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Research Article

Pseudo-totally umbilical lightlike submanifolds

Samuel Ssekajja

School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa

Abstract

We introduce the geometry of pseudo-totally umbilical lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$. In line with the above, we give a complete classification of pseudo-totally umbilical 1-lightlike submanifolds, such as the lightlike hypersurfaces and half-lightlike submanifolds. Furthermore, pseudo-totally umbilical screen distributions are also investigated, with a complete classification for any lightlike hypersurfaces whose screen distributions are pseudo-totally umbilical. Closely linked to the above we also show, under some geometric conditions, that some pseudo-totally umbilical leaves M^*, of the screen distribution over M, as non-degenerate submanifolds of $\bar{M}$, are either contained in semi-Euclidean spheres or the hyperbolic spaces. Moreover, tangible examples are constructed in this case. Finally, we introduce the notion of mean lightlike sectional curvatures and relate them to the well-known tensors used in the characterisation of lightlike hypersurfaces in space-times.

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1. Introduction

Let (M, g) be a non-degenerate submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, where g and $\bar{g}$ are semi-Riemannian metric tensors on M and $\bar{M}$. Then, M is called pseudo-totally umbilical [5–7, 14] if there exist a smooth function f such that

$$\bar{g}(h(X, Y), \zeta) = fg(X, Y),$$

for any X and Y tangent to M, where h denotes the second fundamental form of M in $\bar{M}$, and ζ is the mean curvature vector field of M, given by $\zeta = \frac{1}{\dim M}\text{trace}_g h$. It is obvious that any minimal, i.e. $\zeta \equiv 0$, semi-Riemannian submanifold is trivially pseudo-totally umbilical with $f = 0$. Pseudo-totally umbilical submanifolds are an important class of non-degenerate submanifolds, with many interesting results been discovered about them. For example, see Theorems 1, 2 and 3 in [5], and also Theorems 1 and 2 of [14], and many more results in the literature.

In this paper, we extend the notion of pseudo-totally umbilical submanifolds to lightlike submanifolds of a semi-Riemannian manifold. Lightlike submanifolds differs significantly

Email addresses: samuel.ssekajja@wits.ac.za, ssekajja.samuel.buwaga@aims-senegal.org
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from non-degenerate submanifolds in that, for a lightlike submanifold, the normal bundle to a lightlike submanifold is a subbundle of the tangent bundle of the submanifold. This posses a challenge to the study of lightlike submanifolds since the classical Gauss-Weingarten equations do not apply to lightlike submanifolds. K. L. Duggal and A. Bejancu [8] (also see [11]) have overcame this challenge by introducing a non-degenerate distribution over the submanifold, called the screen distribution. Although screen distributions are, in general, not unique, they are canonically isomorphic to the factor bundle introduced by D. N. Kupeli in [15]. With such a screen distribution, many interesting results have been discovered on lightlike submanifolds. for an up to-date information on these results, we refer to the book [11], as well as some of these papers: [1–3, 9, 10, 12, 13, 16–20]. In reality lightlike submanifolds in particular lightlike hypersurfaces are an important set of geometric objects. For instance, in General Relativity, lightlike hypersurfaces represent different models of blackhole horizons [8].

The rest of the paper is arranged as follows: In Section 2, we quote some basic notions on lightlike geometry required in the rest of the paper. In Section 3, we introduce pseudo-totally umbilical lightlike submanifolds of a semi-Riemannian manifold. In Section 4, we study lightlike submanifolds whose screen distributions are pseudo-totally umbilical submanifolds. In Section 5, we study pseudo-totally umbilical leaves, and in Section 6 we define the notion of mean lightlike sectional curvatures of a lightlike submanifold of a semi-Riemannian manifold.

2. Preliminaries

Let (M, g) be an m-dimensional lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $m+n$. Then, the radical distribution $\operatorname{Rad} TM = TM \cap TM^{\perp}$ is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r, where $1 \leq r \leq \min\{m, n\}$. Moreover, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $\operatorname{Rad} TM$ in TM and $TM^{\perp}$ respectively, called the *screen* and *co-screen* distributions on M, such that $TM = \operatorname{Rad} TM \perp_{orth} S(TM)$ and $TM^{\perp} = \operatorname{Rad} TM \perp_{orth} S(TM^{\perp})$, where $\perp_{orth}$ denotes the orthogonal direct sum. More often, such a lightlike submanifold is denoted by $M = (M, g, S(TM), S(TM^{\perp}))$. Let $\operatorname{tr}(TM)$ and $l\operatorname{tr}(TM)$ denote complementary (but not orthogonal) vector bundlesto TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $l\operatorname{tr}(TM)$ consisting of smooth sections of $S(TM)^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$, $\bar{g}(N_i, N_j) = 0$, $1 \leq i, j \leq r$, where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\operatorname{Rad} TM$. Then, we have the decomposition

$$\begin{aligned} T\bar{M}_{|M} &= TM \oplus \operatorname{tr}(TM) \\ &= \{\operatorname{Rad} TM \oplus l\operatorname{tr}(TM)\} \perp_{orth} S(TM) \perp_{orth} S(TM^{\perp}), \end{aligned}$$

where $\oplus$ denotes the direct (non-orthogonal) sum.

A lightlike submanifold (M, g) is called:

(1) r-lightlike if $1 \leq r < \min\{m, n\}$;
(2) co-isotropic if $1 \leq r = n < m$, $\quad S(TM^{\perp}) = \{0\}$;
(3) isotropic if $1 \leq r = m < n$, $\quad S(TM) = \{0\}$;
(4) totally lightlike if $r = n = m$, $\quad S(TM) = \{0\} = S(TM^{\perp})$.

Throughout this paper, we denote by $F(M)$ the algebra of smooth functions on M and $\Gamma(\Xi)$ the $F(M)$ module of smooth sections of a vector bundle Ξ (same notation for any other vector bundle) over M. For all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\operatorname{tr}(TM))$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$ respectively. Further, ∇ and ∇^t are linear connections on M and $\mathrm{tr}(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Moreover, from (2.1) and (2.2), we have (cf. [11, p. 196–198]):

$$\bar{\nabla}_X Y = \nabla_X Y + h^\ell(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\ell_X N + D^s(X,N), \tag{2.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^\ell(X,W), \tag{2.5}$$

for any X, Y tangent to M, N tangent to $l\mathrm{tr}(TM)$ and W tangent to $S(TM^\perp)$. Furthermore, A_N and A_W are called the shape operators of M, while h^ℓ and h^s are called the lightlike second fundamental form and the screen second fundamental form, respectively. Also, ∇^ℓ and ∇^s are, respectively, linear connections on $l\mathrm{tr}(TM)$ and $S(TM^\perp)$, called the lightlike connection and the screen Otsuki connections on $l\mathrm{tr}(TM)$ and $S(TM^\perp)$, respectively.

Denote the projection of TM onto $S(TM)$ by P. Then, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, we have

$$\begin{aligned} \nabla_X PY &= \nabla^*_X PY + h^*(X,PY), \\ \nabla_X \xi &= -A^*_\xi X + \nabla^{*t}_X \xi, \end{aligned} \tag{2.6}$$

where ∇^* and A^*_ξ are, respectively, the linear connection and shape operator of $S(TM)$. Furthermore, h^* and ∇^{*t} are the second fundamental form and a linear connection on $\mathrm{Rad}\, TM$, respectively. Furthermore, by using (2.3)-(2.6), we obtain, for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}\, TM)$, $N, N' \in \Gamma(l\mathrm{tr}(TM))$ and $W \in \Gamma(S(TM^\perp))$:

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^\ell(X,W)) = g(A_W X, Y), \tag{2.7}$$

$$\bar{g}(h^\ell(X,Y),\xi) + \bar{g}(h^\ell(X,\xi),Y) = -g(\nabla_X \xi, Y), \tag{2.8}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X), \quad \bar{g}(h^\ell(X,PY),\xi) = g(A^*_\xi X, PY), \tag{2.9}$$

$$\bar{g}(h^\ell(X,\xi),\xi) = 0, \quad \bar{g}(h^*(X,PY),N) = g(A_N X, PY). \tag{2.10}$$

Since $\bar{\nabla}$ is a metric connection, by a direct calculations, using (2.3) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^\ell(X,Y),Z) + \bar{g}(h^\ell(X,Z),Y), \tag{2.11}$$

for all $X, Y, Z \in \Gamma(TM)$. It is important to note that ∇^* is a metric connection on $S(TM)$. Let R be the curvature tensor of M. Then, we have (cf. [8, p. 171]):

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^\ell(X,Z)}Y - A_{h^\ell(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\tilde{\nabla}_X h^\ell)(Y,Z) - (\tilde{\nabla}_Y h^\ell)(X,Z) + D^\ell(X, h^s(Y,Z)) \\ & - D^\ell(Y, h^s(X,Z)) + (\tilde{\nabla}_X h^s)(Y,Z) - (\tilde{\nabla}_Y h^s)(X,Z) \\ & + D^s(X, h^\ell(Y,Z)) - D^s(Y, h^\ell(X,Z)), \end{aligned} \tag{2.12}$$

$$\bar{g}(R(X,Y)PZ,N) = \bar{g}((\nabla_X h^*)(Y,PZ) - (\nabla_Y h^*)(X,PZ),N), \tag{2.13}$$

where $\tilde{\nabla} h^\ell$, $\tilde{\nabla} h^s$ and ∇h^* are given by

$$(\tilde{\nabla}_X h^\ell)(Y,Z) = \nabla^\ell_X h^\ell(Y,Z) - h^\ell(\nabla_X Y, Z) - h^\ell(Y, \nabla_X Z), \tag{2.14}$$

$$(\tilde{\nabla}_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z), \tag{2.15}$$

$$(\nabla_X h^*)(Y,PZ) = \nabla^{*t}_X h^*(Y,PZ) - h^*(\nabla_X Y, PZ) - h^*(Y, \nabla^*_X PZ), \tag{2.16}$$

for all $X, Y, Z \in \Gamma(TM)$, $N \in \Gamma(l\mathrm{tr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Taking the inner product of (2.12) with respect to W and ξ, in turns, we have

$$\bar{g}(\bar{R}(X,Y)Z,W) = \bar{g}((\tilde{\nabla}_X h^s)(Y,Z) - (\tilde{\nabla}_Y h^s)(X,Z),W) + \bar{g}(D^s(X,h^\ell(Y,Z)) - D^s(Y,h^\ell(X,Z)),W), \quad (2.17)$$

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = \bar{g}((\tilde{\nabla}_X h^\ell)(Y,Z) - (\tilde{\nabla}_Y h^\ell)(X,Z),\xi) + \bar{g}(D^\ell(X,h^s(Y,Z)) - D^\ell(Y,h^s(X,Z)),\xi). \quad (2.18)$$

A semi-Riemannian manifold of constant curvature c, everywhere, is called a space form and denoted by $\bar{M}(c)$. Moreover, the curvature tensor $\bar{R}$ satisfies the relation

$$\bar{R}(\bar{X},\bar{Y})\bar{Z} = c\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\}, \quad (2.19)$$

for any $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ tangent to $\bar{M}$. Let $\bar{M}(c)$ be a semi-Riemannian space form, then (2.12), (2.13), (2.17), (2.18) and (2.19) leads to:

$$\begin{aligned}
&\bar{g}((\nabla_X h^*)(Y,PZ),N) - c\bar{g}(Y,PZ)\bar{g}(X,N) \\
&\quad + \bar{g}(h^\ell(Y,PZ),A_N X) - \bar{g}(h^s(Y,PZ),D^s(X,N)) \\
&\quad = \bar{g}((\nabla_Y h^*)(X,PZ),N) - c\bar{g}(X,PZ)\bar{g}(Y,N) \\
&\quad + \bar{g}(h^\ell(X,PZ),A_N Y) - \bar{g}(h^s(X,PZ),D^s(Y,N)),
\end{aligned} \quad (2.20)$$

$$\begin{aligned}
&\bar{g}((\tilde{\nabla}_X h^s)(Y,Z),W) + \bar{g}(D^s(X,h^\ell(Y,Z)),W) \\
&\quad = \bar{g}((\tilde{\nabla}_Y h^s)(X,Z),W) + \bar{g}(D^s(Y,h^\ell(X,Z)),W),
\end{aligned} \quad (2.21)$$

$$\begin{aligned}
&\bar{g}((\tilde{\nabla}_X h^\ell)(Y,Z),\xi) + \bar{g}(D^\ell(X,h^s(Y,Z)),\xi) \\
&\quad = \bar{g}((\tilde{\nabla}_Y h^\ell)(X,Z),\xi) + \bar{g}(D^\ell(Y,h^s(X,Z)),\xi).
\end{aligned} \quad (2.22)$$

for any X, Y and Z tangent to M.

Definition 2.1. Let (M,g) be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then:

(1) M is said to be totally umbilical [11, Definition 5.3.1], in $\bar{M}$, if there exist a smooth transversal vector field $\bar{H} \in \Gamma(\mathrm{tr}(TM))$, such that

$$h(X,Y) = g(X,Y)\bar{H}, \quad (2.23)$$

for any X and Y tangent to M. In case $\bar{H} = 0$ (resp. $\bar{H} \neq 0$) then M is said to be totally geodesic (resp. proper totally umbilical). Moreover, from (2.23), it is easy to see that M is totally umbilical, if and only if on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $H^\ell \in \Gamma(l\mathrm{tr}(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^\ell(X,Y) = g(X,Y)H^\ell \quad \text{and} \quad h^s(X,Y) = g(X,Y)H^s. \quad (2.24)$$

It is well-known that this definition does not depend on the screen distribution and the screen transversal vector bundle of M.

(2) $S(TM)$ is totally umbilical [9, Definition 2], in M, if there is a smooth vector field $K' \in \Gamma(\mathrm{Rad}\, TM)$ on M, such that

$$h^*(X,PY) = g(X,PY)K', \quad (2.25)$$

for any X and Y tangent to M. In case $K' = 0$ (resp. $K' \neq 0$) we say that $S(TM)$ is totally geodesic (resp. proper totally umbilical).

For more information on the general theory of lightlike submanifolds, we refer the reader to the books [8] and [11].

3. Pseudo-totally umbilical submanifolds

In this section, we introduce the notion of a pseudo-totally umbilical r-lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$. First, let us consider two vector fields L and S of $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$, respectively, given by

$$L = \frac{1}{m-r}\mathrm{trace}_{|S(TM)}\, h^{\ell} = \frac{1}{m-r}\sum_{i=1}^{m-r} \varepsilon_i h^{\ell}(E_i, E_i), \tag{3.1}$$

$$S = \frac{1}{m-r}\mathrm{trace}_{|S(TM)}\, h^{s} = \frac{1}{m-r}\sum_{i=1}^{m-r} \varepsilon_i h^{s}(E_i, E_i), \tag{3.2}$$

where $\{E_i, \ldots, E_{m-r}\}$ is an orthonormal basis of $S(TM)$, and $\varepsilon_i = g(E_i, E_i)$. Note that, for a totally umbilical lightlike submanifold M, the vector fields H^{ℓ} and H^s in (2.24) coincides, respectively, with the vector fields L and S above.

The transversal vector field $\bar{H} = L + S$ is called the mean curvature vector field of M in $\bar{M}$. In [11, p. 221], K. L. Duggal and B. Sahin have defined a minimal lightlike submanifold isometrically immersed in a semi-Riemannian manifold $\bar{M}$ as one satisfying the conditions: (a) $h^s = 0$ on $\mathrm{Rad}\, TM$ and (b) $\mathrm{trace}_{|S(TM)} h = 0$. From (3.1) and (3.2), we note that the second condition in this definition, i.e. (b), is equivalent to $\bar{H} = 0$. We say that M has parallel mean curvature vector field, $\bar{H}$, if $\nabla^t \bar{H} = 0$. Note that this is equivalent to $\nabla^{\ell} L = 0$ and $\nabla^s S = 0$.

Next, assume that $S(TM)$ is an integrable distribution over M. In this case, it is well-known (see [11, Theorem 5.1.5, p. 200]), that h^* is symmetric on $S(TM)$. Now, consider the vector field $K \in \Gamma(\mathrm{Rad}\, TM)$, given by

$$K = \frac{1}{m-r}\mathrm{trace}_{|S(TM)}\, h^{*} = \frac{1}{m-r}\sum_{i=1}^{m-r} \varepsilon_i h^{*}(E_i, E_i), \tag{3.3}$$

where $\varepsilon_i = g(E_i, E_i)$. Note that, for a totally umbilical screen distribution $S(TM)$, the vector field K' in (2.25) coincides with the vector field K above. The vector field K, in (3.3), is called the mean curvature vector field of $S(TM)$ in M. We say that $S(TM)$ is minimal, in M, if $K = 0$. For example, any lightlike submanifold with a parallel screen distribution, with respect to ∇ (cf. [11, Theorem 5.1.6, p. 200]), is trivially minimal in M since $h^* = 0$. We, also, say that K is parallel if $\nabla^{*t} K = 0$.

Next, let us define two smooth functions ρ and σ, on M, by:

$$\rho = \bar{g}(\bar{H}, K) = \bar{g}(K, L), \quad \sigma = \bar{g}(\bar{H}, S) = \bar{g}(S, S).$$

Then, we shall call the sum $\rho + \sigma$ the mean curvature function of M. Furthermore, M has constant curvature if the function $\rho + \sigma$ is constant on M.

Now, with the help of the functions ρ and σ above, we define a *pseudo-totally umbilical* lightlike submanifold of a semi-Riemannian manifold as follows:

Definition 3.1. Let (M, g) be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. We say that M is *pseudo-totally umbilical* if on every coordinate neighbourhood $\mathcal{U}$, of M, there exists smooth functions ρ and σ such that

$$\bar{g}(h^*(X, PY), L) = \rho g(X, PY) \quad \text{and} \quad \bar{g}(h^s(X, Y), S) = \sigma g(X, Y), \tag{3.4}$$

for any X and Y tangent to M. In case $\rho = \sigma = 0$, we say that M is *pseudo-totally geodesic.*

Example 3.2. A minimal lightlike submanifold is clearly pseudo-totally umbilical with $\rho = \sigma = 0$, i.e. pseudo-totally geodesic.

Example 3.3. Let us consider the future-directed lightlike cone Λ_0 of R^4_1 with signature $(-,+,+,+)$, with respect to a canonical basis $\{\partial_0, \partial_1, \partial_2, \partial_3\}$, given by

$$\Lambda_0 = \left\{(x_0, x_1, x_2, x_3) : x_0 = \sqrt{x_1^2 + x_2^2 + x_3^2} > 0\right\}.$$

By a direct calculation we have $\mathrm{Rad}\, TM = \mathrm{Span}\{\xi\}$, where

$$\xi = \partial_0 + \frac{x_1}{x_0}\partial_1 + \frac{x_2}{x_0}\partial_2 + \frac{x_3}{x_0}\partial_3. \tag{3.5}$$

On the other hand, $S(TM) = \mathrm{Span}\{X, Y\}$, where

$$X = X_1\partial_1 + X_2\partial_2 + X_3\partial_3, \quad Y = Y_1\partial_1 + Y_2\partial_2 + Y_3\partial_3, \tag{3.6}$$

such that

$$\sum_{i=1}^{3} x_i X_i = \sum_{i=1}^{3} x_i Y_i = 0. \tag{3.7}$$

Using (3.5), (3.6) and (3.7)

$$\bar{\nabla}_\xi \xi = 0 \quad \text{and} \quad \bar{\nabla}_X \xi = \frac{1}{x_0} X, \tag{3.8}$$

for all X tangent to $S(T\Lambda_0)$, where $\bar{\nabla}$ is the connection on R^4_1. It follows from (3.8), (2.3) and (2.6) that

$$\begin{aligned} &\nabla_\xi \xi = 0, \quad h^\ell(\xi, \xi) = 0, \quad \nabla_X \xi = 0, \quad h^\ell(X, \xi) = 0, \\ \text{and} \quad &\nabla^{*t}_\xi \xi = 0, \quad A^*_\xi \xi = 0, \quad \nabla^{*t}_X \xi = 0, \quad A^*_\xi X = -\frac{1}{x_0} X. \end{aligned} \tag{3.9}$$

Thus, we see that ξ is a geodesic lightlike vector field and Λ_0 is totally umbilical. Next, $l\mathrm{tr}(TM)$ is spanned by N, where

$$N = \frac{1}{2}\left\{-\partial_0 + \frac{x_1}{x_0}\partial_1 + \frac{x_2}{x_0}\partial_2 + \frac{x_3}{x_0}\partial_3\right\}. \tag{3.10}$$

From (3.10), (3.6), (3.5) and (3.7), we find

$$\bar{\nabla}_\xi N = 0 \quad \text{and} \quad \bar{\nabla}_X N = \frac{1}{2x_0} X, \tag{3.11}$$

for all X tangent to $S(T\Lambda_0)$. Hence, from (2.4) and (3.11), we have

$$\nabla^\ell_\xi N = 0, \quad A_N \xi = 0, \quad \nabla^\ell_X N = 0, \quad A_N X = -\frac{1}{2x_0} X. \tag{3.12}$$

Therefore, $S(T\Lambda_0)$ is totally umbilical. The mean curvature vector field L of Λ_0, in $\mathbb{R}^4_1$, follows from (3.9) as

$$L = \frac{1}{2}\mathrm{trace}_{|S(TM)} h^\ell = \frac{1}{2}\left(\mathrm{trace}_{|S(TM)} A^*_\xi\right) N = -\frac{1}{x_0} N. \tag{3.13}$$

On the other hand, from (3.12), we have

$$h^*(X, Y) = -\frac{1}{2x_0} g(X, Y)\xi. \tag{3.14}$$

It follows from (3.14) and (3.13) that

$$\bar{g}(h^*(X, Y), L) = \frac{1}{2x_0^2} g(X, Y),$$

showing that Λ_0 is pseudo-totally umbilical with $\rho = \frac{1}{2x_0^2}$ and $\sigma = 0$.

In general, we have the following example:

Example 3.4. A totally umbilical lightlike submanifold, with a totally umbilical screen distribution is pseudo-totally umbilical, such that $\rho = \bar{g}(K', H^\ell)$ and $\sigma = \bar{g}(H^s, H^s)$.

Remark 3.5. Although every totally umbilical lightlike submanifold, with a totally umbilical screen distribution is also pseudo-totally umbilical, we note that the converse is generally not true.

Taking into account of (2.7), (2.10) and (3.4), we note that on any pseudo-totally umbilical lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$, the following holds:

Proposition 3.6. *A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is pseudo-totally umbilical if and only if on each coordinate neighbourhood $\mathcal{U}$ there exists smooth functions ρ and σ such that the following holds:*

$$\bar{g}(h^*(\xi, PX), L) = 0, \quad \bar{g}(h^s(\xi, X), S) = 0,$$
$$D^\ell(X, S) = 0, \quad PA_L X = \rho PX, \quad PA_S X = \sigma PX,$$

for any X tangent to M and ξ tangent to $\operatorname{Rad} TM$.

Proposition 3.7. *Let M be a pseudo-totally umbilical lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, such that h^* and h^s are parallel, i.e. $\nabla h^* = 0$ and $\tilde{\nabla} h^s = 0$. If $h^*(K, PX) = h^s(K, X) = 0$, for any X tangent to M, then M is either pseudo-totally geodesic, i.e. $\rho = \sigma = 0$ or the screen shape operator A^*_ξ satisfies $A^*_K = 0$.*

Proof. Suppose that $\nabla h^* = 0$ and $\tilde{\nabla} h^s = 0$, then (2.15) and (2.16), together with the assumptions $h^*(K, PX) = h^s(K, X) = 0$, gives

$$h^*(\nabla_X K, PY) = 0, \quad h^s(\nabla_X K, Y) = 0, \tag{3.15}$$

for any X and Y tangent to M. Taking the inner product of the equations in (3.15) with L and S, respectively, and then apply (3.4) and (2.6), we get

$$\bar{g}(h^*(\nabla_X K, PY), L) = \rho g(\nabla_X K, PY) = -\rho g(A^*_K X, PY) = 0,$$
$$\bar{g}(h^s(\nabla_X K, Y), S) = \sigma g(\nabla_X K, Y) = -\sigma g(A^*_K X, Y) = 0,$$

from which our result follows. □

Proposition 3.8. *Let M be a 1-lightlike pseudo-totally umbilical lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then, either $L = 0$ or $S(TM)$ is totally umbilical in M.*

Proof. As $\dim \operatorname{Rad} TM = 1$, we may write the vector fields in (3.1) and (3.3) as $K = \alpha\xi$ and $L = \beta N$, where ξ and N are the lightlike vector fields spanning the lightlike distributions $\operatorname{Rad} TM$ and $l\mathrm{tr}(TM)$, respectively. Here, α and β are smooth functions on M. Hence, $\rho = \bar{g}(K, L) = \alpha\beta$, and from (3.4) we have

$$\beta\{\bar{g}(h^*(X, PY), N) - \alpha g(X, PY)\} = 0, \tag{3.16}$$

for any X and Y tangent to M. Therefore, from (3.16), we see that either $\beta = 0$ which means that $L = \beta N = 0$ or $\bar{g}(h^*(X, PY), N) = \alpha g(X, PY)$, which implies that $h^*(X, PY) = g(X, PY)(\alpha\xi) = g(X, PY)K$, which shows that $S(TM)$ is totally umbilical in M. □

In particular, if M is a lightlike hypersurface, Proposition 3.8 leads to the following result:

Theorem 3.9. *Any pseudo-totally umbilical lightlike hypersurface M of a semi-Riemannian manifold $\bar{M}$ is either minimally immersed in $\bar{M}$, i.e. $L = 0$, or a non-minimal hypersurface whose screen distribution $S(TM)$ is totally umbilical in M, i.e. $h^* = g \otimes K$.*

Proposition 3.10. *Let M be a pseudo-totally umbilical lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, such that $\dim S(TM^\perp) = 1$. Then, either $S = 0$ or $h^s = g \otimes S$.*

Proof. As $\dim S(TM^{\perp}) = 1$, we put $S = \gamma W$, where γ is some smooth function and W the vector field spanning $S(TM^{\perp})$. It follows that $\sigma = \bar{g}(S,S) = \epsilon\gamma^2$, where $\epsilon = \bar{g}(W,W)$. Then, from the second relation in (3.4), we have

$$\gamma\{\bar{g}(h^s(X,Y),W) - \epsilon\gamma g(X,Y)\} = 0, \tag{3.17}$$

for any X and Y tangent to M. Then, from (3.17), we have $\gamma = 0$ and thus, $S = \gamma W = 0$ or $\bar{g}(h^s(X,Y),W) = \epsilon\gamma g(X,Y)$, which leads to $h^s(X,Y) = g(X,Y)(\gamma W) = g(X,Y)S$, completing the proof. □

Let M be a half-lightlike submanifold, i.e. one in which $\dim \mathrm{Rad}\, TM = 1$ and $\dim S(TM^{\perp}) = 1$. Then, in view of Propositions 3.8 and 3.10, we have the following result:

Theorem 3.11. *Let M be a pseudo-totally umbilical half-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then, M falls in one of the following categories:*

(1) *A non-totally umbilical pseudo-totally geodesic submanifold such that $L = S = 0$, and with a nowhere totally umbilical screen distribution;*
(2) *A non-minimal submanifold satisfying $L = 0$ and $h^s = g \otimes S$;*
(3) *A non-minimal submanifold whose screen distribution $S(TM)$ is totally umbilical i.e. $h^* = g \otimes K$ and $S = 0$;*
(4) *A non-minimal submanifold whose screen distribution $S(TM)$ is totally umbilical i.e. $h^* = g \otimes K$ and $h^s = g \otimes S$.*

Next, we give some examples in support of Theorem 3.11.

Example 3.12. Let us denote by $(\mathbb{R}^4_1, \langle,\rangle)$, the Minkowski space with the signature $(+,+,+,-)$ with respect to the canonical basis $(\partial_1,\dots,\partial_4)$. Let $\mathbb{S}^3_1 = \{p \in \mathbb{R}^4_1 | \langle p,p\rangle = 1\}$ be the 3-dimensional unit pseudosphere of index 1, which is a Lorentzian hypersurface of $\mathbb{R}^4_1$. Now, denote by $\bar{M} = (\mathbb{S}^3_1 \times \mathbb{R}^2_1, \bar{g})$ the semi-Riemannian cross product, where $\mathbb{R}^2_1$ is semi-Euclidean space with the signature $(+,-)$, with respect to the canonical basis $\{\partial_5,\partial_6\}$ and g is the inner product of $\mathbb{R}^6_2 = \mathbb{R}^4_1 \times \mathbb{R}^2_1$ restricted to $\bar{M}$. Then, the half-lightlike submanifold $(M, \bar{g}_M, S(TM), S(TM^{\perp}))$, given by $M = \mathbb{S}^1 \times \mathcal{H} \times \mathbb{R} = \{(p,t,t) \in \mathbb{S}^3_1 \times \mathbb{R}^2_1 | t \in \mathbb{R}\}$, where $p = \frac{\sqrt{2}}{2}(\cos\theta, \sin\theta, \cosh\omega, \sinh\omega) \in \mathbb{R}^4_1$ such that $\theta \in [0,2\pi]$ and $\omega \in \mathbb{R}$ and $\mathcal{H}$ is a hyperbola, is known to be minimal [11, Example 9, p. 221]. Therefore, this half-lightlike submanifold is pseudo-totally umbilical and falls in the first category in Theorem 3.11, since $L = S = 0$.

Example 3.13. Let M be a surface of the Minkowski space$\mathbb{R}^4_1$, and given by the equations $x^1 = x^3$ and $x^2 = (1-(x^4)^2)^{1/2}$. Then, by a direct calculation, we find that TM is spanned by $\xi = \partial_1 + \partial_3$ and $E = -x^4\partial_2 + x^2\partial_4$, while $TM^{\perp}$ is spanned by ξ and $W = x^2\partial_2 + x^4\partial_4$. It follows that $\mathrm{Rad}\, TM = \mathrm{Span}\{\xi\}$, $S(TM) = \mathrm{Span}\{E\}$ and $S(TM^{\perp}) = \mathrm{Span}\{W\}$. On the other hand, $l\mathrm{tr}(TM) = \mathrm{Span}\{N\}$, where $N = \frac{1}{2}\{-\partial_1 + \partial_3\}$. It has been shown in [11, Example 5, p. 189] that $A^*_{\xi}E = A_N E = 0$, $h^{\ell} = 0$, $h^s(\xi,X) = 0$, for any X tangent to M, and $h^s(E,E) = g(E,E)W$. Clearly, M is a non-minimal pseudo-totally umbilical half-lightlike submanifold such that $L = 0$ and $h^s = g \otimes W$, and hence a submanifold of the second category in Theorem 3.11.

Lemma 3.14. *On any pseudo-totally umbilical r-lightlike submanifold M of a semi-Riemannian manifold $\bar{M}(c)$, the following holds:*

$$\begin{aligned} X\rho - (c+\rho)\bar{g}(X,L) - \bar{g}(K, \nabla^{\ell}_X L) - \bar{g}(D^s(X,L),S) \\ = \frac{1}{m-r}PX\rho - \frac{1}{m-r}\sum_{i=1}^{m-r}\varepsilon_i\{\bar{g}(h^*(X,E_i), \nabla^{\ell}_{E_i}L) \\ - \bar{g}(A_L E_i, h^{\ell}(X,E_i)) + \bar{g}(h^s(X,E_i), D^s(E_i,L))\}, \end{aligned}$$

$$X\sigma - 2\sigma\bar{g}(X,L) + 2\bar{g}(D^s(X,L),S)$$
$$= \frac{2}{m-r}PX\sigma - \frac{2}{m-r}\sum_{i=1}^{m-r}\varepsilon_i\{\bar{g}(h^s(X,E_i),\nabla^s_{E_i}S) - \bar{g}(D^s(E_i,h^\ell(X,E_i)),S)\},$$

for all X tangent to M.

Proof. By a direct calculation, while considering (3.4) and (2.11), we derive

$$\bar{g}((\nabla_X h^*)(Y,PZ),L) = (X\rho)g(Y,PZ) - \bar{g}(h^*(Y,PZ),\nabla^\ell_X L) + \rho\bar{g}(h^\ell(X,PZ),Y), \quad (3.18)$$

$$\bar{g}((\tilde{\nabla}_X h^s)(Y,Z),S) = (X\sigma)g(Y,Z) - \bar{g}(h^s(Y,Z),\nabla^s_X S) + \sigma\{\bar{g}(h^\ell(X,Z),Y) + \bar{g}(h^\ell(X,Y),Z)\}, \quad (3.19)$$

for all X, Y and Z tangent to M. Then, from (3.18), (3.19), (2.21) and (2.20) that

$$\begin{aligned}(X\rho)g(Y,PZ) &- \rho\bar{g}(h^\ell(Y,PZ),X) - c\bar{g}(Y,PZ)\bar{g}(X,L)\\ &- \bar{g}(h^*(Y,PZ),\nabla^\ell_X L) + \bar{g}(A_L X,h^\ell(Y,PZ))\\ - \bar{g}(h^s(Y,PZ),D^s(X,L)) &= (Y\rho)g(X,PZ) - \rho\bar{g}(h^\ell(X,PZ),Y)\\ - c\bar{g}(X,PZ)\bar{g}(Y,L) &- \bar{g}(h^*(X,PZ),\nabla^\ell_Y L) + \bar{g}(A_L Y,h^\ell(X,PZ))\\ &- \bar{g}(h^s(X,PZ),D^s(Y,L)),\end{aligned} \quad (3.20)$$

and for σ, we have

$$\begin{aligned}(X\sigma)g(Y,Z) &- \sigma\bar{g}(h^\ell(Y,Z),X) - \bar{g}(h^s(Y,Z),\nabla^s_X S)\\ +\bar{g}(D^s(X,h^\ell(Y,Z)),S) &= (Y\sigma)g(X,Z) - \sigma\bar{g}(h^\ell(X,Z),Y)\\ &- \bar{g}(h^s(X,Z),\nabla^s_Y S) + \bar{g}(D^s(Y,h^\ell(X,Z)),S).\end{aligned} \quad (3.21)$$

Then, Lemma 3.14 follows from (3.20) and (3.21) by tracing over Y and Z, with respect to $S(TM)$, and using (3.1) and (3.3). □

Then, from Lemma 3.14, the following is immediate:

Corollary 3.15. *Let M be a pseudo-totally umbilical lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$. Then, ρ and σ of (3.4) satisfies the following partial differential equations:*

$$\begin{aligned}\xi\rho - (c+\rho)\bar{g}(\xi,L) &- \bar{g}(K,\nabla^\ell_\xi L) - \bar{g}(D^s(\xi,L),S)\\ = -\frac{1}{m-r}\sum_{i=1}^{m-r}\varepsilon_i\{&\bar{g}(h^*(\xi,E_i),\nabla^\ell_{E_i}L) - \bar{g}(A_L E_i,h^\ell(\xi,E_i))\\ &+ \bar{g}(h^s(\xi,E_i),D^s(E_i,L))\},\end{aligned}$$

$$\begin{aligned}\xi\sigma &- 2\sigma\bar{g}(\xi,L) + 2\bar{g}(D^s(\xi,L),S)\\ &= -\frac{2}{m-r}\sum_{i=1}^{m-r}\varepsilon_i\{\bar{g}(h^s(\xi,E_i),\nabla^s_{E_i}S) - \bar{g}(D^s(E_i,h^\ell(\xi,E_i)),S)\},\end{aligned}$$

$$\begin{aligned}(m-r-1)PX\rho &- (m-r)\bar{g}(K,\nabla^\ell_{PX}L) - (m-r)\bar{g}(D^s(PX,L),S)\\ = -\sum_{i=1}^{m-r}\varepsilon_i\{&\bar{g}(h^*(PX,E_i),\nabla^\ell_{E_i}L) - \bar{g}(A_L E_i,h^\ell(PX,E_i))\\ &+ \bar{g}(h^s(PX,E_i),D^s(E_i,L))\},\end{aligned}$$

$$(m-r-2)PX\sigma + 2(m-r)\bar{g}(D^s(PX,L),S)$$
$$= -2\sum_{i=1}^{m-r} \varepsilon_i\{\bar{g}(h^s(PX,E_i),\nabla^s_{E_i}S) - \bar{g}(D^s(E_i,h^\ell(PX,E_i)),S)\},$$

for any X tangent to M and ξ tangent to $\mathrm{Rad}\,TM$.

Theorem 3.16. *There does not exist any non-minimal pseudo-totally umbilical lightlike hypersurface M in a semi-Riemannian space form $\bar{M}(c) : c \neq 0$ such that $S(TM)$ is totally geodesic in M.*

Proof. As M is a lightlike hypersurface, we have $K = \alpha\xi$ and $L = \beta N$, for some smooth functions α and β. It follows that $\rho = \bar{g}(K,L) = \alpha\beta$. On the other hand, Corollary 3.15 gives $\xi\rho - (c+\rho)\bar{g}(\xi,L) - \bar{g}(K,\nabla^\ell_\xi L) = 0$, which reduces to

$$\beta\{\xi\alpha - (c+\alpha\beta) - \alpha\bar{g}(\xi,\nabla^\ell_\xi N)\} = 0. \tag{3.22}$$

But, from Theorem 3.9, we note that either M is minimal i.e. $L = \beta N = 0$, which means $\beta = 0$ or $S(TM)$ is totally umbilical with $K' = K = \alpha\xi$. Thus, if M is non-minimal, that is $\beta \neq 0$ and $S(TM)$ totally geodesic, that is $K' = 0$, equivalently $\alpha = 0$, we see from (3.22) that $c = 0$. This contradiction completes the proof. □

Theorem 3.17. *There does not exist any pseudo-totally umbilical half-lightlike submanifold M of the third kind in Theorem 3.11 of a semi-Riemannian space form $\bar{M}(c) : c \neq 0$ such that $S(TM)$ is totally geodesic.*

Proof. Let M be of the third type in Theorem 3.11, then M is non-minimal, with a totally umbilical screen distribution and $S = 0$. Thus, if we let $K = \alpha\xi$, $L = \beta N$, for some smooth functions α and $\beta \neq 0$ (since M is non-minimal), then we have $\rho = \bar{g}(K,L) = \alpha\beta$. Hence, as $S = 0$, Corollary 3.15 leads to $\xi\alpha - (c+\alpha\beta) - \alpha\bar{g}(\xi,\nabla^\ell_\xi N) = 0$. It, then, follows that if $S(TM)$ is totally geodesic, that is $K' = K = \alpha\xi = 0$, we get $c = 0$. This contradiction completes the proof. □

A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is called *irrotational* [11, p. 245] if for any X tangent to M and ξ tangent to $\mathrm{Rad}\,TM$, one has $\bar{\nabla}_X\xi$ tangent to M. This, further, implies that $h^\ell(X,\xi) = 0$ and $h^s(X,\xi) = 0$.

Theorem 3.18. *Let M be a pseudo-totally umbilical irrotational r-lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$. If the mean curvature vector fields $\bar{H} = L + S$ and K are parallel, then either $\rho = 0$ or M has constant mean curvature $-c$.*

Proof. By the assumption $\bar{H} = L+S$ and K parallel, we deduce that $\nabla^\ell_X L = 0$, $\nabla^s_X S = 0$ and $\nabla^{*t}_X K = 0$, for any X tangent to M. It then follows that $X\rho = X\cdot\bar{g}(K,L) = \bar{g}(\nabla^{*t}_X K, L) + \bar{g}(K,\nabla^\ell_X L) = 0$ and $X\sigma = X\cdot\bar{g}(S,S) = 2\bar{g}(\nabla^s_X S,S) = 0$. Now, using the first two differential equations in Corollary 3.15, we get $(c+\rho)\bar{g}(\xi,L) + \bar{g}(D^s(\xi,L),S) = 0$ and $\sigma\bar{g}(\xi,L) - \bar{g}(D^s(\xi,L),S) = 0$, for any ξ tangent to $\mathrm{Rad}\,TM$. From these two relations, we get $(c+\rho+\sigma)\bar{g}(\xi,L) = 0$. With $\xi = K$ in the last relation, we have $(c+\rho+\sigma)\rho = 0$, which proves our result. □

Corollary 3.19. *Under the same hypothesis as in Theorem 3.18, if M is co-isotropic, then M is pseudo-totally geodesic or has constant mean curvature $-c$.*

4. Pseudo-totally umbilical screen distribution

Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, and let K be the mean curvature vector field of $S(TM)$ as given in (3.3). Then, we have the following definition:

Definition 4.1. Let (M, g) be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. We say that the screen distribution $S(TM)$ is pseudo-totally umbilical, in M, if on each coordinate neighbourhood $\mathcal{U}$ of M there exists a smooth function λ such that, for any X and Y tangent to M, the following holds:

$$\bar{g}(h^{\ell}(X, Y), K) = \lambda g(X, Y), \tag{4.1}$$

where K is the smooth vector field of $\operatorname{Rad} TM$, called the mean curvature vector field of $S(TM)$ in M, and given by (3.3). In case $\lambda = 0$ (resp. $\lambda \neq 0$), we say that $S(TM)$ is pseudo-totally geodesic (resp. proper pseudo-totally umbilical).

Example 4.2. A lightlike submanifold of a semi-Riemannian manifold with a minimal screen distribution, i.e. $K = 0$, carries a pseudo-totally geodesic screen distribution, i.e. $\lambda = 0$.

Example 4.3. The lightlike cone of Example 3.2 has a pseudo totally umbilical screen distribution with $\lambda = \frac{1}{2x_0^2}$. In fact, using (3.3) and (3.14), we have $K = -\frac{1}{2x_0}\xi$. On the other hand, using (3.9), we have $h^{\ell}(X, Y) = -\frac{1}{x_0} g(X, Y)N$, for any X and Y tangent to M. It then follows from (4.1) that $\bar{g}(h^{\ell}(X, Y), K) = \frac{1}{2x_0^2} g(X, Y)$, which confirms our claims.

Example 4.4. We may generalise Example 4.3 as follows: A totally umbilical lightlike submanifold of a semi-Riemannian manifold has a pseudo-totally umbilical screen distribution, such that $\lambda = \bar{g}(H^{\ell}, K)$.

Remark 4.5. Although every totally umbilical lightlike submanifold carries a pseudo-totally umbilical screen distribution, we stress that the converse is generally not true.

In view of relations (2.8), (2.9) and (4.1), we have the following:

Proposition 4.6. *A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ has a screen distribution $S(TM)$ which is pseudo-totally umbilical if and only if on each coordinate neighbourhood $\mathcal{U}$ there exists a smooth function λ such that $h^{\ell}(X, K) = 0$ and $A_K^* X = \lambda PX$, for any X tangent to M.*

Theorem 4.7. *Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, such that $S(TM)$ is pseudo-totally umbilical in M. Suppose that h^{ℓ} is parallel, i.e. $\tilde{\nabla} h^{\ell} = 0$, then $S(TM)$ is pseudo-totally geodesic, i.e. $\lambda = 0$.*

Proof. Suppose that $\tilde{\nabla} h^{\ell} = 0$. Then, relation (2.14) gives

$$\nabla_X^{\ell} h^{\ell}(Y, Z) - h^{\ell}(\nabla_X Y, Z) - h^{\ell}(Y, \nabla_X Z) = 0, \tag{4.2}$$

for any X, Y and Z tangent to M. Taking $Z = K$ in (4.2), and using Proposition 4.6, we have $-h^{\ell}(Y, \nabla_X K) = 0$. It then follows from this relation and the relation in (2.6), that

$$-h^{\ell}(Y, \nabla_X K) = h^{\ell}(Y, A_K^* X) - h^{\ell}(Y, \nabla_X^{*t} K) = 0. \tag{4.3}$$

Tanking the inner product of (4.3) with respect to K, and then using (4.1) and Proposition 4.6, we have

$$\lambda^2 g(X, Y) - \bar{g}(h^{\ell}(Y, \nabla_X^{*t} K), K) = 0. \tag{4.4}$$

On the other hand, using (2.8) and Proposition 4.6, we have

$$-\bar{g}(h^{\ell}(Y, \nabla_X^{*t} K), K) = \bar{g}(h^{\ell}(Y, K), \nabla_X^{*t} K) = 0. \tag{4.5}$$

Finally, from (4.4) and (4.5), we have $\lambda^2 g(X, Y) = 0$, which gives $\lambda = 0$. □

Corollary 4.8. *There are no any lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$, with a proper pseudo-totally umbilical screen distribution such that h^{ℓ} is parallel.*

Proposition 4.9. *Let M be a 1-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, with a pseudo-totally umbilical screen distribution $S(TM)$. Then, either*

(1) *the screen distribution $S(TM)$ is minimal in M, i.e. $K = 0$, and therefore pseudo-totally geodesic, or*
(2) *there exist a smooth transversal vector field H^ℓ of $l\mathrm{tr}(TM)$, such that $h^\ell = g \otimes H^\ell$.*

Proof. Assume that $\dim \mathrm{Rad}\, TM = 1$, we may write $K = \alpha\xi$ and $L = \beta N$, where ξ spans $\mathrm{Rad}\, TM$ and N spans $l\mathrm{tr}(TM)$, α and β are smooth functions on M. It follows that $\lambda = \bar{g}(L, K) = \alpha\beta$. Then, from (4.1), we get

$$\alpha\{\bar{g}(h^\ell(X,Y), \xi) - \beta g(X,Y)\} = 0. \tag{4.6}$$

So, from(4.6), we either have $\alpha = 0$, which means that $K = \alpha\xi = 0$ and $\lambda = \alpha\beta = 0$ or $\bar{g}(h^\ell(X,Y), \xi) = \beta g(X,Y)$. It follows from the last relation that $h^\ell(X,Y) = g(X,Y)\beta N = g(X,Y)H^\ell$, with $H^\ell = \beta N$. □

Corollary 4.10. *Any lightlike hypersurface of a semi-Riemannian manifold with a pseudo-totally umbilical screen distribution has either a pseudo-totally geodesic screen distribution, i.e. $\lambda = 0$, which is minimal in M, i.e. $K = 0$, or is totally umbilical, i.e. $h^\ell = g \otimes H^\ell$.*

Corollary 4.11. *The only lightlike hypersurfaces of a semi-Riemannian manifold with a proper pseudo-totally umbilical screen distributions are the proper totally umbilical ones.*

Lemma 4.12. *On a lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ with a pseudo-totally umbilical screen distribution, the following holds:*

$$\begin{aligned} X\lambda - \lambda\bar{g}(X,L) - \bar{g}(\nabla^{*t}_X K, L) + \bar{g}(K, D^\ell(X,S)) &= \frac{1}{m-r} PX\lambda \\ - \frac{1}{m-r}\sum_{i=1}^{m-r} \varepsilon_i \left\{\bar{g}(\nabla^{*t}_{E_i} K, h^\ell(X,E_i)) - \bar{g}(K, D^\ell(E_i, h^s(X,E_i)))\right\}&, \end{aligned}$$

for any X tangent to M.

Proof. Using (4.1) and (2.11), we have

$$\begin{aligned} \bar{g}((\tilde{\nabla}_X h^\ell)(Y,Z), K) &= (X\lambda) g(Y,Z) - \bar{g}(\nabla^{*t}_X K, h^\ell(Y,Z)) \\ &\quad + \lambda\{\bar{g}(h^\ell(X,Z), Y) - \bar{g}(h^\ell(Y,Z), X)\}, \end{aligned} \tag{4.7}$$

for any X, Y and Z tangent to M. It then follows from (4.7) and (2.22) that

$$\begin{aligned} (X\lambda) g(Y,Z) - \lambda\bar{g}(X, h^\ell(Y,Z)) - \bar{g}(\nabla^{*t}_X K, h^\ell(Y,Z)) + \bar{g}(K, D^\ell(X, h^s(Y,Z))) \\ = (Y\lambda) g(X,Z) - \lambda\bar{g}(Y, h^\ell(X,Z)) - \bar{g}(\nabla^{*t}_Y K, h^\ell(X,Z)) \\ + \bar{g}(K, D^\ell(Y, h^s(X,Z))). \end{aligned} \tag{4.8}$$

Then, letting $Y = Z = E_i$ in (4.8) and summing over all $i \in \{1, \ldots, m-r\}$, we get Lemma 4.12. □

In view of Lemma 4.12, we have the following:

Proposition 4.13. *Let M be a lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$. If $S(TM)$ is pseudo-totally umbilical, then λ satisfy the following partial differential* *equations:*

$$\begin{aligned}
&\xi\lambda - \lambda\bar{g}(\xi, L) - \bar{g}(\nabla^{*t}_{\xi} K, L) + \bar{g}(K, D^{\ell}(\xi, S)) \\
&\quad = \frac{1}{m-r}\sum_{i=1}^{m-r} \varepsilon_i \left\{\bar{g}(\xi, h^{\ell}(\nabla^{*t}_{E_i} K, E_i)) + \bar{g}(K, D^{\ell}(E_i, h^{s}(\xi, E_i)))\right\}, \\
&(m-r-1)PX\lambda - (m-r)\bar{g}(\nabla^{*t}_{PX} K, L) + (m-r)\bar{g}(K, D^{\ell}(PX, S)) \\
&\quad = -\sum_{i=1}^{m-r} \varepsilon_i \left\{\bar{g}(\nabla^{*t}_{E_i} K, h^{\ell}(PX, E_i)) - \bar{g}(K, D^{\ell}(E_i, h^{s}(PX, E_i)))\right\},
\end{aligned}$$

for any ξ tangent to $\mathrm{Rad}\,TM$ *and X tangent to M.*

When M is irrotational, we see from relation (2.7) that $\bar{g}(\xi, D^{\ell}(X, W)) = 0$, for any X tangent to M. It follows from this relation that $D^{\ell} = 0$. With this fact, we have the following:

Corollary 4.14. *With the same hypothesis as in Proposition 4.13, if M is irrotational then:*

$$\xi\lambda - \lambda\bar{g}(\xi, L) - \bar{g}(\nabla^{*t}_{\xi} K, L) = 0,$$

$$(m-r-1)PX\lambda - (m-r)\bar{g}(\nabla^{*t}_{PX} K, L) + \sum_{i=1}^{m-r} \bar{g}(\nabla^{*t}_{E_i} K, h^{\ell}(PX, E_i)) = 0,$$

for any ξ tangent to $\mathrm{Rad}\,TM$ *and X tangent to M.*

By a direct calculation, while using (2.3), (2.4) and (2.6), we have

$$X\lambda = X \cdot \bar{g}(K, L) = \bar{g}(\nabla^{*t}_{X} K, L) + \bar{g}(K, \nabla^{\ell}_{X} L),$$

for all X tangent to M. Now, if M is irrotational, we see from the first relation in Corollary 4.14 and the above relation, with $X = \xi$, that $\bar{g}(K, \nabla^{\ell}_{\xi} L) = \lambda\bar{g}(\xi, L)$, for any ξ tangent to $\mathrm{Rad}\,TM$. Taking $\xi = K$ in this relation, we get $\bar{g}(K, \nabla^{\ell}_{K} L) = \lambda\bar{g}(K, L) = \lambda^2$. Therefore, we have the following result:

Theorem 4.15. *Let M be an irrotational lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$, such that $S(TM)$ is pseudo-totally umbilical. If L is a parallel vector field, then $\lambda = 0$, i.e. $S(TM)$ is pseudo-totally geodesic.*

Next, suppose that the mean curvature vector field K is a parallel, i.e. $\nabla^{*t} K = 0$, then Corollary 4.14 gives $\xi\lambda - \lambda\bar{g}(\xi, L) = 0$ and $(m-r-1)PX\lambda = 0$, for any ξ tangent to $\mathrm{Rad}\,TM$ and X tangent to M. These equations suggest the following:

Theorem 4.16. *Let M be an irrotational lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$, such that $S(TM)$ is pseudo-totally umbilical. If K is a parallel vector field, then $K\lambda - \lambda^2 = 0$. Moreover, either* $\dim S(TM) = 1$ *or λ is constant function along $S(TM)$.*

5. Pseudo-totally umbilical leaves of $S(TM)$

Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Through out this section we assume that $S(TM)$ is integrable and the connections D^{ℓ} and D^{s} vanishes on $S(TM)$. Now, we have the following lemma:

Lemma 5.1. *Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$, such that $S(TM)$ is integrable. Let M^* be a leaf of $S(TM)$, immersed as an $(m-r)$-dimensional submanifold of $\bar{M}$. Then, the following holds:*

$$\bar{\nabla}_X Y = \nabla^{*}_{X} Y + h'(X, Y), \quad \bar{\nabla}_X U = -A'_U X + \nabla^{*\perp}_{X} U, \tag{5.1}$$

for any X and Y tangent to M^* and U tangent to $TM^{*\perp}$. Here, h', A'_U and $\nabla^{*\perp}$ denotes the second fundamental form, the shape operator and normal connction of M^*, and given by

$$h'(X,Y) = h^*(X,Y) + h^{\ell}(X,Y) + h^s(X,Y), \tag{5.2}$$

$$A'_U X = A^*_{U^r}X + A_{U^\ell}X + A_{U^s}X,$$

$$\nabla^{*\perp}_X U = \nabla^{*t}_X U^r + \nabla^{\ell}_X U^{\ell} + \nabla^s_X U^s, \tag{5.3}$$

where U^r, U^ℓ and U^s are the components of U tangent to $\mathrm{Rad}\,TM$, $\mathrm{ltr}(TM)$ *and* $S(TM^\perp)$, *respectively.*

Proof. The relations in the lemma follows directly from (2.3), (2.5) and (2.6). □

Using relations (5.2), (3.2), (3.2) and (3.3) we see that the mean curvature vector field H^* of M^*, in $\bar{M}$, is given by

$$H^* = \mathrm{trace}_{|S(TM)} h' = K + L + S. \tag{5.4}$$

We say that M^* is minimal, in $\bar{M}$, if $H^* = 0$. Obviously this is equivalent to $K = L = S = 0$. On the other hand, we say that H^* is parallel if $\nabla^{*\perp}H^* = 0$. Using (5.3) and (5.4), this is equivalent to $\nabla^{*t}_X K = \nabla^{\ell}_X L = \nabla^s_X S = 0$, for any X tangent to M^*. Next, we denote by $\mathbb{S}^d_q(c_0, r_0) = \{x \in \mathbb{R}^{d+1}_q | \bar{g}(x - c_0, x - c_0) = r_0^2\}$ and $\mathbb{H}^d_q(c_0, r_0) = \{x \in \mathbb{R}^{d+1}_{q+1} | \bar{g}(x - c_0, x - c_0) = -r_0^2\}$ the d-dimensional semi-Euclidean sphere and hyperbolic spaces of constant curvatures r_0^{-2} and $-r_0^{-2}$, respectively, and $c_0 \in \mathbb{R}^d_q$ is the center. Then, we have the following result:

Theorem 5.2. *Let (M, g) be a pseudo-totally umbilical lightlike submanifold of a semi-Riemannian space form $\mathbb{R}^{m+n}_q$. Suppose that $S(TM)$ is also pseudo-totally umbilical. Then, each $(m - r)$-dimensional leaf M^* of $S(TM)$ is pseudo-totally umbilica in $\mathbb{R}^{m+n}_q$, i.e. for any X and Y tangent to M^*, we have $\bar{g}(h'(X,Y), H^*) = \varphi g(X,Y)$, where φ is some smooth function. Moreover, if $\varphi \neq 0$ and the mean curvature vector fields K and $\bar{H} = L + S$ are parallel, then M^* is either contained in $\mathbb{S}^{m+n-1}_q(c_0, r_0)$ or in $\mathbb{H}^{m+n-1}_{q-1}(c_0, r_0)$ as a minimal submanifold, for some $c_0 \in \mathbb{R}^{m+n}_q$ and $r_0 > 0$.*

Proof. From (3.4), (4.1), (5.2) and (5.4), we have

$$\begin{aligned}\bar{g}(h'(X,Y), H^*) &= \bar{g}(h^*(X,Y), H^*) + \bar{g}(h^{\ell}(X,Y), H^*) + \bar{g}(h^s(X,Y), H^*)\\ &= \bar{g}(h^*(X,Y), L) + \bar{g}(h^{\ell}(X,Y), K) + \bar{g}(h^s(X,Y), S)\\ &= (\rho + \lambda + \sigma) g(X,Y),\end{aligned} \tag{5.5}$$

for any X and Y tangent to M^*. It follows from (5.5) that each leaf M^* is pseudo-totally umbilic in $\mathbb{R}^{m+n}_q$, with

$$\varphi = \rho + \lambda + \sigma = \bar{g}(H^*, H^*). \tag{5.6}$$

Next, as K and $L + S$ are parallel, we see that H^* is parallel too. The rest of the proof will follow exactly as given in Lemma 2 of [4, p. 361]. In fact, from (5.6), we have (5.1) to derive

$$X\varphi = X \cdot \bar{g}(H^*, H^*) = 2\bar{g}(\bar{\nabla}_X H^*, H^*) = 2\bar{g}(\nabla^{*\perp}_X H^*, H^*) = 0,$$

for any X tangent to M^*. Thus, φ is a constant function on the leaves M^*. Set

$$\varphi = \frac{\epsilon}{r_0^2}, \tag{5.7}$$

where $\epsilon = \pm 1$. On the other hand, as M^* is pseudo-totally umbilic, it follows from (5.5) and (5.1), that

$$A'_{H^*}X = (\rho + \lambda + \sigma)X = \varphi X. \tag{5.8}$$

Next, let us set $v = x + \epsilon r_0^2 H^*$, where x is the position vector of M^* in $\mathbb{R}_q^{m+n}$. Then, in view of (5.1) and (5.8), we have

$$\bar{\nabla}_X v = \bar{\nabla}_X x + \epsilon r_0^2 \bar{\nabla}_X H^* = X - \epsilon r_0^2 A'_{H^*} X = 0, \tag{5.9}$$

for any X tangent to M^*. Relation (5.9) shows that v is a constant vector along the leaves in $\mathbb{R}_q^{m+n}$. Denote this vector by c_0. Hence, we have $\bar{g}(x - c_0, x - c_0) = r_0^4 \bar{g}(H^*, H^*) = \epsilon r_0^2$, in which we have sed (5.6) and (5.7). Thus, M^* lies in either $\mathbb{S}_q^{m+n-1}(c_0, r_0)$ or in $\mathbb{H}_{q-1}^{m+n-1}(c_0, r_0)$, and by Lemma 1 of [4, p. 360], M^* is minimal in these spaces. □

Example 5.3. Consider the lightlike cone of Example 3.3. Clearly $S(T\Lambda_0)$ is integrable and its leaves are totally umbilical in $\mathbb{R}_1^4$. Furthermore, it is clear that these leaves are pseudo-totally umbilical in $\mathbb{R}_1^4$, with $\varphi = \frac{1}{x_0^2}$. As $\nabla^{*t}\xi = \nabla^{\ell} N = 0$ and $PXx_0 = 0$, we see that the mean curvature vector field $H^* = -\frac{1}{x_0}\left(\frac{1}{2}\xi + N\right)$ of these leaves is parallel, i.e.

$$\nabla^{*\perp}_{PX} H^* = \frac{PXx_0}{x_0^2}\left(\frac{1}{2}\xi + N\right) - \frac{1}{x_0}\left(\frac{1}{2}\nabla^{*t}_{PX}\xi + \nabla^{\ell}_{PX} N\right) = 0,$$

for any X tangent to Λ_0. Therefore, the leaves of $S(T\Lambda_0)$ are contained in the pseudo-Euclidean sphere $\mathbb{S}_1^3(c_0, x_0)$. Furthermore, it is easy to show that these leaves are minimally immersed in $\mathbb{S}_1^3(c_0, x_0)$.

6. Mean lightlike sectional curvatures

Let $x \in M$ and ξ be a lightlike vector of $T_x\bar{M}$. A plane Π of $T_x\bar{M}$ is called a lightlike plane directed by ξ if it contains ξ, $\bar{g}_x(\xi, E) = 0$ for any $E \in \Pi$ and there exist $E_0 \in \Pi$ such that $\bar{g}(E_0, E_0) \neq 0$. Then, the lightlike sectional curvature [11] of Π with respect to ξ and $\bar{\nabla}$ as a real number

$$\bar{\Phi}_\xi(\Pi) = \frac{\bar{g}(\bar{R}(E, \xi)\xi, E)}{g(E, E)}, \tag{6.1}$$

where E is an arbitrary non-lightlike vector in Π. In a similar way, we define lightlike sectional curvature $\Phi_\xi(\Pi)$ of the lightlike plane Π of the tangent space T_xM with respect to ξ and ∇, as a real number

$$\Phi_\xi(\Pi) = \frac{g(R(E, \xi)\xi, E)}{g(E, E)}. \tag{6.2}$$

It is well-known that both lightlike sectional curvatures in (6.1) and (6.2) above are independent of the non-lightlike section E, but quadratically dependent on the lightlike section ξ.

Definition 6.1. Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. We define the mean lightlike sectional curvatures $\bar{\Omega}_\xi[\Pi]$ and $\Omega_\xi[\Pi]$ of $\bar{M}$ and M, respectively, directed by a lightlike vector field ξ, as

$$\bar{\Omega}_\xi[\Pi] = \frac{1}{m-r}\sum_{i=1}^{m-r} \bar{\Phi}_\xi(\Pi_i) \quad \text{and} \quad \Omega_\xi[\Pi] = \frac{1}{m-r}\sum_{i=1}^{m-r} \Phi_\xi(\Pi_i),$$

where the sum is over all lightlike planes Π_i spanned by the lightlike vector field ξ and non-lightlike orthonormal vector fields E_i tangent to $S(TM)$.

Geometrically, the mean lightlike sectional curvatures $\bar{\Omega}_\xi[\Pi]$ and $\Omega_\xi[\Pi]$ are, up to a multiplicative constant, the Ricci tensors $\bar{\mathrm{R}}\mathrm{ic}(\xi, \xi)$ and $\mathrm{Ric}(\xi, \xi)$ of $\bar{M}$ and M, respectively, restricted to $S(TM)$. These tensors play a fundamental role in the characterisation of lightlike hypersurfaces in sapcetimes (see, for instance, K. L. Duggal and B. Sahin [11, Theorem 3.1.8, p. 106]). Furthermore, we note that the vanishing of $\bar{\Phi}_\xi(\Pi_i)$ (resp. $\Phi_\xi(\Pi_i)$) implies the vanishing of $\bar{\Omega}_\xi[\Pi]$ (resp. $\Omega_\xi[\Pi]$), but the converse is, generally, not true.

Proposition 6.2. *Let (M, g) be an irrotational lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. If $S(TM)$ is pseudo-totally umbilical then $\bar{\Omega}_K[\Pi]$ and $\Omega_K[\Pi]$ satisfy:*

$$\bar{\Omega}_K[\Pi] = K\lambda - \lambda^2 - \bar{g}(\nabla^{*t}_K K, L) = \Omega_K[\Pi],$$

where K is the mean curvature vector field in (3.3).

Proof. As M is irrotational, we see from from (2.12) that

$$\bar{g}(\bar{R}(X,\xi)\xi, PY) = g(R(X,\xi)\xi, PY), \tag{6.3}$$

for any X and Y tangent to M, and ξ tangent to $\mathrm{Rad}\,TM$. On the other hand, using the curvature relation in [11, p. 218], we have

$$g(R(X,\xi)\xi, PY) = \bar{g}((\tilde{\nabla}_\xi h^\ell)(X, PY), \xi) - \bar{g}((\tilde{\nabla}_X h^\ell)(\xi, PY), \xi). \tag{6.4}$$

Taking $\xi = K$ in (6.4) and using (4.1) and (2.11), we derive

$$g(R(X,K)K, PY) = \{K\lambda - \lambda^2\} g(X, PY) - \bar{g}(\nabla^{*t}_K K, h^\ell(X, PY)). \tag{6.5}$$

Tracing (6.5) over X and Y, with respect to $S(TM)$, and using (6.3) together with Definition 6.1, we obtain our result. □

Corollary 6.3. *With the same hypothesis as in Proposition 6.2, the mean lightlike sectional curvatures $\bar{\Omega}_K[\Pi]$ and $\Omega_K[\Pi]$ vanishes if and only if λ satisfy the partial differential equation $K\lambda - \lambda^2 - \bar{g}(\nabla^{*t}_K K, L) = 0$.*

When $\bar{M}$ is a space of constant curvature c, then $\bar{\Phi}_\xi(\Pi_i) = \Phi_\xi(\Pi_i) = 0$, and hence we have the following:

Theorem 6.4. *Let M be an irrotational lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$. If $S(TM)$ is pseudo-totally umbilical then the mean lightlike sectional curvatures $\bar{\Omega}_K[\Pi]$ and $\Omega_K[\Pi]$ vanishes.*

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