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On the Topology of δ_ω-Open Sets and Related Topics

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Abstract

The main purpose of this paper is to study the notion of the δ_ω-open sets defined by Al-Jarrah et al via δ_ω-closure operator in [4]. We give various properties of the notions of δ_ω-closure operator and δ_ω-open set. Also, we introduce the notions of δ_ω-continuity, ω-δ-continuity and weakly δ_ω-continuity by means of δ_ω-open sets [4]. Furthermore, we obtain several relationships, examples and counter-examples related to new classes of functions.

Keywords: *δ_ω-open; δ_ω-continuity; ω-δ-continuity; weakly δ_ω-continuity.*
2010 Mathematics Subject Classification: *54C08, 54C10*

1. Introduction

The forms of weak and strong of the notion of open set in topological spaces have been defined and studied by many authors. For instance, in 1982, Hdeib [9] introduced the concept of ω-open set which is weaker than the concept of open set in topological spaces. Also, they proved that the family of all ω-open sets in a space X is a topology which is weaker than the old one. Recently, Al-Zoubi and Al-Nashef [5] have advanced and studied the notion of ω-open set. In 2017, Al Ghour [2], defined the concept of θ_ω-open set which is stronger than the concept of open set. They studied some of its basic properties and obtained characterizations. Moreover, they showed that the family of all θ_ω-open sets in a space X is a topology which is stronger than the old one.

In this paper, we study various properties of the notion of δ_ω-open set defined by Al-Jarrah et al [4]. Although this notion is weaker than the notion of δ-open set defined by Velićko [15], it is stronger than the notion of open set. Also, we define and study the notion of δ_ω-continuous which is weaker than θ_ω-continuous defined by Al Ghour [2]. Furthermore, we obtain several characterizations of δ_ω-continuous functions and investigate their some fundamental properties. Finally, we investigate the relationships among the notions of weakly δ_ω-continuous, ω-δ-continuous and separation axioms.

2. Preliminaries

Throughout this present paper, (X,τ) and (Y,σ) (briefly X and Y) represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open, clopen) sets of (X,τ) is denoted $C(X)$ (resp. $O(X)$ or τ, $CO(X)$) and the family of all closed (resp. open) sets of X containing a point x of X is denoted by $C(X,x)$ (resp. $O(X,x)$). The cocountable topology on X, τ_{coc}; the topology whose open sets are the empty set and complements of subsets of X which are at most countable. The cofinite topology on X, τ_{cof}; the topology whose open sets are the empty set and complements of subsets of X which are at most finite. The indiscrete topology on X, τ_{ind}; the usual topology on $\mathbb{R}$, τ_u.
We recall the following definitions which will be used throughout this paper.

Definition 2.1. *A subset A of a space X is called:*
(a) *regular open [14] if $A = int(cl(A))$. The complement of a regular open set is called regular closed. The family of all regular open sets is denoted by $RO(X)$. A point $x \in X$ is said to be the δ-cluster point [14] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. If $A = \delta$-$cl(A)$, then A is called δ-closed [14]. The complement of a δ-closed set is called δ-open. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by δ-$int(A)$. The family of all δ-open sets of (X,τ) is a topology on X and is denoted by τ_δ.*

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(b) ω-open [9] if for every $x \in A$ there exists an open set U containing x such that $U \setminus A$ is countable. The complement of an ω-open set is said to be ω-closed [9]. The intersection of all ω-closed sets containing A is called the ω-closure of A and is denoted by ω-$cl(A)$. The family of all ω-open (resp. ω-closed) sets in (X,τ) is a topology on X and is denoted by τ_ω (resp. $\omega C(X)$).

Definition 2.2. *A point $x \in X$ is said to be the θ-cluster point [15] of A if $cl(U) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by θ-$cl(A)$. If $A = \theta$-$cl(A)$, then A is called θ-closed [15]. The complement of a θ-closed set is called θ-open. The set $\{x | (\exists U \in O(X,x))(cl(U) \subseteq A)\}$ is called the θ-interior of A and is denoted by θ-$int(A)$. The family of all θ-open sets of (X,τ) is a topology on X and is denoted by τ_θ.*

Lemma 2.3. *[5] A subset A of a space X is ω-open iff for each $x \in A$ there exists $U \in \tau$ such that $x \in U$ and $U \setminus A$ is countable.*

Corollary 2.4. *[5] A subset A of a space X is ω-open iff for each $x \in A$ there exists $U \in \tau$ and a countable set C such that $x \in U \setminus C \subseteq A$.*

Corollary 2.5. *[5] If X is a countable set then in the space X every subset is ω-open.*

Definition 2.6. *Let (X,τ) be a topological space. Then the space X is called:*
(a) Locally indiscrete [10] if $\tau = RO(X)$.
(b) Locally countable [13] if for each $x \in X$, there exists $U \in \tau$ such that $x \in U$ and U is countable.
(c) Anti-locally countable [13] if for each $U \in \tau \setminus \{\emptyset\}$ is uncountable.
(d) ω-regular [1] if for each closed set $F \subseteq X$ and $x \in X \setminus F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.
(e) ω-locally indiscrete [2] if for every open set in X is ω-closed.

Lemma 2.7. *(a) [2] Every locally indiscrete topological space is ω-locally indiscrete.*
(b) [3] Every locally countable topological space is ω-locally indiscrete.
(c) [1] If X is anti-locally countable space, then for all $A \in \tau_\omega$, ω-$cl(A) = cl(A)$.
(d) [1] A topological space X is ω-regular iff for each $U \in \tau$ and each $x \in U$ there is $V \in \tau$ such that $x \in V \subseteq \omega$-$cl(V) \subseteq U$.
(e) [2] A topological space X is locally indiscrete iff every open set in X is closed.

Definition 2.8. *Let A be a subset of a topological space (X,τ).*
(a) A point $x \in X$ is in the θ_ω-closure [2] of A ($x \in \theta_\omega$-$cl(A)$) iff ω-$cl(U) \cap A \neq \emptyset$ for any $U \in \tau$ with $x \in U$.
(b) A set A is called θ_ω-closed [2] iff θ_ω-$cl(A) = A$. A set A is called θ_ω-open [2] iff its complement is θ_ω-closed. Also, the family of all θ_ω-open (resp. θ_ω-closed) sets in (X,τ) is denoted by τ_{θ_ω} (resp. $\theta_\omega C(X)$).

Definition 2.9. *Let A be a subset of a topological space (X,τ).*
(a) A point $x \in X$ is in the δ_ω-closure [4] of A ($x \in \delta_\omega$-$cl(A)$) if $int(\omega$-$cl(U)) \cap A \neq \emptyset$ for any $U \in \tau$ with $x \in U$.
(b) A set A is called δ_ω-closed [4] iff δ_ω-$cl(A) = A$. A set A is called δ_ω-open iff its complement is δ_ω-closed. Also, the family of all δ_ω-open (resp. δ_ω-closed) sets in (X,τ) is denoted by τ_{δ_ω} (resp. $\delta_\omega C(X)$).

Lemma 2.10. *[4] Let A be a subset of a topological space X. Then the following hold. $cl(A) \subseteq \delta_\omega$-$cl(A) \subseteq \delta$-$cl(A)$.*

Corollary 2.11. *[4] Let (X,τ) be a topological space. Then $\tau_\delta \subseteq \tau_{\delta_\omega} \subseteq \tau$.*

Lemma 2.12. *[4] Let A be a subset of a topological space (X,τ). Then the following hold.*
(a) $cl(A) = \delta_\omega$-$cl(A)$ for each $A \in \tau_\omega$.
(b) $cl(A) = \delta_\omega$-$cl(A) = \delta$-$cl(A)$ for each $A \in \tau$.

Lemma 2.13. *[4] Let (X,τ) be a topological space. Then τ_{δ_ω} is a topology on X.*

Theorem 2.14. *[4] Let (X,τ) be a topological space and $A \subseteq X$. Then $A \in \tau_{\delta_\omega}$ if and only if for each $x \in A$, there exists $U \in \tau$ such that $x \in U \subseteq int(\omega$-$cl(U)) \subseteq A$.*

3. On δ_ω-closure operator and δ_ω-open sets

Theorem 3.1. *Let A be a subset of a topological space X. Then the following hold.*
(a) $cl(A) \subseteq \delta_\omega$-$cl(A) \subseteq \theta_\omega$-$cl(A)$,
(b) If A is δ-closed, then A is δ_ω-closed,
(c) If A is δ_ω-closed, then A is closed.

Proof. (a) Let $x \in \delta_\omega$-$cl(A)$.
$x \in \delta_\omega\text{-}cl(A) \Rightarrow (\forall U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A \neq \emptyset) \Rightarrow (\forall U \in O(X,x))(\omega\text{-}cl(U) \cap A \neq \emptyset) \Rightarrow x \in \theta_\omega\text{-}cl(A)$.
(b) Let $A \in \delta C(X)$.
$A \in \delta C(X) \Rightarrow A = \delta\text{-}cl(A) \overset{\text{Lemma 2.10}}{\Rightarrow} A = \delta_\omega\text{-}cl(A) \Rightarrow A \in \delta_\omega C(X)$.
(c) Let $A \in \delta_\omega C(X)$.
$A \in \delta_\omega C(X) \Rightarrow A = \delta_\omega\text{-}cl(A) \overset{\text{Lemma 2.10}}{\Rightarrow} A = cl(A) \Rightarrow A \in C(X)$. □

Theorem 3.2. *Let X be an ω-locally indiscrete topological space and $A \subseteq X$. Then the following hold.*
(a) $cl(A) = \delta_\omega$-$cl(A)$,
(b) If A is closed in X, then A is δ_ω-closed in X.

Proof. (a) Let $x \in \delta_\omega$-$cl(A)$.

$$\left.\begin{array}{r} x \in \delta_\omega\text{-}cl(A) \Rightarrow (\forall U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A \neq \emptyset) \Rightarrow (\forall U \in O(X,x))(\omega\text{-}cl(U) \cap A \neq \emptyset) \\ X \text{ is } \omega\text{-locally indiscrete} \end{array}\right\} \Rightarrow (\forall U \in O(X,x))(U \cap A \neq \emptyset)$$

$\Rightarrow x \in cl(A)$.
Therefore $cl(A) = \delta_\omega\text{-}cl(A)$.
(b) Let $A \in C(X)$.
$A \in C(X) \Rightarrow A = cl(A) \overset{(a)}{\Rightarrow} A = \delta_\omega\text{-}cl(A) \Rightarrow A \in \delta_\omega C(X)$. □

Corollary 3.3. *Let X be locally indiscrete and $A \subseteq X$. Then the following hold.*
(a) $cl(A) = \delta_\omega\text{-}cl(A)$,
(b) *If A is closed in X, then A is δ_ω-closed in X.*

Corollary 3.4. *Let X be locally countable and $A \subseteq X$. Then the following hold.*
(a) $cl(A) = \delta_\omega\text{-}cl(A)$,
(b) *If A is closed in X, then A is δ_ω-closed in X.*

Theorem 3.5. *Let X be anti-locally countable and $A \subseteq X$. Then the following hold.*
(a) $\delta\text{-}cl(A) = \delta_\omega\text{-}cl(A)$,
(b) *If A is δ_ω-closed in X, then A is δ-closed in X.*

Proof. (a) Let $x \in \delta\text{-}cl(A)$.
$x \in \delta\text{-}cl(A) \Rightarrow (\forall U \in O(X,x))(int(cl(U)) \cap A \neq \emptyset) \overset{\text{Hypothesis}}{\Rightarrow} (\forall U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A \neq \emptyset) \Rightarrow x \in \delta_\omega\text{-}cl(A)$
Then $\delta\text{-}cl(A) \subseteq \delta_\omega\text{-}cl(A)$. Thus $\delta\text{-}cl(A) = \delta_\omega\text{-}cl(A)$.

(b) Let $A \in \delta_\omega C(X)$.
$A \in \delta_\omega C(X) \Rightarrow A = \delta_\omega\text{-}cl(A) \overset{(a)}{\Rightarrow} A = \delta\text{-}cl(A) \Rightarrow A \in \delta C(X)$. □

Theorem 3.6. *Let X be a topological space. Then the following hold.*
(a) *If $A \subseteq B \subseteq X$, then $\delta_\omega\text{-}cl(A) \subseteq \delta_\omega\text{-}cl(B)$.*
(b) *$\delta_\omega\text{-}cl(A \cup B) = \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B)$ for each subsets $A, B \subseteq X$.*
(c) *$\delta_\omega\text{-}cl(A)$ is closed in X for each subset $A \subseteq X$.*

Proof. (a) Let $x \in \delta_\omega\text{-}cl(A)$ and $A \subseteq B$.
$$\left.\begin{array}{r} x \in \delta_\omega\text{-}cl(A) \Rightarrow (\forall U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A \neq \emptyset) \\ A \subseteq B \end{array}\right\} \Rightarrow (\forall U \in O(X,x))(int(\omega\text{-}cl(U)) \cap B \neq \emptyset) \Rightarrow x \in \delta_\omega\text{-}cl(B).$$
(b) Let A and B be subsets of X.
$$\left.\begin{array}{l} A \subseteq A \cup B \overset{(a)}{\Rightarrow} \delta_\omega\text{-}cl(A) \subseteq \delta_\omega\text{-}cl(A \cup B) \\ B \subseteq A \cup B \overset{(a)}{\Rightarrow} \delta_\omega\text{-}cl(B) \subseteq \delta_\omega\text{-}cl(A \cup B) \end{array}\right\} \Rightarrow \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B) \subseteq \delta_\omega\text{-}cl(A \cup B) \ldots (1)$$
Let $x \notin \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B)$.
$x \notin \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B) \Rightarrow (x \notin \delta_\omega\text{-}cl(A))(x \notin \delta_\omega\text{-}cl(B))$
$$\left.\begin{array}{r} \Rightarrow (\exists U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A = \emptyset)(\exists V \in O(X,x))(int(\omega\text{-}cl(V)) \cap B = \emptyset) \\ W = U \cap V \end{array}\right\} \Rightarrow$$
$\Rightarrow (W \in O(X,x))(int(\omega\text{-}cl(W)) \cap A = \emptyset)(int(\omega\text{-}cl(W)) \cap B = \emptyset)$
$\Rightarrow (W \in O(X,x))(int(\omega\text{-}cl(W)) \cap (A \cup B) = \emptyset)$
$\Rightarrow x \notin \delta_\omega\text{-}cl(A \cup B)$
It means that $\delta_\omega\text{-}cl(A \cup B) \subseteq \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B) \ldots (2)$
$(1), (2) \Rightarrow \delta_\omega\text{-}cl(A \cup B) = \delta_\omega\text{-}cl(A) \cup \delta_\omega\text{-}cl(B)$.
(c) Suppose that $x \notin \delta_\omega\text{-}cl(A)$.
$$\begin{array}{rcl} x \notin \delta_\omega\text{-}cl(A) & \Rightarrow & (\exists U \in O(X,x))(int(\omega\text{-}cl(U)) \cap A = \emptyset) \\ & \Rightarrow & (\exists U \in O(X,x))(U \cap A = \emptyset) \\ & \Rightarrow & (\exists U \in O(X,x))(U \cap \delta_\omega\text{-}cl(A) = \emptyset \vee U \cap \delta_\omega\text{-}cl(A) \neq \emptyset) \\ & \Rightarrow & (\exists U \in O(X,x))(U \cap \delta_\omega\text{-}cl(A) = \emptyset) \vee \underbrace{(\exists U \in O(X,x))(U \cap \delta_\omega\text{-}cl(A) \neq \emptyset)}_{False} \\ & \Rightarrow & (\exists U \in O(X,x))(U \cap \delta_\omega\text{-}cl(A) = \emptyset) \\ & \Rightarrow & x \notin cl(\delta_\omega\text{-}cl(A)) \end{array}$$
Therefore $\delta_\omega\text{-}cl(A) = cl(\delta_\omega\text{-}cl(A))$ which means that $\delta_\omega\text{-}cl(A) \in C(X)$. □

Corollary 3.7. *Every open ω-closed set in a topological space is δ_ω-open.*

Proof. Let $A \in O(X) \cap \omega C(X)$ and $x \in A$.
$x \in A \in O(X) \cap \omega C(X) \Rightarrow (A \in O(X,x))(A \in \omega C(X))$
$$\left.\begin{array}{r} \Rightarrow \omega\text{-}cl(A) = A \in O(X,x) \Rightarrow x \in A = int(A) = int(\omega\text{-}cl(A)) \\ U = A \end{array}\right\} \Rightarrow (U \in O(X,x))(int(\omega\text{-}cl(U)) \subseteq A).$$ □

Corollary 3.8. *Every countable open set in a topological space is δ_ω-open.*

Proof. Let $U \in O(X)$ and $|U| \leqslant \aleph_0$.
$$\left.\begin{array}{r} |U| \leqslant \aleph_0 \Rightarrow U \in \omega C(X) \\ U \in O(X) \end{array}\right\} \overset{\text{Corollary 3.7}}{\Rightarrow} U \in \delta_\omega O(X).$$ □

Lemma 3.9. *[6] Let (X,τ) and (Y,σ) be two topological spaces.*
(a) $(\tau\times\sigma)_\omega \subseteq \tau_\omega\times\sigma_\omega$,
(b) If $A\subseteq X$ and $B\subseteq Y$, then $\omega\text{-}cl(A)\times\omega\text{-}cl(B)\subseteq\omega\text{-}cl(A\times B)$.

Theorem 3.10. *Let (X,τ) and (Y,σ) be two topological spaces. If $G\in(\tau\times\sigma)_{\delta_\omega}$, then $\pi_X(G)\in\tau_{\delta_\omega}$ and $\pi_Y(G)\in\sigma_{\delta_\omega}$.*

Proof. Let $x\in\pi_X(G)$ and $(x,y)\in G$.

$$(x,y)\in G\in(\tau\times\sigma)_{\delta_\omega} \overset{\text{Theorem 2.14}}{\Rightarrow} (\exists H\in O(X\times Y,(x,y)))(H\subseteq int(\omega\text{-}cl(H))\subseteq G)$$
$$\Rightarrow (\exists U\in O(X,x))(\exists V\in O(Y,y))(U\times V\subseteq H\subseteq int(\omega\text{-}cl(H))\subseteq G)$$
$$\begin{aligned}\overset{\text{Lemma 3.9}}{\Rightarrow} (U\in O(X,x))(V\in O(Y,y))(U\times V &\subseteq int(\omega\text{-}cl(U))\times int(\omega\text{-}cl(V))\\ &= int(\omega\text{-}cl(U)\times\omega\text{-}cl(V))\\ &\subseteq int(\omega\text{-}cl(U\times V))\\ &\subseteq int(\omega\text{-}cl(H))\subseteq G)\end{aligned}$$
$$\Rightarrow (U\in O(X,x))(U\subseteq int(\omega\text{-}cl(U))\subseteq\pi_X(G)).$$

Then we have $\pi_X(G)\in\tau_{\delta_\omega}$. Similarly $\pi_Y(G)\in\sigma_{\delta_\omega}$. □

Definition 3.11. *A function $f:(X,\tau)\to(Y,\sigma)$ is called contra open [7] if $f[U]$ is closed in Y for each open subset U in X.*

Theorem 3.12. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. If $f:(X,\tau)\to(Y,\sigma)$ is open and $f:(X,\tau)\to(Y,\sigma_\omega)$ is contra open, then $f:(X,\tau_\delta)\to(Y,\sigma_{\delta_\omega})$ is open.*

Proof. Let $A\in\tau_\delta$ and $y\in f[A]$.

$$\left.\begin{array}{r} y\in f[A]\Rightarrow(\exists x\in A)(y=f(x))\\ A\in\tau_\delta\subseteq\tau\end{array}\right\}\Rightarrow \left.\begin{array}{r}(\exists U\in O(X,x))(U\subseteq A)\\ f\text{ is }(\tau\text{-}\sigma)\text{ open}\end{array}\right\}\Rightarrow \left.\begin{array}{r}(f[U]\in O(Y,f(x))=O(Y,y))(f[U]\subseteq f[A])\\ f\text{ is }(\tau\text{-}\sigma_\omega)\text{ contra open}\end{array}\right\}\Rightarrow$$
$$\Rightarrow (f[U]\in O(Y,y))(f[U]=int(\omega\text{-}cl(f[U]))\subseteq f[A]).$$ □

Theorem 3.13. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. If $f:(X,\tau)\to(Y,\sigma)$ is open and $f:(X,\tau_\omega)\to(Y,\sigma_\omega)$ is contra open, then $f:(X,\tau_{\delta_\omega})\to(Y,\sigma_{\delta_\omega})$ is open.*

Proof. Let $A\in\tau_{\delta_\omega}$ and $y\in f[A]$.

$$\left.\begin{array}{r} y\in f[A]\Rightarrow(\exists x\in A)(y=f(x))\\ A\in\tau_{\delta_\omega}\end{array}\right\}\Rightarrow \left.\begin{array}{r}(\exists U\in O(X,x))(U\subseteq int(\omega\text{-}cl(U))\subseteq A)\\ f\text{ is }(\tau\text{-}\sigma)\text{ open}\end{array}\right\}\Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(f[U]\in O(Y,y))(f[U]\subseteq f[int(\omega\text{-}cl(U))]\subseteq f[A])\\ f\text{ is }(\tau_\omega\text{-}\sigma_\omega)\text{ contra open}\end{array}\right\}\Rightarrow \left.\begin{array}{r}(f[U]\in O(Y,y))(f[U]=int(\omega\text{-}cl(f[U]))\subseteq f[int(\omega\text{-}cl(U))])\subseteq f[A])\\ V=f[U]\end{array}\right\}\Rightarrow$$
$$\Rightarrow (V\in O(Y,y))(V=int(\omega\text{-}cl(V))\subseteq f[A]).$$ □

Definition 3.14. *A function $f:(X,\tau)\to(Y,\sigma)$ is called contra continuous [8] if $f^{-1}[V]$ is closed in X for each open subset V in Y.*

Theorem 3.15. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. If $f:(X,\tau)\to(Y,\sigma)$ is continuous and $f:(X,\tau_\omega)\to(Y,\sigma_\omega)$ is contra continuous, then $f:(X,\tau_{\delta_\omega})\to(Y,\sigma_{\delta_\omega})$ is continuous.*

Proof. Let $B\in\tau_{\delta_\omega}$ and $x\in f^{-1}[B]$.

$$\left.\begin{array}{r} x\in f^{-1}[B]\Rightarrow f(x)\in B\\ B\in\tau_{\delta_\omega}\end{array}\right\}\Rightarrow \left.\begin{array}{r}(\exists V\in O(Y,f(x)))(V\subseteq int(\omega\text{-}cl(V))\subseteq B)\\ f\text{ is }(\tau\text{-}\sigma)\text{ continuous}\end{array}\right\}\Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(f^{-1}[V]\in O(X,x))(f^{-1}[V]\subseteq f^{-1}[int(\omega\text{-}cl(V))]\subseteq f^{-1}[B]\\ f\text{ is }(\tau_\omega\text{-}\sigma_\omega)\text{ contra continuous}\end{array}\right\}\Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(f^{-1}[V]\in O(X,x))(f^{-1}[V]=int(\omega\text{-}cl(f^{-1}[V]))\subseteq f^{-1}[int(\omega\text{-}cl(V))]\subseteq f^{-1}[B])\\ U=f^{-1}[V]\end{array}\right\}\Rightarrow$$
$$\Rightarrow (U\in O(X,x))(U=int(\omega\text{-}cl(U))\subseteq f^{-1}[B]).$$ □

Definition 3.16. *A topological space (X,τ) is said to be ω-T_2 [2] if for any pair (x,y) of distinct points in X, there exist $U\in\tau$, $V\in\tau_\omega$ such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$.*

Theorem 3.17. *Let (X,τ) be a ω-locally indiscrete space. Then (X,τ) is ω-T_2 space if and only if $\delta_\omega\text{-}cl(\{x\})=\{x\}$ for each $x\in X$.*

Proof. $(\Rightarrow)$: Let (X,τ) be an ω-T_2 space. Suppose that $x\in X$ and $\delta_\omega\text{-}cl(\{x\})\neq\{x\}$.

$$\left.\begin{array}{r}\delta_\omega\text{-}cl(\{x\})\neq\{x\}\Rightarrow(\exists y\in X)(y\in\delta_\omega\text{-}cl(\{x\})\setminus\{x\})\Rightarrow(y\in\delta_\omega\text{-}cl(\{x\}))(y\neq x)\\ (X,\tau)\text{ is }\omega\text{-}T_2\text{ space}\end{array}\right\}\Rightarrow$$
$$\Rightarrow (y\in\delta_\omega\text{-}cl(\{x\}))(\exists U\in\omega O(X,x))(\exists V\in O(Y,y))(U\cap V=\emptyset)$$
$$\Rightarrow (int(\omega\text{-}cl(V))\cap\{x\}\neq\emptyset)(U\in\omega O(X,x))(V\subseteq\setminus U)$$
$$\Rightarrow (x\in int(\omega\text{-}cl(V)))(U\in\omega O(X,x))(int(\omega\text{-}cl(V))\subseteq int(\omega\text{-}cl(X\setminus U))=int(X\setminus U)\subseteq X\setminus U)$$
$$\Rightarrow (x\in int(\omega\text{-}cl(V)))(U\in\omega O(X,x))(int(\omega\text{-}cl(V))\cap U=\emptyset)$$
$$\Rightarrow (int(\omega\text{-}cl(V))\cap U\neq\emptyset)(int(\omega\text{-}cl(V))\cap U=\emptyset)$$

This is a contradiction.

$(\Leftarrow)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ \text{Hypothesis}\end{array}\right\}\Rightarrow \left.\begin{array}{r} x\notin \delta_\omega\text{-}cl(\{y\})\Rightarrow(\exists U\in O(X,x))(int(\omega\text{-}cl(U))\cap\{y\}=\emptyset)\\ (X,\tau)\text{ is }\omega\text{-locally indiscrete}\end{array}\right\}\Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U\in O(X,x))(int(\omega\text{-}cl(U))\in\omega C(X))(int(\omega\text{-}cl(U))\cap\{y\}=\emptyset)\\ V=X\setminus int(\omega\text{-}cl(U))\end{array}\right\}\Rightarrow(U\in O(X,x))(V\in\omega O(X,y))(U\cap V=\emptyset).$$
□

4. On δ_ω-continuity

Definition 4.1. *A function $f : X \to Y$ is said to be δ-continuous [12] if for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f[int(cl(U))] \subseteq int(cl(V))$.*

Definition 4.2. *A function $f : X \to Y$ is said to be δ_ω-continuous if for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f[int(cl(U))] \subseteq int(\omega\text{-}cl(V))$.*

Theorem 4.3. *Let $f : X \to Y$ be a function. If f is δ_ω-continuous, then it is δ-continuous.*

Proof. Straightforward. □

Remark 4.4. *Every δ-continuous function need not be δ_ω-continuous as shown by the following example.*

Example 4.5. *Consider the function $f : (\mathbb{N}, \tau_{ind}) \to (\mathbb{N}, \tau_{cof})$, where $\mathbb{N}$ is the set of all natural numbers, defined as $f(x) = x$. The function f is δ-continuous but not δ_ω-continuous.*

Theorem 4.6. *If $f : X \to Y$ is δ-continuous and Y is an anti-locally countable space, then f is δ_ω-continuous.*

Proof. Let $x \in X$ and let V be any open subset of Y containing $f(x)$.

$$\left.\begin{array}{r}(x\in X)(V\in O(Y,f(x)))\\ f\text{ is }\delta\text{-continuous}\end{array}\right\}\Rightarrow\left.\begin{array}{r}(\exists U\in O(X,x))(f[int(cl(U))]\subseteq int(cl(V)))\\ Y\text{ is anti-locally countable}\end{array}\right\}\Rightarrow(\exists U\in O(X,x))(f[int(cl(U))]\subseteq int(\omega\text{-}cl(V)).$$
□

The following two examples show that the notions of continuity and δ_ω-continuity are independent.

Example 4.7. *Consider the function $f : (\mathbb{R}, \tau_{ind}) \to (\mathbb{R}, \tau_{cof})$ defined as $f(x) = x$.*
It is obvious that f is δ_ω-continuous but not continuous.

Example 4.8. *Consider the function $f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau)$ where $\tau = \{\mathbb{N}, \emptyset, \{1\}\}$ and $f(x) = x$. It is not difficult to see that f is continuous but not δ_ω-continuous.*

Theorem 4.9. *Let $f : X \to Y$ be a function. If f is δ_ω-continuous and Y is ω-regular, then f is continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{l}V\in O(Y,f(x))\\ Y\text{ is }\omega\text{-regular}\end{array}\right\}\Rightarrow\left.\begin{array}{r}(\exists H\in O(Y,f(x)))(H\subseteq\omega\text{-}cl(H)\subseteq V)\\ f\text{ is }\delta_\omega\text{-continuous}\end{array}\right\}\Rightarrow$$
$$\Rightarrow(\exists U\in O(X,x))(f[U]\subseteq f[int(cl(U))]\subseteq int(\omega\text{-}cl(H))\subseteq int(V)=V).$$
□

Definition 4.10. *A function $f : X \to Y$ is said to be weakly continuous [11] if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists an open subset U in X containing x such that $f[U] \subseteq cl(V)$.*

Definition 4.11. *A function $f : X \to Y$ is said to be ω-δ-continuous if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists an open subset U in X containing x such that $f[int(\omega\text{-}cl(U))] \subseteq cl(V)$.*

Theorem 4.12. *Let $f : X \to Y$ be a function. If f is ω-δ-continuous function, then it is weakly continuous.*

Proof. Straightforward. □

Theorem 4.13. *Let $f : X \to Y$ be a function. If f is weakly continuous and X is ω-locally indiscrete, then f is ω-δ-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x\in X)(V\in O(Y,f(x)))\\ f\text{ is weakly continuous}\end{array}\right\}\Rightarrow\left.\begin{array}{r}(\exists U\in O(X,x))(f[U]\subseteq cl(V))\\ X\text{ is }\omega\text{-locally indiscrete}\end{array}\right\}\Rightarrow(\exists U\in O(X,x))(f[U]=f[int(\omega\text{-}cl(U))]\subseteq cl(V)).$$
□

Corollary 4.14. *If $f : X \to Y$ is weakly continuous and X is locally indiscrete, then f is ω-δ-continuous.*

Proof. By Lemma 2.7 and Theorem 4.13. □

Corollary 4.15. *If $f : X \to Y$ is weakly continuous and X is locally countable, then f is ω-δ-continuous.*

Proof. By Lemma 2.7 and Theorem 4.13. □

Theorem 4.16. *Let $f : X \to Y$ be a function. If f is weakly continuous and X is ω-regular, then f is ω-δ-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is weakly continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists H \in O(X,x))(f[H] \subseteq cl(V)) \\ X \text{ is } \omega\text{-regular}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in O(X,x))(f[int(\omega\text{-}cl(U))] \subseteq f[\omega\text{-}cl(U)] \subseteq f[H] \subseteq cl(V)).$$
□

Theorem 4.17. *Let $f : X \to Y$ be a function. If f is δ-continuous, then it is ω-δ-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.
$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is } \delta\text{-continuous}\end{array}\right\} \Rightarrow (\exists U \in O(X,x))(f[int(cl(U))] \subseteq int(cl(V)))$$
$$\Rightarrow (\exists U \in O(X,x))(f[int(\omega\text{-}cl(U))] \subseteq f[int(cl(U))] \subseteq int(cl(V)) \subseteq cl(V)).$$
□

Remark 4.18. *Every ω-δ-continuous function need not be δ-continuous as shown by the following example.*

Example 4.19. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be identity function. Then f is ω-δ-continuous. On the other hand, it is proved in [[12] Example 4.5] that f is not δ-continuous.*

Theorem 4.20. *Let $f : X \to Y$ be a function. If f is open ω-δ-continuous and X is anti-locally countable, then f is δ-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is } \omega\text{-}\delta\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in O(X,x))(f[int(\omega\text{-}cl(U))] \subseteq cl(V)) \\ X \text{ is anti-locally countable}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U \in O(X,x))(f[int(\omega\text{-}cl(U))] = f[int(cl(U))] \subseteq cl(V)) \\ f \text{ is open}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in O(X,x))(int(f[int(\omega\text{-}cl(U))]) = f[int(cl(U))] \subseteq int(cl(V))).$$
□

Definition 4.21. *A function $f : X \to Y$ is said to be weakly δ_ω-continuous if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists an open subset U in X containing x such that $f[U] \subseteq int(\omega\text{-}cl(V))$.*

Theorem 4.22. *Let $f : X \to Y$ be a function. If f is weakly δ_ω-continuous, then it is weakly continuous.*

Proof. Straightforward. □

Remark 4.23. *Every weakly continuous function need not be δ_ω-weakly continuous as shown by the following example.*

Example 4.24. *Consider the identity function $f : (\mathbb{N},\tau) \to (\mathbb{N},\sigma)$ where $\tau = \{\emptyset,\mathbb{N}\}$ and $\sigma = \{\emptyset,\mathbb{N},\{1\}\}$. Then f is weakly continuous but not weakly δ_ω-continuous.*

Theorem 4.25. *Let $f : X \to Y$ be a function. If f is open weakly continuous and Y is anti-locally countable, then f is weakly δ_ω-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is weakly continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in O(X,x))(f[U] \subseteq cl(V)) \\ Y \text{ is anti-locally countable}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in O(X,x))(f[U] \subseteq cl(V) = \omega\text{-}cl(V)) \\ f \text{ is open}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in O(X,x))(int(f[U]) = f[U] \subseteq int(\omega\text{-}cl(V))).$$
□

Theorem 4.26. *Let $f : X \to Y$ be a function. If f is continuous, then it is weakly δ_ω-continuous.*

Proof. Straightforward. □

Remark 4.27. *Every weakly δ_ω-continuous function need not be continuous as shown by the following example.*

Example 4.28. *Consider the identity function $f : (\mathbb{R},\tau_u) \to (\mathbb{R},\tau_{coc})$. Then f is weakly δ_ω-continuous but not continuous.*

Theorem 4.29. *Let $f : X \to Y$ be a function. If f is weakly δ_ω-continuous and Y is ω-locally indiscrete, then f is continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is weakly } \delta_\omega\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in O(X,x))(f[U] \subseteq int(\omega\text{-}cl(V))) \\ Y \text{ is } \omega\text{-locally indiscrete}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in O(X,x))(f[U] \subseteq int(\omega\text{-}cl(V)) = int(V) = V).$$
□

Corollary 4.30. *If $f : X \to Y$ is weakly δ_ω-continuous and Y is locally indiscrete, then f is continuous.*

Proof. By Lemma 2.7 and Theorem 4.29. □

Corollary 4.31. *If $f : X \to Y$ is weakly δ_ω-continuous and Y is locally countable, then f is continuous.*

Proof. By Lemma 2.7 and Theorem 4.29. □

Theorem 4.32. *Let $f : X \to Y$ be a function. If f is δ_ω-continuous, then it is weakly δ_ω-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is } \delta_\omega\text{-continuous}\end{array}\right\} \Rightarrow \ (\exists U \in O(X,x))(f[int(cl(U))] \subseteq int(\omega\text{-}cl(V)))$$

$$\Rightarrow (\exists U \in O(X,x))(f[U] = f[int(U)] \subseteq f[int(cl(U))] \subseteq int(\omega\text{-}cl(V))).$$ □

Remark 4.33. *Every weakly δ_ω-continuous need not be δ_ω-continuous as shown by the following example.*

Example 4.34. *Consider by Example 4.8, $f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau)$ where $\tau = \{\mathbb{N}, \emptyset, \{1\}\}$ and $f(x) = x$. It is clear that f is continuous. Also, f is weakly δ_ω-continuous from Theorem 4.27. However, f is not δ_ω-continuous as shown in Example 4.8.*

Theorem 4.35. *Let $f : X \to Y$ be a function. If f is weakly δ_ω-continuous and X is locally indiscrete, then f is δ_ω-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f \text{ is weakly } \delta_\omega\text{-continuous}\end{array}\right\} \Rightarrow (\exists U \in O(X,x))(f[U] \subseteq int(\omega\text{-}cl(V))) \\ X \text{ is locally indiscrete}\end{array}\right\} \Rightarrow (\exists U \in O(X,x))(f[int(cl(U))] \subseteq int(\omega\text{-}cl(V))).$$ □

Corollary 4.36. *We have the following diagram from definitions and results obtained above.*

$$\begin{array}{ccccc} \textit{continuous} & \rightarrow & \textit{weakly } \delta_\omega\textit{-continuous} & \rightarrow & \textit{weakly continuous} \\ & \nearrow & & \nearrow & \uparrow \\ \delta_\omega\textit{-continuous} & \rightarrow & \delta\textit{-continuous} & \rightarrow & \omega\text{-}\delta\textit{-continuous} \end{array}$$

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References

[1] S. Al Ghour, Certain covering properties related to paracompactness, Ph.D. thesis, University of Jordan, Amman, Jordan, (1999).
[2] S. Al Ghour and B. Irshedat, The topology of θ_ω-open sets, Filomat 31 (16) (2017), 5369-5377.
[3] S. Al Ghour and B. Irshedat, On θ_ω-continuity, Heliyon, 6 (2) (2020).
[4] H.H. Al-Jarrah, A. Al-Rawshdeh, E.M. Al-Saleh, K.Y. Al-Zoubi, Characterizations of $R_\omega O(X)$ sets by using δ_ω-cluster points, Novi. Sad. J. Math., 2 (2019), 109-122.
[5] K.Y. Al-Zoubi and B. Al-Nashef, The topology of ω-open subsets, Al-Manarah Journal, 9 (2003), 169-179.
[6] K.Y. Al-Zoubi, On generalized ω-open sets, Int. J. Math. Math. Sci., 13 (2005), 2011-2021.
[7] C.W. Baker, Contra-open functions and contra-closed functions, Math. Today, 15 (1997), 19-24.
[8] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, Internat. J. Math. Sci., 19 (1996), 303-310.
[9] H.Z. Hdeib, ω-closed mappings, Rev. Colombiana Mat., 16 (1982), 65-78.
[10] D.S. Jankovic, On locally irreducible spaces, Ann. Soc. Sci. Bruxelles Ser. I, 97 (1983), 59-72.
[11] N. Levine, A decomposition of continuity in topological spaces, Amer. Math. Monthly. 68 (1961), 44-46.
[12] T. Noiri, On δ-continuous Functions, J. Korean Math. Soc. 16 (2) (1979), 161-166.
[13] C.M. Pareek, Hereditarily Lindelöf and hereditarily almost Lindelöf spaces, Math. Japon. 30 (1985), 635-639.
[14] M.H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375-381.
[15] N.V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. 78 (2) (1968), 103-118.