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# New Ostrowski Type Inequalities for Trigonometrically Convex Functions Via Classical Integrals

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**Highlights**
- Trigonometrically convex function is studied.
- Ostrowski type integral inequalities for trigonometrically convex functions were obtained.
- The results of the new type of integral inequalities are determined.

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**Abstract**

In the paper, we introduce the class of trigonometrically convex functions and using the Hölder, Hölder-Işcan, Power-mean and Improved power-mean integral inequality together with an identity we establish some new inequalities of Ostrowski-type for functions whose second derivatives are trigonometrically convex which is a special case of $h$-convex functions. Some applications for special means are also given.

## 1. INTRODUCTION

In 1938, Alexander Markovich Ostrowski proved the following important integral inequalty [1].

**Theorem 1.** Let $\varphi: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ (the interior the interval $I$), such that $\varphi' \in L[\rho_1, \rho_2]$, where $\rho_1, \rho_2 \in I$ $(\rho_1 < \rho_2)$, then

$$\left|\varphi(\omega) - \frac{1}{\rho_2 - \rho_1}\int_{\rho_1}^{\rho_2} \varphi(u)du\right| < \frac{M}{\rho_2 - \rho_1}\left[\frac{(\omega-\rho_1)^2 + (\rho_2-\omega)^2}{2}\right] \quad (1)$$

for every $\omega \in [\rho_1, \rho_2]$. This inequality is known as Ostrowski inequality. Ostrowski, a addressed the problem of estimation deviation from the integral mean of the function. In recent years ineqality (1) has attracted the attention of many researchers and based on Ostrowski's inequality studies to obtain various extensions, generalizations and applications has been included in the literature. See [2-9] and references therein.

The definition given below is well-known in the mathematical application analysis literature for convex functions.

**Definition 1.** A function $\varphi: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on interval $I$ if the following inequality

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$$\varphi(\tau\,\rho_1+(1-\tau\,)\rho_2)\leq\tau\,\varphi(\rho_1)+(1-\tau\,)\varphi(\rho_2) \tag{2}$$

is valid for every $\rho_1,\rho_2\in I$ and $\tau\in[0,1]$ and $\varphi$ is said to be concave on $I$ if the inequality (2) holds in reversed direction.

The inequalities shown below are known in the field of inequalities theory as Hermite-Hadamard [10].

**Theorem 2.** Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function on $I$ and $\rho_1,\rho_2\in I(\rho_1<\rho_2)$, then two inequalities holds:

$$\varphi\left(\frac{\rho_1+\rho_2}{2}\right)\leq\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du\leq\frac{\varphi(\rho_1)+\varphi(\rho_2)}{2}.$$

Convexity theory formed the basis of many inequalities in the pure and applied fields of mathematics. See [11-12] and the references therein.

**Definition 2.** Let $h: J\to\mathbb{R}$ be non-negative and for all $\rho_1\in J$, $h\neq 0$. When $\varphi$ is non-negative function, $\rho_1,\rho_2\in I$ and $\alpha\in(0,1)$, if we obtain

$$\varphi(\alpha\rho_1+(1-\alpha)\rho_2)\leq h(\alpha)\varphi(\rho_1)+h(1-\alpha)\varphi(\rho_2). \tag{3}$$

We say that $\varphi: I\to\mathbb{R}$ is an $h$-convex functions (or $\varphi\in SX(h,I)$) [13].

**Definition 3.** A non-negative function $\varphi: J\to\mathbb{R}$ is called trigonometrically convex function on interval $[\,\rho_1,\rho_2]$, if for every $\omega,\varpi\in[\,\rho_1,\rho_2]$ and $\tau\in[0,1]$,

$$\varphi(\tau\omega+(1-\tau)\varpi)\leq\sin\left(\frac{\pi\tau}{2}\right)\varphi(\omega)+\cos\left(\frac{\pi\tau}{2}\right)\varphi(\varpi). \tag{4}$$

The class of all trigonometrically convex functions on interval $I$ is denoted $TC(I)$ [14].

More clearly, every non-negative convex function is trigonometrically convex function and every trigonometrically convex function is $h$-convex function with $h(\tau)=sin\,(\frac{\pi\tau}{2})$.

For example, nonnegative constant functions are trigonometrically convex, since $\sin\left(\frac{\pi\tau}{2}\right)+\cos\left(\frac{\pi\tau}{2}\right)\geq 1$ for all $\tau\in[0,1]$.

In [15], a new refinement of the Hölder's integral inequality presented by İşcan is given below:

**Theorem 3. (Hölder-Işcan integral inequality)** Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $\varphi$ and $\gamma$ are real mappings on interval $[\rho_1,\rho_2]$. If $|\varphi|^q$ and $|\,\gamma|^q$ are integrable mappings on $[\rho_1,\rho_2]$, then

$$i)\ \int_{\rho_1}^{\rho_2}|\varphi(\omega)\,\gamma(\omega)|d\omega$$

$$\leq\frac{1}{\rho_2-\rho_1}\left\{\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\varphi(\omega)|^p d\omega\right)^{\frac{1}{p}}\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\,\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right. \tag{5}$$

$$+\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\varphi(\omega)|^p d\omega\right)^{\frac{1}{p}}\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\,\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\Bigg\}$$

$$ii)\quad \frac{1}{\rho_2-\rho_1}\left\{\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\varphi(\omega)|^p d\omega\right)^{\frac{1}{p}}\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\,\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\varphi(\omega)|^p d\omega\right)^{\frac{1}{p}}\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\,\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\int_{\rho_1}^{\rho_2}|\varphi(\omega)|^p d\omega\right)^{\frac{1}{p}}\left(\int_{\rho_1}^{\rho_2}|\,\gamma(\omega)|^q\, d\omega\right)^{\frac{1}{q}}.$$

Hölder-İşcan integral inequality clearly showed a better result than Hölder integral inequality. In [16], a variation of the hölder-işcan inequality is given follows:

**Theorem 4. (Improved power-mean**) Let $q \geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $\varphi$ and $\gamma$ are real mappings on interval $[a,\rho_2]$. If $|\varphi|$ and $|\varphi||\,\gamma|^q$ are integrable mappings on $[a,\rho_2]$, then

$$i)\quad \int_{\rho_1}^{\rho_2}|\varphi(\omega)\gamma(\omega)|d\omega$$

$$\leq\frac{1}{\rho_2-\rho_1}\left\{\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\varphi(\omega)|d\omega\right)^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\varphi(\omega)||\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right. \tag{6}$$

$$\left.+\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\varphi(\omega)|d\omega\right)^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\varphi(\omega)||\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right\}$$

$$ii)\quad \frac{1}{\rho_2-\rho_1}\left\{\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)\,|\varphi(\omega)|d\omega\right)^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_2}(\rho_2-\omega)|\varphi(\omega)|\,|\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\rho_1}^{\rho_2}(\omega-\rho_1)\,|\varphi(\omega)|d\omega\right)^{1-\frac{1}{q}}\left(\int_{a}^{\rho_2}(\omega-\rho_1)|\varphi(\omega)|\,|\gamma(\omega)|^q d\omega\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\int_{\rho_1}^{\rho_2}|\varphi(\omega)|d\omega\right)^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_2}|\varphi(\omega)||\gamma(\omega)|^q\, d\omega\right)^{\frac{1}{q}}.$$

Improved power-mean integral inequality showed a better result than Power-mean integral inequality.

The lemma we need to prove our main results is given below [17]:

**Lemma 1.** Let $I \subseteq \mathbb{R}$, $\varphi: I \to \mathbb{R}$ be a two times differentiable mapping on $I^{\circ}$ with $\varphi'' \in L[\rho_1, \rho_2]$ where $\rho_1, \rho_2 \in I$ with $\rho_1 < \rho_2$, then

$$\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)$$
$$=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2\varphi''(\tau\omega+(1-\tau)\rho_1)d\tau+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2\varphi''(t\omega+(1-\tau)\rho_2)d\tau$$

for each $\omega \in [\rho_1, \rho_2]$.

## 2. MAIN RESULTS

In this section, we have some Ostrowski type inequalities for functions whose second derivatives are trigonometrically convex function by using classical integral inequalities, Hölder, Hölder-İşcan, Power-mean and Improved power-mean integral inequality.

**Theorem 5.** Let $\varphi: I \subseteq \mathbb{R} \to \mathbb{R}$ be two times differentiable function on $I^{\circ}$ assume that $\varphi'' \in L[\rho_1, \rho_2]$, where $\rho_1, \rho_2 \in I$ ( $\rho_1 < \rho_2$ ) . If $|\varphi''|$ is trigonometrically convex on $[\rho_1, \rho_2]$, then

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \tag{7}$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left[\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_1)|\right]$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left[\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_2)|\right] .$$

**Proof:** Because the function $|\varphi''|$ trigonometrically convex, we have

$$|\varphi''(\tau\omega+(1-\tau)\rho_1)|\leq\sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|+\cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|$$

and

$$|\varphi''(\tau\omega+(1-\tau)\rho_2)|\leq\sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|+\cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|.$$

Combining Lemma 1 and trigonometrically convex of $|\varphi''|$, we conclude that

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|d\tau+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|d\tau$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|\right)d\tau$$

$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1 \tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|\right)d\tau$$
$$=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left[\int_0^1 \tau^2 sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|d\tau+\int_0^1 \tau^2 cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|d\tau\right]$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left[\int_0^1 \tau^2 sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|d\tau+\int_0^1 \tau^2 cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|d\tau\right]$$

where

$$\int_0^1 \tau^2\sin\left(\frac{\pi\tau}{2}\right)d\tau=\frac{8\pi-16}{\pi^3}\ ,\qquad \int_0^1 \tau^2\cos\left(\frac{\pi\tau}{2}\right)d\tau=\frac{2\pi^2-16}{\pi^3}.$$

By using the above integral calculations, we get

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left[\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_1)|\right]$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left[\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_2)|\right].$$

This is the desired conclusion.

**Corollary 1.** Under the assumptions of the Theorem 5 for $\omega=\frac{\rho_1+\rho_2}{2}$, we get the following the inequality:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right|$$
$$\leq\frac{(\rho_2-\rho_1)^2}{4}\left[\frac{4\pi-8}{\pi^3}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|+\frac{\pi^2-8}{\pi^3}A(|\varphi''(\rho_1)|,|\varphi''(\rho_2)|)\right],$$

where $A$ is the arithmetic mean.

**Theorem 6.** Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$ be two times differentiable function on $I^\circ$ with $\varphi''\in L[\rho_1,\rho_2]$, where $\rho_1,\rho_2\in I$ ( $\rho_1<\rho_2$). If for $p>1$, $|\varphi''|^q$ is trigonometrically convex on $[\rho_1,\rho_2]$, then the following holds:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}(|\varphi''(\omega)|^q+|\varphi''(\rho_1)|^q)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}(|\varphi''(\omega)|^q+|\varphi''(\rho_2)|^q)^{\frac{1}{q}}. \quad (8)$$

**Proof:** Because the function $|\varphi''|$ is trigonometrically convex, we get

$$|\varphi''(\tau\omega+(1-\tau)\rho_1)|^q \leq \sin\left(\frac{\pi\tau}{2}\right)|\varphi''\omega|^q + \cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q$$

and

$$|\varphi''(\tau\omega+(1-\tau)\rho_2)|^q \leq \sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + \cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q.$$

Combining lemma 1 and using the Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \\
&\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1 \tau^2|\varphi''(\tau\omega+(1-\tau)\rho_1)|d\tau+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1 \tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|d\tau \\
&\leq \frac{(\omega-a)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1|\varphi''(\tau\omega+(1-\tau)\rho_1)|^q d\tau\right)^{\frac{1}{q}} \\
&+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1|\varphi''(\tau\omega+(1-\tau)\rho_2)|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

Because $|\varphi''|^q$ is trigonometrically convex, we have inequalities

$$\begin{aligned}
&\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \\
&\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}} \\
&+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}} \\
&=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}(|\varphi''(\omega)|^q+|\varphi''(\rho_1)|^q)^{\frac{1}{q}} \\
&+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}(|\varphi''(\omega)|^q+|\varphi''(\rho_2)|^q)^{\frac{1}{q}}
\end{aligned}$$

where

$$\int_0^1 \tau^{2p}d\tau=\frac{1}{2p+1} \quad \text{and} \quad \int_0^1 sin\left(\frac{\pi\tau}{2}\right)d\tau=\int_0^1 cos\left(\frac{\pi\tau}{2}\right)d\tau=\frac{2}{\pi}\ .$$

Hence our theorem is proved.

**Corollary 2.** Under the assumptions of the Theorem 6 for $\omega = \frac{\rho_1+\rho_2}{2}$ and $p = q = 2$, we obtain following the inequality:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du - \varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right|$$
$$\leq \frac{(\rho_2-\rho_1)^2}{16}\left(\frac{2}{5\pi}\right)^{\frac{1}{2}}\left(\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|^2 + |\varphi''(\rho_1)|^2\right)^{\frac{1}{2}}$$
$$+\frac{(\rho_2-\rho_1)^2}{16}\left(\frac{2}{5\pi}\right)^{\frac{1}{2}}\left(\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|^2 + |\varphi''(\rho_2)|^2\right)^{\frac{1}{2}}.$$

**Corollary 3.** Under the assumptions of the Corollary 2 for $\varphi''(\rho_1) = \varphi''\left(\frac{\rho_1+\rho_2}{2}\right) = \varphi''(\rho_2)$, we have the following the inequality:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du - \varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right| \leq \frac{(\rho_2-\rho_1)^2}{8}\left(\frac{1}{5\pi}\right)^{\frac{1}{2}}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|.$$

**Theorem 7.** Let $\varphi: I \subset (0,\infty) \to \mathbb{R}$ be two times differentiable function on $I^\circ$ with $\varphi'' \in L[\rho_1,\rho_2]$, where $\rho_1,\rho_2 \in I$ $(\rho_1 < \rho_2)$. If for $p > 1$, $|\varphi''|^q$ is trigonometrically convex on $[\rho_1,\rho_2]$, then the following inequality

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du - \varphi(\omega) + \left(\omega - \frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \tag{9}$$
$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right) + |\varphi''(\rho_1)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right) + |\varphi''(\rho_1)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-a)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right) + |\varphi''(\rho_2)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-a)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right) + |\varphi''(\rho_2)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof:** Combining Hölder-İşcan integral inequality, Lemma 1 and trigonometrically convexity of $|f''|^q$ gives

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du - \varphi(\omega) + \left(\omega - \frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$
$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1 \tau^2|\varphi''(\tau\omega + (1-\tau)\rho_1)|d\tau + \frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1 \tau^2|\varphi''(\tau\omega + (1-\tau)\rho_2)|d\tau$$

$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 (1-\tau)\tau^{2p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 (1-\tau)\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p+1}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \tau\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 (1-\tau)\tau^{2p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 (1-\tau)\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^{2p+1}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \tau\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right)+|\varphi''(\rho_1)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right)+|\varphi''(\rho_2)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$

where

$$\int_0^1 (1-\tau)\tau^{2p}d\tau = \frac{1}{4p^2+6p+2},$$
$$\int_0^1 \tau^{2p+1}d\tau = \frac{1}{2p+2},$$
$$\int_0^1 (1-\tau)\sin\left(\frac{\pi\tau}{2}\right)d\tau = \int_0^1 \tau\cos\left(\frac{\pi\tau}{2}\right)d\tau = \frac{2\pi-4}{\pi^2},$$
$$\int_0^1 (1-\tau)\cos\left(\frac{\pi\tau}{2}\right)d\tau = \int_0^1 \tau\sin\left(\frac{\pi\tau}{2}\right)d\tau = \frac{4}{\pi^2}.$$

Therefore, the proof is completed.

**Remark 1.** Because $h\colon [0,\infty) \to \mathbb{R},\ h(r) = r^s, s \in (0,1]$, is a concave function, for every $u, v \geq 0$, we get

$$h\left(\frac{u+v}{2}\right) = \left(\frac{u+v}{2}\right)^s \geq \frac{h(u)+h(v)}{2} = \frac{u^s+v^s}{2}.$$

Using above inequality, if we consider the right hand-side of the inequality (9) again, we have

$$\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right)+|\varphi''(\rho_1)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$

$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4p^2+6p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{2\pi-4}{\pi^2}\right)+|\varphi''(\rho_2)|^q\left(\frac{4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left[|\varphi''(\omega)|^q\left(\frac{4}{\pi^2}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi-4}{\pi^2}\right)\right]^{\frac{1}{q}}$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}2\left[\frac{|\varphi''(\omega)|^q\left(\frac{2}{\pi}\right)+|\varphi''(\rho_1)|^q\left(\frac{2}{\pi}\right)}{2}\right]^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}2\left[\frac{|\varphi''(\omega)|^q\left(\frac{2}{\pi}\right)+|\varphi''(\rho_2)|^q\left(\frac{2}{\pi}\right)}{2}\right]^{\frac{1}{q}}$$
$$=\frac{(\omega-\rho_1)^3}{(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|\varphi''(\omega)|^q,|\varphi''(\rho_1)|^q)$$
$$+\frac{(\rho_2-\omega)^3}{(\rho_2-\rho_1)}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|\varphi''(\omega)|^q,|\varphi''(\rho_2)|^q).$$

This indicate that the inequality (9) is a better approximation than the inequality (8).

**Theorem 8.** Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$ be two times differentiable function on $I^\circ$ with $\varphi''\in L[\rho_1,\rho_2]$, where $\rho_1,\rho_2\in I$ $(\rho_1<\rho_2)$. If for $q\geq 1$, $|\varphi''|^q$ is trigonometrically convex on $[\rho_1,\rho_2]$, then the following holds:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \tag{10}$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|^q+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_1)|^q\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|^q+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_2)|^q\right)^{\frac{1}{q}}.$$

**Proof:** From the Lemma 1, trigonometrically convexity of $|f''|^q$ and power-mean integral inequality, we have inequality

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_1)|d\tau+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|d\tau$$
$$\leq\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1\tau^2d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_1)|^qd\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1\tau^2d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|^qd\tau\right)^{\frac{1}{q}}$$

$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^2 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1 \tau^2 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q + cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}}$$
$$=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|^q+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_1)|^q\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}|\varphi''(\omega)|^q+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_2)|^q\right)^{\frac{1}{q}}$$

where

$$\int_0^1 \tau^2 d\tau = \frac{1}{3},$$
$$\int_0^1 \tau^2 \sin\left(\frac{\pi\tau}{2}\right)d\tau = \frac{8\pi-16}{\pi^3}$$

and

$$\int_0^1 \tau^2 \cos\left(\frac{\pi\tau}{2}\right)d\tau = \frac{2\pi^2-16}{\pi^3}.$$

We eventually get the desired result and this completes the proof.

**Corollary 4.** Under the assumption of the Theorem 8 for $q = 1$, we get the conclusion of the Theorem 5.

**Corollary 5.** Under the assumption of the Theorem 8 with $\omega = \frac{\rho_1+\rho_2}{2}$ *and* $p = q = 2$, we obtain the inequality:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right|$$
$$\leq\frac{(\rho_2-\rho_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{2}}\left(\frac{8\pi-16}{\pi^3}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|^2+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_1)|^2\right)^{\frac{1}{2}}$$
$$+\frac{(\rho_2-\rho_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{2}}\left(\frac{8\pi-16}{\pi^3}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|^2+\frac{2\pi^2-16}{\pi^3}|\varphi''(\rho_2)|^2\right)^{\frac{1}{2}}.$$

**Corollary 6.** Under the assumption of Corollary 5 with $\varphi''(\rho_1) = \varphi''\left(\frac{\rho_1+\rho_2}{2}\right) = \varphi''(\rho_2)$, we get the following the inequality:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right|\leq\frac{(\rho_2-\rho_1)^2}{8}\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(\frac{\pi^2+4\pi-16}{\pi^3}\right)^{\frac{1}{2}}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|.$$

**Theorem 9.** Let $\varphi: I \subset (0,\infty) \to \mathbb{R}$ be twice differentiable function on $I^{\circ}$ with $\varphi'' \in L[\rho_1,\rho_2]$, where $\rho_1,\rho_2 \in I$ $(\rho_1 < \rho_2)$. If for $q \geq 1$, $|\varphi''|^q$ is trigonometrically convex on $[\rho_1,\rho_2]$, then the inequality holds:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right| \tag{11}$$

$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$

$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}.$$

**Proof:** Combining Lemma 1, the property of the trigonometrically convexity function of $|\varphi''|^q$ and Improved power-mean integral inequality, then we have inequality

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi(\omega)+\left(\omega-\frac{\rho_1+\rho_2}{2}\right)\varphi'(\omega)\right|$$

$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_1)|d\tau+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\int_0^1\tau^2|\varphi''(\tau\omega+(1-\tau)\rho_2)|d\tau$$

$$\leq \frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1(1-\tau)\tau^2d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\tau)\tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q\right.\right.$$

$$\left.\left.+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}}$$

$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\int_0^1\tau^3d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^3\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_1)|^q\right)d\tau\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1(1-\tau)\tau^2d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\tau)\tau^2\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\int_0^1\tau^3d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^3\left(sin\left(\frac{\pi\tau}{2}\right)|\varphi''(\omega)|^q+cos\left(\frac{\pi\tau}{2}\right)|\varphi''(\rho_2)|^q\right)d\tau\right)^{\frac{1}{q}}$$

$$=\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$

$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}$$

where

$$\int_0^1 \tau^3 d\tau=\frac{1}{4} \quad \text{and} \quad \int_0^1 (1-\tau)\tau^2 d\tau=\frac{1}{12},$$
$$\int_0^1 (1-\tau)\tau^2 \sin\left(\frac{\pi\tau}{2}\right)d\tau=\frac{96-4\pi^2-16\pi}{\pi^4},$$
$$\int_0^1 (1-\tau)\tau^2 \cos\left(\frac{\pi\tau}{2}\right)d\tau=\frac{32\pi-96}{\pi^4},$$
$$\int_0^1 \tau^3 \sin\left(\frac{\pi\tau}{2}\right)d\tau=\frac{12\pi^2-96}{\pi^4},$$
$$\int_0^1 \tau^3 \cos\left(\frac{\pi\tau}{2}\right)d\tau=\frac{2\pi^3-48\pi+96}{\pi^4}.$$

**Remark 2.** The inequality (11) gives a better approximation than the inequality (10).

**Proof:** Because the function $h:[0,\infty)\to\mathbb{R}$, $h(r)=r^s$, $s\in(0,1]$ is concave function, if we consider the right-hand-side of the inequality (11) again, we obtain,

$$\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(\omega-\rho_1)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{96-4\pi^2-16\pi}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{32\pi-96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$+\frac{(\rho_2-\omega)^3}{2(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\varphi''(\omega)|^q\left(\frac{12\pi^2-96}{\pi^4}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi^3-48\pi+96}{\pi^4}\right)\right)^{\frac{1}{q}}$$
$$\leq\frac{(\omega-\rho_1)^3}{(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi''(\omega)|^q\left(\frac{8\pi-16}{\pi^3}\right)+|\varphi''(\rho_1)|^q\left(\frac{2\pi^2-16}{\pi^3}\right)}{2}\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\omega)^3}{(\rho_2-\rho_1)}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi''(\omega)|^q\left(\frac{8\pi-16}{\pi^3}\right)+|\varphi''(\rho_2)|^q\left(\frac{2\pi^2-16}{\pi^3}\right)}{2}\right)^{\frac{1}{q}}$$

which is required result and this proves the remark.

**Corollary 7.** Under the assumption of the Thereom 9 for $\omega=\frac{\rho_1+\rho_2}{2}$ and $q=1$ , we get the inequality below:

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\varphi(u)du-\varphi\left(\frac{\rho_1+\rho_2}{2}\right)\right| \leq \frac{(\rho_2-\rho_1)^2}{4}\left[\frac{4\pi-8}{\pi^3}\left|\varphi''\left(\frac{\rho_1+\rho_2}{2}\right)\right|+\frac{\pi^2-8}{\pi^3}A(|\varphi''(\rho_1)|,|\varphi''(\rho_2)|)\right].$$

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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