, the point of support is the nondominated solution $f(\hat{x})$.

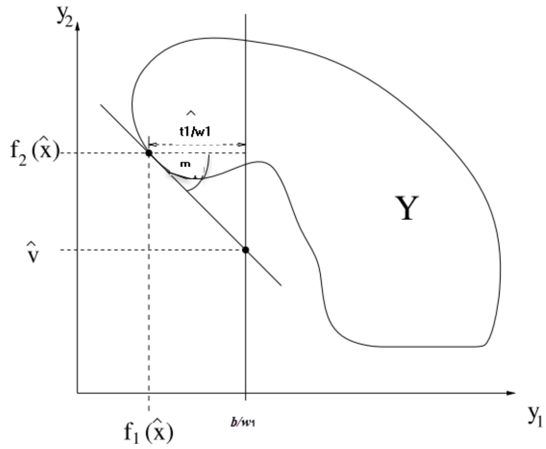


Figure 1. A bicriteria example

In the following subsection we show that the including slack variables to the constraints help us to provide results for proper efficiency. Hence, this modification can resolve the drawback of the weighted-constraint and the objective-constraint approaches. The results presented in the next subsection, demonstrate the main advantages of the suggested scalarized problem, theoretically.

3.1. Theoretical results for $(SOP^k_{\tilde{x}})$

In this subsection, we present some theoretical results for characterizing (weak, proper) efficient solutions of MOP (2.1) in terms of optimal solutions of the scalarized problem $(SOP^k_{\tilde{x}})$. These results establish some necessary and sufficient conditions for optimal solutions of the modified problem to be (weakly, properly) Pareto optimal solutions of MOP (2.1).

In [32], Rizvi provided a sufficient condition for weakly Pareto optimal solution by the hypothesis $w_i = 1/f_i(\bar{x})/\sum_{j=1}^{n} 1/f_j(\bar{x})$. The next theorem generalizes the result of [32] and provides a characterization for weakly Pareto optimal points, only under positive weights assumption. In the following theorem, we claim that optimal solutions of $(SOP^k_{\tilde{x}})$ for some $k \in \{1, 2, \dots, n\}$ are weakly Pareto optimal.

Theorem 3.2. *Let $(\tilde{x}, \tilde{t}) \in Sol^k_{\tilde{x}}$, for some $k \in \{1, 2, \dots, n\}$. Then, the next assertions hold.*

(1) *If $\lambda \geq 0$ and $w > 0$, then $\tilde{x} \in WP(MOP)$.*
(2) *If $\lambda > 0$ and $w \geq 0$, then $\tilde{x} \in WP(MOP)$.*

Proof. (1) Let $\lambda \geq 0$ and $w > 0$. Suppose that $\tilde{x} \notin WP(MOP)$. Then, there exists some $x \in X$ with $f_i(x) < f_i(\tilde{x})$, $\forall i \in \{1, 2, \dots, n\}$. Since $w > 0$, this implies

$w_i f_i(x) < w_i f_i(\tilde{x}),\ \forall i \in \{1,\dots,n\}$. Therefore, $w_i f_i(\tilde{x}) = w_i f_i(x) + d_i$ with $d_i > 0$. Then

$$w_k f_k(\bar{x}) \geqslant w_i f_i(\tilde{x}) + \tilde{t}_i = w_i f_i(x) + d_i + \tilde{t}_i = w_i f_i(x) + t_i, \quad i \neq k,$$

where $d_i + \tilde{t}_i = t_i$. Thus (x,t) is a feasible solution for $(SOP^k_{\bar{x}})$. Also, since $w_k f_k(x) < w_k f_k(\tilde{x})$ we have,

$$w_k f_k(x) - \sum_{i\neq k} \lambda_i t_i < w_k f_k(\tilde{x}) - \sum_{i\neq k} \lambda_i \tilde{t}_i.$$

This contradicts optimality of $(\tilde{x},\tilde{t})$ for $(SOP^k_{\bar{x}})$.

(2) Assume that $\lambda > 0$ and $w \geq 0$. If $\tilde{x} \notin WP(MOP)$, then there exists some $x \in X$ with $f_i(x) < f_i(\tilde{x}), \forall i \in \{1,\dots,n\}$. Thus, $w_i f_i(x) \leqslant w_i f_i(\tilde{x})$. Hence, there exists $d_i \geqslant 0$ such that

$$w_i f_i(\tilde{x}) = w_i f_i(x) + d_i, \quad \forall i \in \{1,2,\dots,n\}.$$

Thus

$$w_k f_k(\bar{x}) \geqslant w_i f_i(\tilde{x}) + \tilde{t}_i = w_i f_i(x) + d_i + \tilde{t}_i,\ i \neq k.$$

By assuming $t_i = d_i + \tilde{t}_i$ for each $i \neq k$, we obtain $w_k f_k(\bar{x}) \geqslant w_i f_i(x) + t_i$. Therefore (x,t) is a feasible solution of $(SOP^k_{\bar{x}})$.

Now we distinguish two cases $w_k = 0$ and $w_k \neq 0$. If $w_k = 0$, then there is some index $i \neq k$ such that $t_i > \tilde{t}_i$. If $w_k \neq 0$, then $w_k f_k(x) < w_k f_k(\tilde{x})$. In both cases it follows that

$$w_k f_k(x) - \sum_{i\neq k} \lambda_i t_i < w_k f_k(\tilde{x}) - \sum_{i\neq k} \lambda_i \tilde{t}_i,$$

contradicting the assumption.

□

The following theorem shows that, unlike some other approaches [7, 12, 32], to ensure a solution generated by $(SOP^k_{\bar{x}})$ is Pareto optimal, it is not necessary to solve k different problems. In [32], utilizing the objective-constraint problem (2.3) for all k, a sufficient condition was attained for Pareto optimal solutions of MOP (2.1). Now, we apply the scalarized problem (3.1) for some k and obtain a sufficient condition for Pareto optimal solutions. This result actually shows that incorporating slack variables to the scalarized problem (2.3) can cause to obtain stronger results.

Theorem 3.3. *Let $(\tilde{x},\tilde{t}) \in Sol^k_{\bar{x}}$, for some $k \in \{1,2,\dots,n\}$ with $\lambda > 0$ and $w > 0$. Then $\tilde{x} \in P(MOP)$.*

Proof. By contradiction, suppose that $\tilde{x} \notin P(MOP)$. Hence, there is some $x \in X$ with $f_i(x) \leqslant f_i(\tilde{x}), \forall i \in \{1,2,\dots,n\}$ and strict inequality for at least one index j. So, $w_i f_i(x) \leqslant w_i f_i(\tilde{x})$ for all $i \neq k$ and

$$w_k f_k(\bar{x}) \geqslant w_i f_i(\tilde{x}) + \tilde{t}_i = w_i f_i(x) + d_i + \tilde{t}_i = w_i f_i(x) + t_i, \quad i \neq k,$$

where $d_i + \tilde{t}_i = t_i$ and $d_i \geqslant 0$.

Let us now consider two cases $j = k$ and $j \neq k$. The first statement obviously causes a contradiction. The second statement confirms that $\tilde{t}_j < t_j$ for some index $j \neq k$, which implies

$$w_k f_k(x) - \sum_{i\neq k} \lambda_i t_i < w_k f_k(\tilde{x}) - \sum_{i\neq k} \lambda_i \tilde{t}_i.$$

These contradictions yield the result. □

Note that optimal solutions of $(SOP^k_{\bar{x}})$ with $\lambda \geq 0$ and $w > 0$ are not necessarily Pareto optimal for MOP (2.1), as the following example demonstrates.

Example 3.4. Assume that the feasible set of a bi-objective programming problem is as

$$X = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \leqslant 0.8,\ (x_1 - 0.5)(x_2 - 0.5) \leqslant 0\}$$

and the objective functions are $(f_1, f_2) = (x_1, x_2)$. Set $\bar{x} = (0.5, 0.4)$, $\lambda = (1, 0)$ and $w = (0.4, 0.5)$. Problem $(SOP^1_{\bar{x}})$ can be written as:

$$\begin{array}{ll} \min & 0.4x_1 \\ s.t. & 0.5x_2 + t_2 \leqslant 0.2, \\ & x \in X. \end{array}$$

Here, $(\tilde{x}, \tilde{t})$ with $\tilde{t}_2 = 0$ and $\tilde{x} = (0.5, 0.4)$ is an optimal point of $(SOP^1_{\bar{x}})$. However, $\tilde{x}$ is not a Pareto optimal solution.

Theorem 3.5. *If $(\tilde{x}, \tilde{t}) \in Sol^k_{\bar{x}}$, for all $k \in \{1, \ldots, n\}$ with $\lambda \geq 0$ and $w > 0$, then $\tilde{x} \in P(MOP)$.*

Proof. Assume that $\tilde{x} \notin P(MOP)$. Then, there is a feasible solution $x \in X$ with $f_i(x) \leqslant f_i(\tilde{x})$ for all $i \in \{1, 2, \ldots, n\}$ and $f_k(x) < f_k(\tilde{x})$ for some index $k \in \{1, 2, \ldots, n\}$. Therefore, by a procedure analogous to that of Theorem 3.3, we conclude that

$$w_k f_k(x) - \sum_{i \neq k} \lambda_i t_i < w_k f_k(\tilde{x}) - \sum_{i \neq k} \lambda_i \tilde{t}_i.$$

This concludes that $\tilde{x} \notin Sol^k_{\bar{x}}$, which is a contradiction. □

By a method similar to that of Theorem 3.5, we can also see that an optimal solution of $(SOP^k_{\bar{x}})$ for all $k \in \{1, 2, \ldots, n\}$ with $\lambda > 0$ and $w \geq 0$, implies a Pareto optimal solution of MOP (2.1).

Corollary 3.6. *If $(\tilde{x}, \tilde{t}) \in Sol^k_{\bar{x}}$, for all $k \in \{1, \ldots, n\}$ with $\lambda > 0$ and $w \geq 0$, then $\tilde{x} \in P(MOP)$.*

To certify that a solution generated by the objective-constraint problem (2.3) is Pareto optimal, it is necessary to solve k different optimization problems or we must attain a unique optimal solution of the scalarized problem for some k. Generally, uniqueness is not simple to check. Optimal points of the objective-constraint problem (2.3) are not Pareto optimal, in general. However, Theorems 3.3 and 3.5 and Corollary 3.6 clarify that optimal solutions of the scalarized problem (3.1) with weights greater than zero, are always Pareto optimal.

Since the objective functions are bounded below on the feasible set X, w.l.o.g., we can suppose that $f_i(x) > 0\ \forall i \in \{1, \ldots, n\}$ and all $x \in X$. By considering this fact, in the next result we obtain a sufficient condition for Pareto optimal solutions of MOP (2.1).

Theorem 3.7. *If $\tilde{x} \in P(MOP)$, then there exist $w > 0$, $\lambda \geqq 0$ and $\tilde{t}$ such that $(\tilde{x}, \tilde{t}) \in Sol^k_{\bar{x}}$.*

Proof. Let us consider $\tilde{t} = 0$ and $\lambda = 0$. Assume that there is a feasible point (x, t) that is an optimal solution of $(SOP^k_{\bar{x}})$ for some $k \in \{1, 2, \ldots, n\}$. We define

$$w_r = \begin{cases} \frac{1}{f_r(\tilde{x})}, & \forall\ r \neq k, \\ \\ \frac{1}{f_r(\bar{x})}, & \text{for } r = k. \end{cases}$$

Since (x, t) is a feasible solution and $w_k = 1/f_k(\bar{x})$, we conclude

$$w_i f_i(x) \leqslant w_i f_i(x) + t_i \leqslant w_k f_k(\bar{x}) = 1 = w_i f_i(\tilde{x}),\ \forall i \neq k.$$

Therefore,

$$f_i(x) \leqslant f_i(\tilde{x}),\ \ \forall i \neq k. \tag{3.2}$$

From optimality of (x, t) we derive

$$w_k f_k(x) - \sum_{i \neq k} \lambda_i t_i < w_k f_k(\tilde{x}) \Longrightarrow f_k(x) < f_k(\tilde{x}). \tag{3.3}$$

From (3.2) and (3.3), it follows that $\tilde{x} \notin P(MOP)$. Thus, the result follows. □

In the next result, in terms of optimal points of the scalarized problem (3.1), we present a necessary and sufficient condition for Pareto optimal solutions.

Theorem 3.8. *$\tilde{x} \in P(MOP)$ if and only if there are $w > 0$, $\lambda \geqq 0$ and $\tilde{t}$ such that $(\tilde{x}, \tilde{t}) \in Sol_{\bar{x}}^{k}$, for all $k \in \{1, 2, \dots, n\}$.*

Proof. By Theorems 3.5 and 3.7, the result can be derived. □

We apply Lemma 3.1 to get a condition for characterizing properly Pareto optimal points among the optimal solutions of $(SOP_{\bar{x}}^{k})$. For this aim, we will now provide a connection between properly Pareto optimal solutions of MOP with two feasible sets. Assume that

$$X_{\bar{x}}^{k} = \{x \in X | w_i f_i(x) \leqslant w_k f_k(\bar{x}),\ i \neq k\}$$

Lemma 3.9. *If $\tilde{x} \in PP(MOP)$ with the feasible set $X_{\bar{x}}^{k}$ and $w_i f_i(\tilde{x}) < w_k f_k(\bar{x})$ for each $i \neq k$ and $0 < w_i < \infty$ for all $i \in \{1, 2, \dots, n\}$, then $\tilde{x} \in PP(MOP)$ with the feasible set X.*

Proof. Assume that $\tilde{x}$ is not a properly Pareto optimal solution for MOP (2.1) with feasible set X. Hence, there is $x \in X$ and an index $i \in \{1, 2, \dots, n\}$ with $f_i(\tilde{x}) > f_i(x)$ and

$$f_i(\tilde{x}) - f_i(x) > M\big(f_j(x) - f_j(\tilde{x})\big)$$

for every $M > 0$ and for all j with $f_j(\tilde{x}) < f_j(x)$. Assume that $M_\beta > 0$ is an unbounded sequence of positive real scalars. We suppose that for every real sequence M_β, there exists $x_\beta \in X$ and an index $i \in \{1, \dots, n\}$ such that $f_i(\tilde{x}) > f_i(x_\beta)$ and

$$\frac{f_i(\tilde{x}) - f_i(x_\beta)}{f_j(x_\beta) - f_j(\tilde{x})} > M_\beta, \tag{3.4}$$

for all $j \neq i$ with $f_j(\tilde{x}) < f_j(x_\beta)$. With no loss of generality, one may suppose that there exists a fixed index i for every β such that the above relation is satisfied. Selecting a subsequent, we assume that $D = \{j : f_j(\tilde{x}) < f_j(x_\beta)\}$ is constant for every β.
We distinguish the following statements.
The first statement is $f_i(\tilde{x}) - f_i(x_\beta) \to \infty$ for $\beta \to \infty$. This yields a contradiction to boundedness of $f(X)$.
The second statement is $f_j(x_\beta) - f_j(\tilde{x}) \to 0$ for $\beta \to \infty$. We define

$$\bar{D} = \{j : f_j(x_\beta) \to f_j(\tilde{x}) \text{ for } \beta \to \infty \text{ and } f_j(\tilde{x}) < f_j(x_\beta)\}.$$

If $j \in \bar{D}$, then $w_j f_j(x_\beta) < w_k f_k(\bar{x})$ for all $j \in \bar{D} \setminus \{k\}$. Since we have

$$\lim_{\beta \to \infty} w_j f_j(x_\beta) = w_j f_j(\tilde{x}) < w_k f_k(\bar{x}), \ \ \forall j \in \bar{D} \setminus \{k\}.$$

Therefore, there is $\tilde{\beta} > 0$ such that

$$w_j f_j(x_\beta) < w_k f_k(\bar{x}), \ \ \forall j \in \bar{D} \setminus \{k\}, \ \ \forall \beta > \tilde{\beta}.$$

Now, suppose that there exists some index $j \notin \bar{D}$ and $w_j f_j(x_\beta) > w_k f_k(\bar{x})$ for infinitely many β. According to relation (3.4), we have $w_j f_j(x_\beta) > w_k f_k(\bar{x})$ for only finitely many β. Hence, there exists some $\bar{\beta}$ such that $w_j f_j(x_\beta) \leqslant w_k f_k(\bar{x})$ for all $j \in \{1, \dots, n\} \setminus \bar{D} \cup \{k\}$ and for every $\beta > \bar{\beta}$. Therefore, we have

$$w_j f_j(x_\beta) \leqslant w_k f_k(\bar{x}), \ \ \forall \beta > \max\{\tilde{\beta}, \bar{\beta}\}, \ \ \forall j \in \{1, 2, \dots, n\} \setminus \{k\}. \tag{3.5}$$

Thus relations (3.4) and (3.5) yield a contradiction to $\tilde{x}$ being a properly Pareto optimal solution of MOP (2.1) with the feasible set $X^k_{\bar{x}}$. Hence, $\tilde{x}$ is a properly Pareto optimal point of MOP (2.1) with the feasible set X. $\square$

Next we state two theorems which establish the connection between optimal points of $(SOP^k_{\bar{x}})$ and properly Pareto optimal solutions of MOP (2.1). As pointed in [21], the main motivation of defining properly efficient solutions is to remove those Pareto optimal solutions from investigation, that a small improvement in one objective function can cause an unbounded deterioration of at least one other criterion. A related economic interpretation, is that properly Pareto optimal solutions must prohibit unbounded trade-offs. The provided sufficient and necessary conditions for proper efficiency are established for general MOPs without any additional assumption, like differentiability or convexity. In [32], Rizvi provided a necessary condition for Karush-Kuhn-Tucker properly Pareto optimal solutions [28] of a multicriteria programming problem, using the optimal solutions of the objective-constraint approach [32]. However, that result is valid under some conditions like differentiability of the functions and constraints and regularity conditions. In addition, conditions for clarifying properly Pareto optimal points among the solutions of the scalarized problems are not mentioned in [7] and [6]. In the next result, we state a sufficient condition for proper efficiency.

Theorem 3.10. *Let $(\tilde{x}, \tilde{t}) \in Sol^k_{\bar{x}}$ with $\tilde{t}_i > 0$, $\lambda_i > 0$ for all $i \neq k$ and $0 < w_i < \infty$, $\forall i \in \{1, 2, \ldots, n\}$. Then, $\tilde{x} \in PP(MOP)$.*

Proof. Assume that $(\tilde{x}, \tilde{t}) \in Sol^k_{\bar{x}}$. According to Lemma 3.1, at optimality we have $\tilde{t}_i = w_k f_k(\bar{x}) - w_i f_i(\tilde{x})$ for each $i \neq k$. Therefore, the objective function can be rewritten as follows:

$$w_k f_k(\tilde{x}) - \sum_{i \neq k} \lambda_i \tilde{t}_i = w_k f_k(\tilde{x}) + \sum_{i \neq k} \lambda_i w_i f_i(\tilde{x}) - \sum_{i \neq k} \lambda_i w_k f_k(\bar{x}). \tag{3.6}$$

If we define

$$\tilde{w}_i = \begin{cases} w_i \lambda_i, & i \in \{1, 2, \ldots, n\} \backslash \{k\}, \\ w_i, & i = k, \end{cases}$$

then, from relation (3.6) we have:

$$w_k f_k(\tilde{x}) - \sum_{i \neq k} \lambda_i \tilde{t}_i = \sum_{i=1}^{n} \tilde{w}_i f_i(\tilde{x}) - w_k f_k(\bar{x}) \sum_{i \neq k} \lambda_i.$$

$(\sum_{i \neq k} \lambda_i) w_k f_k(\bar{x})$ is constant, therefore we conclude that $\tilde{x}$ is an optimal point of the weighted sum problem with positive weights and feasible set $X^k_{\bar{x}}$. From Theorem 1 in [21], it follows that $\tilde{x}$ is a properly Pareto optimal point of MOP (2.1) with feasible set $X^k_{\bar{x}}$. From Lemma 3.9 we see that if $\tilde{t}_i > 0$ $\forall i \neq k$, then $\tilde{x}$ is a properly Pareto optimal solution of MOP (2.1) with feasible set X. $\square$

Our next theorem shows a necessary condition for properly Pareto optimal points of MOP (2.1). We may w.l.o.g. assume that $f_r(x) > 0$ for all $r \in \{1, \ldots, n\}$ and all $x \in X$.

Theorem 3.11. *If $\tilde{x} \in PP(MOP)$, then there are $\tilde{t}$, $\lambda > 0$ and $w > 0$ such that $(\tilde{x}, \tilde{t}) \in Sol^k_{\tilde{x}}$, for all $k \in \{1, 2, \ldots, n\}$.*

Proof. Suppose that $\tilde{x} \in PP(MOP)$. We distinguish the following statements.
At first statement, assume that $\nexists\, x \in X$ and $i \in \{1, 2, \ldots, n\}$ such that $f_i(x) < f_i(\tilde{x})$, i.e. $f_i(\tilde{x}) \leqslant f_i(x)$ $\forall x \in X$ and $i \in \{1, 2, \ldots, n\}$. In this statement, we assume that $\tilde{t} = 1$. We define

$$w_i = \begin{cases} \frac{1}{\max_{x \in X} f_i(x)}, & \forall i \neq k, \\ \\ \frac{1}{f_i(\tilde{x})}, & \text{for } i = k. \end{cases} \tag{3.7}$$

We presume that (x,t) is an arbitrary feasible vector of $(SOP_{\tilde{x}}^k)$. By relation (3.7) it follows that $w_i f_i(x) + t_i \leqslant 1 = w_k f_k(\bar{x})$ for each $i \neq k$. Therefore, $w_i f_i(x) \leqslant 1$ and $t_i \leqslant 1$. Hence, we get $w_k f_k(\tilde{x}) \leqslant w_k f_k(x)$ and $w_k f_k(\tilde{x}) - \sum_{i\neq k} \lambda_i \tilde{t}_i \leqslant w_k f_k(x) - \sum_{i\neq k} \lambda_i t_i$ for each $\lambda > 0$. So (x,t) does not imply a better objective function value than $(\tilde{x},\tilde{t})$ for $(SOP_{\tilde{x}}^k)$.
At the second statement, we assume that there are $x \in X$ and $i \in \{1,2,\ldots,n\}$ with $f_i(x) < f_i(\tilde{x})$. Since $\tilde{x}$ is properly Pareto optimal solution, for all $x \in X$ and $i \in \{1,2,\ldots,n\}$ with $f_i(x) < f_i(\tilde{x})$, there exists a scalar $M > 0$ and an index $j \in \{1,2,\ldots,n\}$ such that $f_j(\tilde{x}) < f_j(x)$ and

$$\frac{f_i(\tilde{x}) - f_i(x)}{f_j(x) - f_j(\tilde{x})} \leqslant M. \tag{3.8}$$

The following appropriate weights are defined for $(SOP_{\tilde{x}}^k)$ that are positive and finite:

$$w_r = \begin{cases} \frac{1}{f_r(\tilde{x})}, & r \neq k, \\ \\ \frac{1}{f_r(\bar{x})}, & r = k. \end{cases} \tag{3.9}$$

Suppose that $(\tilde{x},\tilde{t}) \notin Sol_{\tilde{x}}^k$, for some $k \in \{1,2,\ldots,n\}$. Then, there exists some feasible point (x,t) of $(SOP_{\tilde{x}}^k)$ with

$$w_k f_k(x) - \sum_{i\neq k} \lambda_i t_i < w_k f_k(\tilde{x}) - \sum_{i\neq k} \lambda_i \tilde{t}_i. \tag{3.10}$$

Since (x,t) is feasible, from relation (3.9) we attain $w_i f_i(x) + t_i \leqslant 1 = w_k f_k(\bar{x})$ for each $i \neq k$. Thus $w_i f_i(x) \leqslant 1$ and $t_i \leqslant 1$ for each $i \neq k$. We claim that only for $j = k$, the inequality $f_j(\tilde{x}) < f_j(x)$ is strict. By contradiction, assume that $j \neq k$, then

$$f_j(\tilde{x}) < f_j(x) \Rightarrow 1 < \frac{f_j(x)}{f_j(\tilde{x})} \Rightarrow 1 < w_j f_j(x).$$

This contradicts the feasibility of (x,t) for $(SOP_{\tilde{x}}^k)$.
Hence, we have

$$f_k(x) - f_k(\tilde{x}) \geqslant \frac{1}{M}(f_i(\tilde{x}) - f_i(x)). \tag{3.11}$$

Set $\tilde{t} = 0$, $E = \{r : f_r(\tilde{x}) = f_r(x)\}$ and

$$\lambda_i = \frac{w_k}{(n-1-|E|)Mw_i}, \quad i \in \{1,2,\ldots,n\} \setminus E \cup \{k\}. \tag{3.12}$$

From optimality of (x,t) for $(SOP_{\tilde{x}}^k)$, by Lemma 3.1, we have

$$w_i f_i(x) + t_i = w_k f_k(\bar{x}) = w_i f_i(\tilde{x}) \Rightarrow t_i = w_i f_i(\tilde{x}) - w_i f_i(x), \;\; i \neq k. \tag{3.13}$$

According to relations (3.11) and (3.12), we conclude

$$\lambda_i = \frac{w_k}{(n-1-|E|)Mw_i} \leqslant \frac{w_k}{(n-1-|E|)w_i} \times \frac{f_k(x) - f_k(\tilde{x})}{f_i(\tilde{x}) - f_i(x)}, \;\; i \in \{1,2,\ldots,n\} \setminus E \cup \{k\}.$$

This implies that

$$w_i \lambda_i (f_i(\tilde{x}) - f_i(x)) \leqslant \frac{w_k(f_k(x) - f_k(\tilde{x}))}{n-1-|E|}, \;\; i \in \{1,2,\ldots,n\} \setminus E \cup \{k\}$$

Now by summation over $i \in \{1,2,\ldots,n\} \setminus E \cup \{k\}$, we obtain

$$\sum_{i\in\{1,2,\ldots,n\}\setminus E\cup\{k\}} \lambda_i (w_i f_i(\tilde{x}) - w_i f_i(x)) \leqslant \sum_{i\in\{1,2,\ldots,n\}\setminus E\cup\{k\}} \frac{w_k(f_k(x) - f_k(\tilde{x}))}{n-1-|E|} = w_k(f_k(x) - f_k(\tilde{x})).$$

Based on relation (3.13), we immediately get

$$\sum_{i\in\{1,2,\ldots,n\}\setminus E\cup\{k\}} \lambda_i t_i \leqslant w_k f_k(x) - w_k f_k(\tilde{x}).$$

Since $t_i = w_i f_i(\tilde{x}) - w_i f_i(x) = 0$ for all $i \in E$, we get

$$\sum_{i \neq k} \lambda_i t_i \leqslant w_k f_k(x) - w_k f_k(\tilde{x}), \tag{3.14}$$

which is a contradiction with relation (3.10). Hence, we deduce that $(\tilde{x}, \tilde{t}) \in Sol^k_{\tilde{x}}$. □

4. Numerical examples

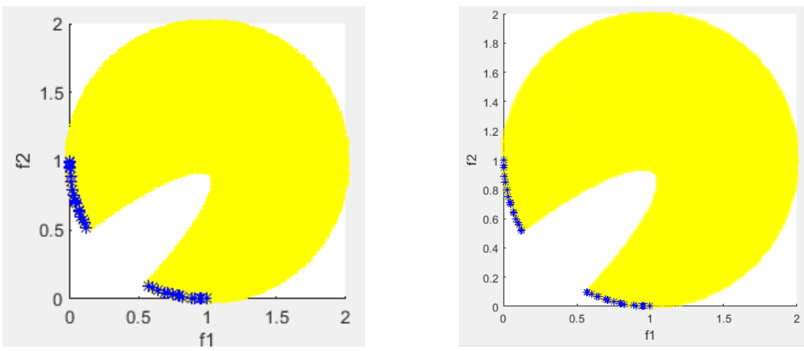
In this section, in order to justify the suggested methodology, numerical examples are solved. We test and compare our method with the proposed algorithms in [7, 32].

We pay special attention to problems with complicated structures including a disconnected or nonconvex feasible set or Pareto front. The numerical evaluations show that the proposed approach are able to address successfully the complexities arising from the cases when the domain or the Pareto front is nonconvex or even disconnected. The gained approach is performed to two test problems and a real application in engineering design. All examples are executed within MATLAB (R2015a). We implemented the problems on a laptop with a core i5 processor at 4 GB RAM and 2.5 GHz running Windows 7 Home Basic Operating system.

Example 4.1. Consider the following nonconvex bi-objective programming problem that is a modified form of a bi-objective problem in [32].

$$\begin{array}{ll} \min & (x_1, x_2) \\ s.t. & (1 - x_1)^2 + (1 - x_2)^2 \leqslant 1, \\ & 1 - x_2^2 - 4(x_2 - x_1)^2 \leqslant 0.1, \\ & x_1, x_2 \geqslant 0. \end{array}$$

This problem has a disconnected Pareto front and its feasible region is nonconvex. We run Algorithm 1 in [32], which implements the objective-constraint problem (2.3), with $N = 22$. We also apply this algorithm for our method to obtain Pareto points. Steps $1-6$ of Algorithm 1 in [32] are used, only in Step 4 our proposed scalarized problem (3.1) is solved. Furthermore, for the algorithm that implements our scalarized problem (3.1), we select the nonnegative weights $\lambda_1 = w_1/50$ and $\lambda_2 = w_2/50$. Figure 2 indicates the generated Pareto solutions with $N = 22$ and $u = (-3, -3)$.



(a) The approximation produced by our scalarized problem (3.1).

(b) The approximation produced by the scalarized problem (2.3).

Figure 2. Pareto front approximation of Example 4.1 with $N = 22$.

Our scalarized problem (3.1) is solved in order to approximate the Pareto front. The attained Pareto points are evenly distributed in the Pareto front, as shown in Figure 2a. Moreover, the proposed procedure generates all the end points of the Efficient curve.

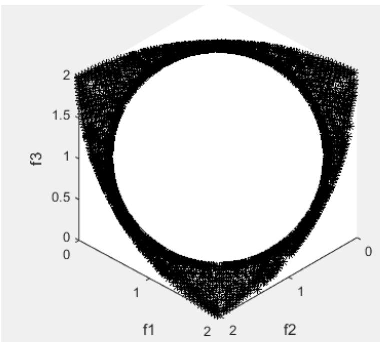
In Figure 2b, with $N = 22$, the Pareto front obtained by the the objective-constraint problem (2.3) is shown.

We illustrate the advantages of our method by means of an example with three-objective functions.

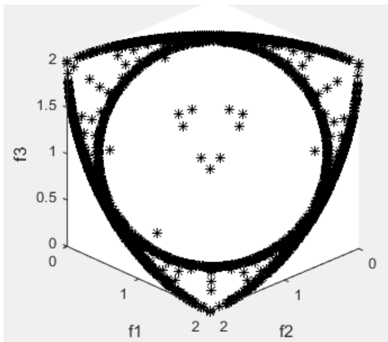
Example 4.2. We distinguish a three-objective optimization problem by an efficient front with a complicated boundary. Here, the feasible set is the intersection of the complement of an ellipsoid with a sphere. The Pareto front obtained by our method, is multiply connected with a hole and bounded. This test problem, from Burachik et al. [7] and Akbari et al. [2], is expressed as follows:

$$\begin{array}{ll} \min & (x_1, x_2, x_3) \\ s.t. & (2-x_1)^2 + (2-x_2)^2 + (2-x_3)^2 \leqslant 4, \\ & (1.05)^2(x_1+x_2+x_3)^2 - 4(x_1^2+x_2^2+x_3^2) + 7.18 \leqslant 0, \\ & 0 \leqslant x_1 \leqslant 4, \\ & 0 \leqslant x_2 \leqslant 4. \end{array}$$

We implement Algorithm 1 in [7], which uses the objective-constraint problem. Also, for the presented technique, steps $1-5$ of Algorithm 1 in [7] are applied and our scalarized problem (3.1) is utilized in Step 4. We distinguish this test problem for two cases of scalarized problem with $N = 50$ and $u = (0, 0, 0)$. For the case that our scalarized problem is solved, the nonnegative weights $\lambda_i, i = 1, 2, 3$ are chosen as $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The generated Pareto points are depicted in Figure 3.



(a) The approximation produced by our scalarized problem (3.1).



(b) The approximation produced by the scalarized problem (2.3).

Figure 3. Pareto front approximation of Example 4.2 with $N = 50$.

This figure shows the Pareto points generated by our proposed approach applied in the given algorithm, is extremely successful than Algorithm 1 in [7]. As it can be seen, our approach does not generate non-Pareto points. The proposed technique produces Pareto points, that are spaced relatively evenly in the approximation of the Pareto front, and among these produced points all the end points of the Pareto front, outer and inner ones are generated, as shown in the figure. The distribution of points obtained by our method and the proposed approach in [7], is depicted in Figure 3. By comparing the Pareto front generated by our technique and the technique in [7], it is concluded that the approximation obtained by the our approach covers the whole Pareto front and this shows the strengths and advantages of the introduced method.

5. Application to an engineering design problem

We display the usage of the proposed procedure to an engineering design problem. As it was stated in the introduction, many of the engineering applications have a multicriteria structure. Nevertheless, these multiple objectives are often treated as a single-objective programming problem, in practice. We consider an engineering problem that includes designing a four-bar plane truss as depicted in Figure 4. This test problem is shown as a bi-criteria optimization problem. The first criterion (f_1) is the volume V of the truss and the second one (f_2) is the displacement Δ of the joint. Three forces with the magnitudes F and $2F$, cause stress on the truss as seen in Figure 4. The Youngs modulus of elasticity (denoted by E), the length of each bar (denoted by L) and the stress component (denoted by σ) are selected as constants in this test problem. The value of the elasticity is $E = 2 \times 10^5$ kN/cm^2, the acting force is equal to $F = 10$ kN, the stress component is equal to $\sigma = 10$ kN/cm^2 and we set the length as $L = 200$ cm. The cross-sectional areas x_1, x_2, x_3 and x_4 of the four bars are subject to several physical conditions that define by

$$X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : F \leqslant \sigma x_1, \sigma x_4 \leqslant 3F, \sqrt{2}F \leqslant \sigma x_2, \sigma x_3 \leqslant 3F\} \tag{5.1}$$

The objective functions that are going to minimize simultaneously, are
$f_1(x) = 2L(x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}x_3 + \frac{1}{2}x_4)$ and $f_2(x) = \frac{FL}{E}(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{1}{x_4})$.

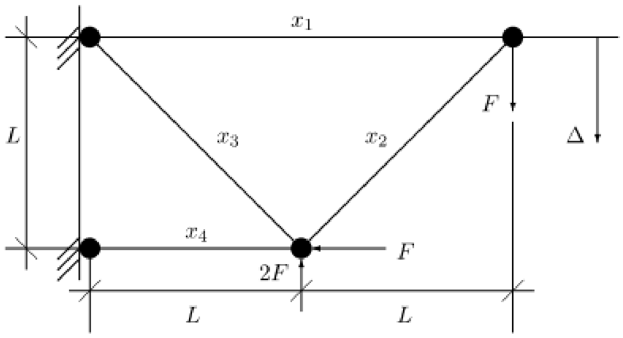
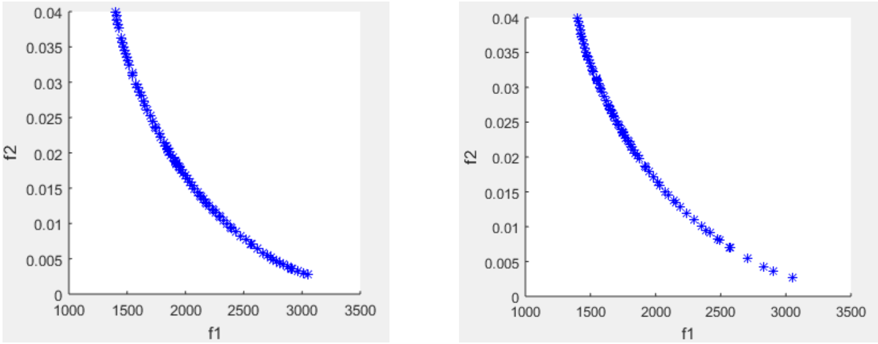


Figure 4. A four-bar plane truss.

We must solve the next multiobjective programming problem [19].

$$\begin{array}{ll} \min & (f_1(x), f_2(x)) \\ \\ s.t. & F \leqslant \sigma x_1 \leqslant 3F, \\ & \sqrt{2}F \leqslant \sigma x_2 \leqslant 3F, \\ & \sqrt{2}F \leqslant \sigma x_3 \leqslant 3F, \\ & F \leqslant \sigma x_4 \leqslant 3F. \end{array}$$

We apply Algorithm 1 in [32], in which the objective-constraint problem (2.3), is solved. We solve the scalarized problem for getting approximation points of the Pareto front. As well, for the proposed technique, the steps $1-6$ of Algorithm 1 in [32] are applied and in Step 4, the scalarized problem (3.1) is utilized. In both algorithms we take $u = (0, 0)$ and $N = 50$. For the algorithm in which our scalarized problem (3.1) is addressed, the nonnegative weights $\lambda_1 = \lambda_2 = 1$. The performance of the Pareto curve is illustrated in Figure 5. As it can be seen, the algorithm in which the scalarized problem (2.3) is utilized, can not construct the lower parts of the Pareto front. Furthermore, the Pareto points are not distributed uniformly by this algorithm. Hence, the Pareto front is approximated poorly by this algorithm. However, if we use our scalarized problem (3.1) in this algorithm, the produced Pareto points are evenly distributed in the Pareto front. This figure illustrates that the interval of the possible truss volumes range is between about 1500 cm^3 and 3500 cm^3 with the joint displacements less than 0.045 cm.



(a) The approximation produced by our scalarized problem (3.1).

(b) The approximation produced by the scalarized problem (2.3).

Figure 5. Pareto front approximation of the engineering design problem with $N = 50$.

6. Conclusions

In this article, we proposed a modification of the objective-constraint scalarization approach for solving multiobjective programming problems. We proved necessary and sufficient conditions for various types of efficiency, in particular for proper efficiency. We showed that there is no gap between necessary and sufficient conditions for (proper) Pareto efficiency. The proposed approach was applied to solve problems with convex, nonconvex, connected and disconnected Pareto fronts or feasible sets. The performance of this approach was illustrated by test problems. The results showed that the proposed method can attain comparable performance with the other algorithms. Future research will focus on performing the suggested technique in an algorithmic procedure to solve multiobjective optimization problems.

Acknowledgment. The authors would like to express their gratitude to the handling editor and anonymous referees for their helpful comments and suggestions that improved the quality of the paper.

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