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Research Article

On the existence of weak solutions for a class of singular reaction diffusion systems

Salim Mesbahi

LMFN Laboratory, Department of Mathematics, Faculty of Sciences, Ferhat Abbas University, Setif, Algeria

Abstract

We study the existence of weak solutions for a parabolic reaction diffusion model applied in Quenching endowed with singular production terms by reaction. The singularity is due to a potential occurrence of quenching localized to the domain boundary. The techniques used are based on energy estimates to approach nonsingular problems and uniform control on the set where singularities are localizing.

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1. Introduction

The applications of reaction diffusion systems are numerous, particularly in the modeling of real diffusion phenomena in biology, chemistry and engineering, see Levenspiel [19] and Murray [27,28]. A very interesting case is that of quenching. We recall that quenching is the rapid cooling of a hot object in the air or in a liquid such as water or oil to obtain certain properties. The study of quenching phenomena began in 1975 with an article by Kawarada [15], where he studied a model in one space dimension. This article initiated a broad study of the quenching problem by many scientific researchers, including work on existence, the structure of quenching points, the asymptotic behaviour of solutions, etc. Since then, this notion has been widely studied and developed in hundreds of books and scientific articles. For a detailed survey, we refer also to Levine [20].

The quenching phenomena have been studied by many researchers and therefore we find a lot of works in this field. This is due to its many and varied applications in the fields of science, especially engineering. We find a lot of models in Constantin et *al.* [6], Kiselev and Zlatos [16] and Marion [22]. For example, quenching can increase the hardness of metallic or plastic materials, to see other examples we refer to Aris [3], Muntean [26] and references given there. In this kind of problems, singularities can appear in the solutions of the systems formulated. This can complicate the theoretical and numerical mathematical analysis of the problem.

In this work, we are interested in the study of a singular reaction diffusion system. For modeling and mathematical analysis of this type of problem, several methods have been

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proposed; see for example, [1,2], [4], [5], [7], [9–14], [21–24], [29–33], [37] and corresponding references therein.

The study of specialized behaviour for reaction diffusion systems has been an active area of research for decades. Two types of specialized behaviour, solution explosion and solution quenching, have been of particular interest more recently. In explosion problems, the solution becomes unbounded in finite time, i.e. when t approaches the explosion instant T, which is finite. Naturally, the reason for this behaviour of the solution is due to the singularity of the nonlinearities. In quenching problems, the solution remains bounded while the first order time derivative becomes unbounded in finite time. We refer also to Salin's works [30–33], Selçuk [34] and to references therein for more detailed information and a mathematical analysis of models involving singular terms.

So, the objective here is the study of a notion which one calls "quenching" generalizing that of the explosion. We are then interested in the study of the existence of weak solutions of the following reaction diffusion system with singular nonlinearities :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = F(t,x,u,v) & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - \Delta v = G(t,x,u,v) & \text{in } Q_T \end{cases} \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(0,x) = u_0(x) & \text{in } \Omega \\ v(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.2}$$

and the following boundary conditions

$$\begin{cases} u(t,x) = 0 & \text{on } \Gamma_1 \times (0,T) \\ v(t,x) = 0 & \text{on } \Gamma_2 \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \times (0,T) \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,T) \end{cases} \tag{1.3}$$

with

$$\begin{cases} F(t,x,r,s) = f(t,x)\dfrac{s}{r^\gamma} \\ G(t,x,r,s) = g(t,x)\dfrac{s}{r^\gamma} \end{cases} \tag{1.4}$$

where $Q_T = \Omega \times (0,T)$, Ω is a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$. γ is a real parameter such as $0 < \gamma \leq 1$. Γ_1 and Γ_2 are such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \phi$. The Haussdorff measure of Γ_1 and Γ_2 does not vanish. Here ν denotes the outer normal to $\partial\Omega$. The functions $f,\ g : Q_T \to \mathbb{R}$ and satisfy the following conditions :

$$f,\ g \in L^1(Q_T) \quad , \quad f \geq 0 \ \text{ and } \ f + g \leq 0 \tag{1.5}$$

Note that the functions F and G are singular at $r = 0$, and we assume that the functions u_0 and v_0 are such that

$$u_0,\ v_0 \geq 0 \ \text{ and } \ u_0,\ v_0 \in L^\infty(\Omega) \tag{1.6}$$

We will study our problem in the following spaces :

$$V = \{\varphi \in H^1(\Omega) : \varphi = 0 \quad \text{on } \Gamma_1\}$$

$$W = \{\psi \in H^1(\Omega) : \psi = 0 \quad \text{on } \Gamma_2\}$$

The dual spaces of V and W, respectively, are denoted by V^* and W^*.

The problem $(1.1) - (1.4)$ consists of a weakly coupled system whose right term is singular in the variable u. We hear that the nonlinearities $F(t,x,r,s)$ and $G(t,x,r,s)$ can become unbounded to the neighborhood of $r = 0$.

The main contribution is to study a reaction diffusion system with singular and nonregular nonlinearities F and G dependent on the independent variables t and x and on the unknown functions u and v. The functions f and g verify simple properties, this allows us to choose them from a wide range.

We confirm that the studied model is not only applied to the quenching phenomenon, but can also be applied to other singular reaction diffusion models in different scientific fields.

Before stating the main result of this work, it is worth mentioning that several mathematicians have dealt with this type of problem using various analytical and numerical techniques and methods, under different hypotheses as appropriate, see for example, [4], [7], [9–14] and [37].

This document is organized as follows : In the next section, we state our main result. In the third section, we give a result concerning the nonsingular approximating problem. In the fourth section, we give important a priori estimates. The fifth section is devoted to some important results of convergence and compactness. In the sixth section, we prove certain properties relating to our problem near the singularity. The last section is devoted to prove the main result. The paper ends with a conclusion and some perspectives.

The difficulties in this work are similar to those in [9–14], and the techniques are of the same spirit, but specific new difficulties due to the nature of the system must be handled.

2. The main result

We first introduce the notion of weak solution to the problem (1.1) – (1.4) used here.

Definition 2.1. A weak solution to problem (1.1) – (1.4) is a nonnegative couple

$$(u,v) \in [L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)] \times [L^2(0,T;W) \cap L^\infty(0,T;L^2(\Omega))]$$

with

$$(u_t,v_t) \in [L^2(0,T;V^*) + L^1(0,T;L^1_{loc}(\Omega))] \times [L^2(0,T;W^*) + L^1(0,T;L^1_{loc}(\Omega))]$$

such that

$$u\left(0,x\right) = u_0\left(x\right) \ , \ v\left(0,x\right) = v_0\left(x\right) \quad \text{a.e. } x \in \Omega$$

$$\int_{Q_T} f\frac{v}{u^\gamma}\phi < +\infty \quad , \quad \int_{Q_T} g\frac{v}{u^\gamma}\eta < +\infty$$

and

$$\begin{aligned} -\int_\Omega u_0(x)\varphi(0,x) - \int_{Q_T} u\frac{\partial \varphi}{\partial t} + \int_{Q_T} \nabla u \nabla \varphi &= \int_{Q_T} F\varphi \\ -\int_\Omega v_0(x)\psi(0,x) - \int_{Q_T} v\frac{\partial \psi}{\partial t} + \int_{Q_T} \nabla v \nabla \psi &= \int_{Q_T} G\psi \end{aligned}$$

for all $\phi,\ \eta,\ \varphi,\ \psi \in C_0^\infty([0,T) \times \Omega)$.

Now, we can state the main result of this work :

Theorem 2.2. *Assume* (1.4) – (1.6)*, then there exists a weak solution* (u,v) *to problem* (1.1) – (1.4) *in the sense of Definition 2.1.*

3. Nonsingular approximating problem

To study the problem (1.1) – (1.4), we consider the nonsingular approximating problem. Essentially, we are truncating in such a way as to eliminate the singularity. In this case, we define the sequences of functions f_n and g_n, such that

$$\begin{aligned} f_n\left(t,x\right) &= \min\left\{n, f\left(t,x\right)\right\} \\ g_n\left(t,x\right) &= -\min\left\{n, -g\left(t,x\right)\right\} \end{aligned}$$

It is easily seen that f_n and g_n satisfy the same properties as f and g, moreover,

$$\begin{array}{ccc} 0 \le f_n \le f & , & \lim_{n\to+\infty} f_n = f \\ g \le g_n \le 0 & , & \lim_{n\to+\infty} g_n = g \end{array}$$

The approximate problem is the following :

$$\text{Find } (u_n, v_n) \text{ in } [L^2(0,T;V)\cap L^\infty(Q_T)]\times[L^2(0,T;W)\cap L^\infty(Q_T)]$$

such that

$$\begin{cases} (u_n)_t - \Delta u_n = F_n(t,x,u_n,v_n) & \text{in } Q_T \\ (v_n)_t - \Delta v_n = G_n(t,x,u_n,v_n) & \text{in } Q_T \\ u_n(0,x) = u_{0,n}(x) & \text{in } \Omega \\ v_n(0,x) = v_{0,n}(x) & \text{in } \Omega \\ u_n(t,x) = 0 & \text{on } \Gamma_1\times(0,T) \\ v_n(t,x) = 0 & \text{on } \Gamma_2\times(0,T) \\ \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on } \Gamma_2\times(0,T) \\ \dfrac{\partial v_n}{\partial \nu} = 0 & \text{on } \Gamma_1\times(0,T) \end{cases} \tag{3.1}$$

where

$$F_n(t,x,u_n,v_n) = \begin{cases} f_n(t,x)\dfrac{v_n}{(u_n+\frac{1}{n})^\gamma} & \text{if } u_n\ge 0 \text{ and } v_n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$G_n(t,x,u_n,v_n) = \begin{cases} g_n(t,x)\dfrac{v_n}{(u_n+\frac{1}{n})^\gamma} & \text{if } u_n\ge 0 \text{ and } v_n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

while $u_{0,n},\ v_{0,n} \in L^\infty(\Omega)\cap H_0^1(\Omega)$ are suitable regularizations of the initial data obtained by a standard convolution technique (see [8]) such that

$$\lim_{n\to\infty}\frac{1}{n}\|u_{0,n}\|_{H_0^1(\Omega)} = 0 \tag{3.2}$$

$$\lim_{n\to\infty}\frac{1}{n}\|v_{0,n}\|_{H_0^1(\Omega)} = 0 \tag{3.3}$$

We have the following important result :

Lemma 3.1. *Problem* (3.1) *admits a nonnegative couple of solutions :*

$$(u_n, v_n) \in [L^2(0,T;V)\cap L^\infty(Q_T)]\times[L^2(0,T;W)\cap L^\infty(Q_T)]$$

such that

$$-\int_\Omega u_{0,n}(x)\varphi(0,x) - \int_{Q_T} u_n\frac{\partial\varphi}{\partial t} + \int_{Q_T}\nabla u_n\nabla\varphi = \int_{Q_T} F_n(t,x,u_n,v_n)\varphi \tag{3.4}$$

$$-\int_\Omega v_{0,n}(x)\psi(0,x) - \int_{Q_T} v_n\frac{\partial\psi}{\partial t} + \int_{Q_T}\nabla v_n\nabla\psi = \int_{Q_T} G_n(t,x,u_n,v_n)\psi \tag{3.5}$$

for every $\varphi,\ \psi \in C_0^\infty(\Omega\times[0,T))$.

Proof. For simplicity, we suppose $u_{0,n} = 0$ and $v_{0,n} = 0$. Then, by a direct application of the method of Stampacchia in [36], we can prove the positivity of the solution by taking as test function in the first equation of the problem (3.1) the function $\varphi = -u_n^-$, where

$$u_n = u_n^+ - u_n^- \quad , \quad u_n^+ = \max\{u_n, 0\} \quad , \quad u_n^- = \max\{-u_n, 0\}$$

Since $u_n^+ = 0$ on the support of u_n^-, we have that the right hand side of (3.4) is zero because

$$F_n(t,x,u_n,v_n) = \begin{cases} f_n(t,x)\dfrac{v_n}{(u_n+\frac{1}{n})^\gamma} & \text{if } u_n \geq 0 \text{ and } v_n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so we have

$$\int_{Q_T} (u_n)_t(-u_n^-) + \int_{Q_T} \nabla u_n \nabla(-u_n^-) = 0$$

which give

$$\int_{Q_T} (u_n^+ - u_n^-)_t(-u_n^-) + \int_{Q_T} \nabla(u_n^+ - u_n^-)\nabla(-u_n^-) = 0$$

We observe that on the support of u_n^- we have $u_n^+.u_n^- = 0$, he comes

$$\frac{1}{2}\int_\Omega (u_n^-)^2(t) + \int_{Q_\tau} \left|\nabla(u_n^-)\right|^2 = 0, \text{ for all } t \in [0,T]$$

and we deduce that

$$u_n^- = 0 \quad \text{a.e. in } Q_T$$

i.e. that $u_n \geq 0$ a.e. in Ω and for all $t \in [0,T)$. In the same way, we prove that $v_n \geq 0$, by choosing as test function $\psi = -v_n^-$. □

In everything that follows, we denote with C a generic constant. Usually C is thought to be independent of n, if not otherwise mentioned. Before giving the proof of our result, let us denote by T_k the truncation function

$$T_k(s) = \max\{-k, \min\{k,s\}\}, \quad k \geq 0,\ s \in \mathbb{R}$$

and by $\mathcal{G}_k$ the function

$$\mathcal{G}_k(s) = s - T_k(s)$$

In the following, we will denote by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V (and also between W^* and W).

4. A priori uniform estimates

4.1. Uniform estimate for (u_n, v_n) in $L^\infty(Q_T)$

Proposition 4.1. *There exist positive constants M_1 and M_2, independent of n, such that*

$$\|u_n\|_{L^\infty(Q_T)} \leq M_1 \tag{4.1}$$

$$\|v_n\|_{L^\infty(Q_T)} \leq M_2 \tag{4.2}$$

Proof. The uniform estimate (4.1) for the sequence $\{u_n\}$ follows directly by Proposition 2.13 in [11] with some abbreviations that go along with our problem. For simplicity we suppose $v_{0,n}(x) = 0$. To handle the equation solved by v_n we choose as test function

$$\psi = \mathcal{G}_{M_2}(v_n) := (v_n - M_2)^+$$

with $M_2 > 1$ fixed, we obtain, with $Q_t := \Omega \times [0,t)$

$$\begin{aligned} &\int_{Q_t} (v_n)_t (v_n - M_2)^+ + \int_{Q_t} \nabla v_n . \nabla (v_n - M_2)^+ \\ = &\int_{Q_t} g_n(t,x) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma} (v_n - M_2)^+ \leq 0 \end{aligned}$$

Neglecting the nonnegative term on the left hand side, it comes

$$(v_n - M_2)^+ = 0 \ \text{ a.e. in } Q_T$$

which proves (4.2). □

4.2. Energy estimate for (u_n, v_n) in $L^2(0,T;V) \times L^2(0,T;W)$

Proposition 4.2. *There exists a positive constant C, independent of n, such that*

$$\|u_n\|_{L^2(0,T;V)} + \|v_n\|_{L^2(0,T;W)} \leq C \tag{4.3}$$

Proof. Choosing as test function $\varphi = u_n \in L^2(0,T;V)$ in the first equation of problem (3.1) solved by u_n and integrating over $\Omega \times [0,t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{Q_T} u_n^2 dt + \int_{Q_T} |\nabla u_n|^2 = \int_{Q_T} f_n(t,x) \frac{v_n u_n}{\left(u_n + \frac{1}{n}\right)^\gamma}$$

which give

$$\frac{1}{2}\int_\Omega u_n^2(t)\,dt + \int_{Q_T} |\nabla u_n|^2 = \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)} + \int_{Q_T} f_n(t,x) \frac{v_n u_n}{\left(u_n + \frac{1}{n}\right)^\gamma}$$

By observing that $\frac{u_n}{\left(u_n+\frac{1}{n}\right)^\gamma} \leq u_n^{1-\gamma}$ and $0 < 1-\gamma < 1$, we obtain

$$\int_{Q_T} f_n(t,x) \frac{v_n u_n}{\left(u_n + \frac{1}{n}\right)^\gamma} \leq \int_{Q_T} f_n(t,x)\, u_n^{1-\gamma} v_n \leq \sup\left|u_n^{1-\gamma} v_n\right| \int_{Q_T} f_n \leq C$$

We then obtain

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

For the second equation of problem (3.1), we choose as test function $\psi = v_n \in L^2(0,T;W)$. We obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega v_n^2(t)\,dt + \int_{Q_T} |\nabla v_n|^2 &= \frac{1}{2}\|v_{0,n}\|^2_{L^2(\Omega)} + \int_{Q_T} g_n(t,x) \frac{v_n u_n}{\left(u_n + \frac{1}{n}\right)^\gamma} \\ &\leq \frac{1}{2}\|v_{0,n}\|^2_{L^2(\Omega)}\end{aligned}$$

which give

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

hence the inequality (4.3). □

Proposition 4.3. *There exists a positive constant C, independent of n, such that*

$$\int_{Q_T} f_n(t,x) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma}\varphi^2(x) \leq C \quad , \quad \textit{for all } n \in \mathbb{N} \tag{4.4}$$

$$\int_{Q_T} |g_n(t,x)| \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma}\psi^2(x) \leq C \quad , \quad \textit{for all } n \in \mathbb{N} \tag{4.5}$$

for every $\varphi, \psi \in C_0^\infty(\Omega)$.

Proof. (i) We multiply the first equation of problem (3.1) by the test function $\varphi^2(x)$, we get

$$\int_0^T \left\langle (u_n)_t, \varphi^2(x)\right\rangle + 2\int_{Q_T} \nabla u_n \varphi \nabla\varphi = \int_{Q_T} f_n(t,x) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma}\varphi^2(x)$$

which give

$$\int_{Q_T} f_n(t,x) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma}\varphi^2(x) \leq 2\int_{Q_T} |\nabla u_n| . |\varphi| . |\nabla\varphi| + C$$

this gives, by applying the Hölder's inequality and the previous Proposition

$$\int_{Q_T} f_n(t,x) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^\gamma}\varphi^2(x) \leq C + 2\left(\int_{Q_T} |\nabla u_n|^2 .\varphi^2\right)^{\frac{1}{2}} . \left(\int_{Q_T} |\nabla\varphi|^2\right)^{\frac{1}{2}} \leq C_2$$

which proves the inequality (4.4).

(ii) In the same way, but this time we multiply the second equation of problem (3.1) by the test function $\psi^2(x)$, we get

$$\int_0^T \left\langle (v_n)_t, \psi^2(x) \right\rangle + 2\int_{Q_T} \nabla v_n \psi \nabla \psi = \int_{Q_T} g_n\left(t,x\right) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \psi^2\left(x\right)$$

which give

$$\left| \int_0^T \left\langle (v_n)_t, \psi^2(x) \right\rangle \right| + 2\left| \int_{Q_T} \nabla v_n \psi \nabla \psi \right| \geq \left| \int_{Q_T} g_n\left(t,x\right) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \psi^2\left(x\right) \right|$$

We choose C so that

$$\int_{Q_T} \left| g_n\left(t,x\right) \right| \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \psi^2\left(x\right) \leq C + 2\int_{Q_T} \left| \nabla v_n \right| . \left| \psi \right| . \left| \nabla \psi \right|$$

this gives, by applying the Hölder's inequality and the previous Proposition

$$\int_{Q_T} \left| g_n\left(t,x\right) \right| \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \psi^2\left(x\right) \leq C + 2\left(\int_{Q_T} \left| \nabla v_n \right|^2 . \psi^2 \right)^{\frac{1}{2}} . \left(\int_{Q_T} \left| \nabla \psi \right|^2 \right)^{\frac{1}{2}} \leq C_2$$

which gives the desired result (4.5). □

4.3. Uniform estimate on critical sets

In this paragraph we will consider the following critical sets

$$\{(t,x) \in Q_T : u_n(t,x) \leq \delta\} \text{ and } \{(t,x) \in Q_T : v_n(t,x) \leq \delta\}$$

These sets are prone to hosting the locations of the singularity. In fact, we wish to avoid a potential blow up of the solutions on these sets. This is ensured by the following Proposition :

Proposition 4.4. *For $\gamma > 0$, we have*

$$\int_{Q_T \cap \{0 \leq u_n \leq \delta\}} f_n\left(t,x\right) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \varphi^2\left(x\right) \leq C\delta \tag{4.6}$$

$$\int_{Q_T \cap \{0 \leq v_n \leq \delta\}} f_n\left(t,x\right) \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \varphi^2\left(x\right) \leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0 < \gamma < 1 \\ C\sqrt{\delta} & \text{if } \gamma = 1 \end{cases} \tag{4.7}$$

$$\int_{Q_T \cap \{0 \leq v_n \leq \delta\}} \left| g_n\left(t,x\right) \right| \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \varphi^2\left(x\right) \leq C\delta \tag{4.8}$$

$$\int_{Q_T \cap \{0 \leq u_n \leq \delta\}} \left| g_n\left(t,x\right) \right| \frac{v_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \varphi^2\left(x\right) \leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0 < \gamma < 1 \\ C\sqrt{\delta} & \text{if } \gamma = 1 \end{cases} \tag{4.9}$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$.

Proof. (i) We prove first (4.6). Following the same ideas as in the proof of Proposition 2.20 in [11], we choose as test function in the equation solved by u_n the function $\varphi_\sigma =$

$\frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^2(x)$, with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geq 0$. Therefore we get

$$\begin{aligned} &\textstyle\int_0^T \left\langle (u_n)_t\,, \dfrac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^2(x)\right\rangle + \dfrac{1}{\sigma}\int_{Q_T} \nabla u_n \nabla\left(T_\sigma(-(u_n-\delta)^-)\right)\varphi^2(x) \\ &\qquad +2\textstyle\int_{Q_T} \nabla u_n \dfrac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi\nabla\varphi \\ &= \textstyle\int_{Q_T} f_n\,(t,x)\,\dfrac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\dfrac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^2(x) \end{aligned} \tag{4.10}$$

First, we show that

$$\int_0^T \langle (u_n)_t, \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^2(x)\rangle \geq -\delta|\Omega| \tag{4.11}$$

where $|\Omega|$ is the Lebesgue measure of Ω. For that, we introduce the function $v_{\sigma,\nu} = \frac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}$, where $u_{n,\nu}$ is, for any fixed $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}$, the solution of the following ordinary differential equation problem

$$\begin{cases} \dfrac{1}{\sigma}\left[u_{n,\nu}\right]_t + u_{n,\nu} = u_n \\ u_{n,\nu}\,(0) = u_{0,n} \end{cases} \tag{4.12}$$

The function $u_{n,\nu}$ satisfies the following properties (see [17] and [18]) :

$$\begin{gathered} u_{n,\nu} \in L^2(0,T;H_0^1(\Omega)), \quad (u_{n,\nu})_t \in L^2(0,T;H_0^1(\Omega)) \\ \|u_{n,\nu}\|_{L^\infty(Q_T)} \leq \|u_n\|_{L^\infty(Q_T)} \\ u_{n,\nu} \to u_n \quad \text{in } L^2(0,T;H_0^1(\Omega)) \quad \text{as } \nu \to +\infty \\ (u_{n,\nu})_t \to (u_n)_t \quad \text{in } L^2(0,T;H_0^{-1}(\Omega)) \text{ as } \nu \to +\infty \end{gathered}$$

So, we have

$$\begin{aligned} &\textstyle\int_0^T \left\langle (u_n)_t\,, \dfrac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^2(x)\right\rangle \\ &= \lim_{\nu\to\infty}\textstyle\int_{Q_T}\left[(u_{n,\nu}-\delta)_t^+\right]\dfrac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}\varphi^2(x) \\ &-\lim_{\nu\to\infty}\textstyle\int_{Q_T}\left[(u_{n,\nu}-\delta)_t^-\right]\dfrac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}\varphi^2(x) \\ &= \lim_{\nu\to\infty}\textstyle\int_{Q_T}\left[(u_{n,\nu}-\delta)_t^-\right]\dfrac{T_\sigma((u_{n,\nu}-\delta)^-)}{\sigma}\varphi^2(x) \end{aligned} \tag{4.13}$$

Introducing now the function $\Phi_\sigma(s) := \int_0^{(s-\delta)^-}\frac{T_\sigma(\rho)}{\sigma}d\rho$, from (4.13), we obtain

$$\begin{aligned} &\lim_{\nu\to\infty}\int_{Q_T}(u_{n,\nu}-\delta)_t^-\frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\delta}\varphi^2(x) \\ &= \lim_{\nu\to\infty}\int_{Q_T}\frac{d}{dt}\Phi_\sigma(u_{n,\nu}) \\ &= \lim_{\nu\to\infty}\int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(T) - \lim_{\nu\to\infty}\int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(0) \\ &\geq -\lim_{\nu\to\infty}\int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(0) = -\int_\Omega \Phi_\sigma(u_n-\delta)^-(0) \geq -\delta|\Omega| \end{aligned}$$

which proves (4.11). By (4.11), observing also that $\frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}=0$ on the set $\{(x,t)\in Q_T : u_n(x,t)\geq\delta\}$, the equality (4.10) becomes

$$\frac{1}{\sigma}\int_{Q_T\cap\{\delta-\sigma\leq u_n\leq\delta\}}|\nabla u_n|^2.\varphi^2(x)+\int_{Q_T}f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\sigma}\varphi^2(x)$$
$$\leq 2\int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|.\varphi.|\nabla\varphi|+\delta|\Omega| \tag{4.14}$$

Using Hölder's inequality in the right hand side of (4.14), we obtain

$$\int_{Q_T}f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\sigma}\varphi^2(x)$$
$$\leq\ 2\left(\int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|^2.\varphi^2\right)^{\frac{1}{2}}\left(\int_{Q_T}|\nabla\varphi|^2\right)^{\frac{1}{2}}+\delta|\Omega|$$

Now, we can prove that

$$\int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|^2\varphi^2(x)\leq C\delta \tag{4.15}$$

Indeed, multiplying problem (3.1) by the test function $-(u_n-\delta)^-\varphi^2(x)$, $\varphi\in C_0^\infty(\Omega)$, $\varphi\geq 0$, we obtain

$$\begin{gathered}\int_0^T\langle(u_n)_t,(-(u_n-\delta)^-)\varphi^2\rangle+\\ \int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|^2\varphi^2-2\int_{Q_T}\nabla u_n(u_n-\delta)^-\varphi\nabla\varphi\leq 0\end{gathered} \tag{4.16}$$

For the first term of (4.16), we follow the same arguments as those used to obtain (4.11), he comes

$$\int_0^T\langle(u_n)_t,(-(u_n-\delta)^-)\varphi^2(x)\rangle\geq-\delta|\Omega| \tag{4.17}$$

By (4.17), the inequality (4.16) becomes

$$\int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|^2\varphi^2\leq 2\int_{Q_T\cap\{u_n<\delta\}}|\nabla u_n|(\delta-u_n)\varphi|\nabla\varphi|+\delta|\Omega|$$

which, by Hölder's inequality and (4.3), leads to

$$\int_{Q_T\cap\{u_n\leq\delta\}}|\nabla u_n|^2\varphi^2\leq 2\delta\left(\int_{Q_T}|\nabla u_n|^2\varphi^2\right)^{\frac{1}{2}}\left(\int_{Q_T}|\nabla\varphi|^2\right)^{\frac{1}{2}}+\delta|\Omega|$$
$$\leq C\delta$$

Thus, (4.15) holds. Finally, we have obtained that

$$\int_{Q_T}f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\delta}\varphi^2(x)\leq C\delta \tag{4.18}$$

Now, we can pass to the limit in (4.18) for $\sigma\to 0$ and n fixed. For this we use Lebesgue dominate convergence Theorem since $\frac{T_\sigma((u_n-\delta)^-)}{\sigma}\to 1$ a.e. on the set $\{(x,t)\in Q_T : u_n(t,x)<\delta\}$. Therefore, we obtain

$$\int_{Q_T\cap\{0\leq u_n\leq\delta\}}f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2\leq C\delta$$

and hence (4.6) holds.

(ii) We are now focusing on estimation (4.7). If $0<\gamma<1$, we consider the decomposition

$$\begin{aligned}
&\int_{Q_T\cap\{0\le v_n\le\delta\}} f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2(x)\\
&=\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\{0\le u_n\le\delta\}} f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2(x)\\
&+\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\{u_n>\delta\}} f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2(x)\\
&\le\int_{Q_T\cap\{0\le u_n\le\delta\}} f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2(x)\\
&+\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\{u_n>\delta\}} f_n(t,x)\frac{v_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\varphi^2(x)\\
&=I_1+I_2
\end{aligned}\tag{4.19}$$

By (4.6), we obtain

$$I_1\le C\delta \tag{4.20}$$

To handle the term I_2, we proceed as follows : Since $v_n\le\delta$ and $\left(\frac{1}{n}+u_n\right)^\gamma>\delta^\gamma$, he comes $\frac{1}{\left(u_n+\frac{1}{n}\right)^\gamma}<\frac{1}{\delta^\gamma}$. Then

$$\begin{aligned}
I_2&\le\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\{u_n>\delta\}} f_n(t,x)\frac{\delta}{\delta^\gamma}\varphi^2(x)=\delta^{1-\gamma}\int_{Q_T} f_n(t,x)\varphi^2(x)\\
&\le\delta^{1-\gamma}\sup_{x\in\Omega}\left|\varphi^2(x)\right|\int_{Q_T} f_n(t,x)\le C\delta^{1-\gamma}
\end{aligned}\tag{4.21}$$

If $\gamma=1$, we consider the decomposition

$$\begin{aligned}
&\int_{Q_T\cap\{0\le v_n\le\delta\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&=\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\left\{0\le u_n\le\sqrt{\delta}\right\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&+\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\left\{u_n>\sqrt{\delta}\right\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&\le\int_{Q_T\cap\left\{0\le u_n\le\sqrt{\delta}\right\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&+\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\left\{u_n>\sqrt{\delta}\right\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&=J_1+J_2
\end{aligned}\tag{4.22}$$

Choosing as test function in the equation solved by u_n the function

$$\phi_\sigma=\frac{T_\sigma\left(-(u_n-\sqrt{\delta})^-\right)}{\sigma}\varphi^2(x)$$

with $\varphi\in C_0^\infty(\Omega),\varphi\ge0$, and repeating the same arguments of the proof of (4.6), we obtain

$$J_1\le C\sqrt{\delta} \tag{4.23}$$

For the term J_2, since $v_n\le\delta$ and $\frac{1}{n}+u_n>\sqrt{\delta}$, he comes $\frac{1}{u_n+\frac{1}{n}}<\frac{1}{\sqrt{\delta}}$, we obtain

$$\begin{aligned}
&\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\left\{u_n>\sqrt{\delta}\right\}} f_n(t,x)\frac{v_n}{u_n+\frac{1}{n}}\varphi^2(x)\\
&\le\sqrt{\delta}\int_{Q_T\cap\{0\le v_n\le\delta\}\cap\left\{u_n>\sqrt{\delta}\right\}} f_n(t,x)\varphi^2(x)\\
&\le\sqrt{\delta}\sup_{x\in\Omega}|\varphi(x)|\int_{Q_T} f_n(t,x)\le C\sqrt{\delta}
\end{aligned}\tag{4.24}$$

Therefore, by (4.18) – (4.24), we finally obtain (4.7).

We proceed as in the previous cases (i) and (ii), we easily arrive at (4.8) and (4.9). □

5. Convergence and compactness results

To pass to the limit as $n \to \infty$ in (3.4) and (3.5) we need strongly convergent subsequences, which ensured by the following Proposition.

Proposition 5.1. *There exists a couple*

$$(u, v) \in \left[L^2(0,T;V) \cap L^\infty(Q_T)\right] \times \left[L^2(0,T;W) \cap L^\infty(Q_T)\right]$$

such that, as $n \to \infty$, we have

$$u_n \rightharpoonup u \quad \textit{weakly in } L^2(0,T;V) \tag{5.1}$$

$$v_n \rightharpoonup v \quad \textit{weakly in } L^2(0,T;W) \tag{5.2}$$

$$u_n \rightharpoonup u \quad \textit{weakly in } L^\infty(Q_T) \tag{5.3}$$

$$v_n \rightharpoonup v \quad \textit{weakly in } L^\infty(Q_T) \tag{5.4}$$

$$u_n \to u \quad \textit{strongly in } L^1(Q_T) \tag{5.5}$$

$$v_n \to v \quad \textit{strongly in } L^1(Q_T) \tag{5.6}$$

$$u_n \to u \quad \textit{a.e. in } Q_T \tag{5.7}$$

$$v_n \to v \quad \textit{a.e. in } Q_T \tag{5.8}$$

up to a subsequence.

Proof. Convergences (5.1) and (5.2) are direct consequences of (4.3). The same thing applies to convergences (5.3) and (5.4). To prove (5.5) and (5.7), we observe that the estimate (4.4) leads to

$$f_n\,(t,x)\,\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\varphi^2 \in L^1(Q_T) \quad , \quad \forall \varphi \in C_0^\infty(\Omega) \tag{5.9}$$

In addition, we have

$$\frac{\partial(u_n\varphi^2)}{\partial t} \quad \text{is bounded in } L^2(0,T;V^*) + L^1(Q_T) \tag{5.10}$$

By (5.10), choosing s such that $s > \dfrac{N}{2} + 1$, using the same argument as Lemma 2.3 in [25], we deduce that $\dfrac{\partial(u_n\varphi)}{\partial t}$ is also bounded in $L^1(0,T;H^{-s})$. Consequently, since $s > \dfrac{N}{2}$, we find that

$$V \subset L^p(\Omega) \subset H^{-s}(\Omega)$$

and the embedding $V \hookrightarrow L^p(\Omega)$ is compact. Applying now Corollary 4 in [35], by (5.10) and the compactness results we deduce that $u_n\varphi$ is relatively compact in $L^2(Q_T)$. Hence, up to a subsequences, convergences (5.5) and (5.7) are satisfied. In the same way for the sequence $\{v_n\}$, we find (5.6) and (5.8). □

Proposition 5.2. *We have*

$$\lim_{n\to\infty}\int_{Q_T} |\nabla\,(u_n - u)|^2 = 0 \tag{5.11}$$

$$\lim_{n\to\infty}\int_{Q_T} |\nabla\,(v_n - v)|^2 = 0 \tag{5.12}$$

Therefore,

$$\nabla u_n \to \nabla u \quad \textit{a.e. in } Q_T \tag{5.13}$$

$$\nabla v_n \to \nabla v \quad \textit{a.e. in } Q_T \tag{5.14}$$

Proof. This result can be obtained as a particular case of Proposition 3.14 in [11]. □

6. Uniform estimate near the singularity

We consider the set $\{(t,x) \in Q_T : u(t,x) = 0 \text{ a.e. in } Q_T\}$. As a consequence of the uniform estimate near the singularity (4.6), we have the following Proposition :

Proposition 6.1. *The couple* (u,v) *as a solution to* (1.1)−(1.4)*, in the sense of Definition 2.1, satisfies*

$$\int_{Q_T \cap \{u=0\}} f(t,x) \frac{v}{u^\gamma} \psi = 0 \tag{6.1}$$

$$\int_{Q_T \cap \{u=0\}} g(t,x) \frac{v}{u^\gamma} \phi = 0 \tag{6.2}$$

for all $\psi, \phi \in C_0^\infty([0,T) \times \Omega)$ *with* $\psi, \phi \geq 0$*. Moreover, it holds*

$$\int_{Q_T} f(t,x) \frac{v}{u^\gamma} \psi = \int_{Q_T \cap \{u>0\}} f(t,x) \frac{v}{u^\gamma} \psi \tag{6.3}$$

$$\int_{Q_T} g(t,x) \frac{v}{u^\gamma} \phi = \int_{Q_T \cap \{u>0\}} g(t,x) \frac{v}{u^\gamma} \phi \tag{6.4}$$

Proof. (i) We consider a test function $\psi \in C_0^\infty([0,T) \times \Omega)$, $\psi \geq 0$, with $\operatorname{supp} \psi = [0,T_1] \times Y$, $T_1 < T$, $Y \subset\subset E \subset\subset \Omega$ and $\varphi \in C_0^1(\Omega)$ with $\varphi(x) = 1$ over Y, $\varphi \geq 0$ with $\operatorname{supp} \varphi = E$. By the uniform estimate (4.6), we obtain

$$\begin{aligned} &\int_{Q_T \cap \{u_n < \delta\}} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(t,x) \\ &\leq \|\psi\|_\infty \int_{[0,T] \times Y} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \\ &\leq \|\psi\|_\infty \int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^2(x) \chi_{\{u_n<\delta\}} \leq C\delta \end{aligned}$$

On the other hand,

$$\begin{aligned} &\textstyle\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \psi(t,x) \\ &= \textstyle\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \chi_{\{u=\delta\}} \psi(t,x) \\ &+ \textstyle\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \chi_{\{u\neq\delta\}} \psi(t,x) \leq C\delta \end{aligned} \tag{6.5}$$

We observe that there exists at most a countable set D_1 such that $\operatorname{meas}\{(t,x) : u(t,x) = \delta\} > 0$. We choose δ outside of this set D_1, so that, in (6.5), the integral

$$\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \chi_{\{u=\delta\}} \psi(t,x) = 0$$

So, we have

$$\begin{aligned} &\textstyle\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \psi(t,x) \\ &= \textstyle\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n<\delta\}} \chi_{\{u\neq\delta\}} \psi(t,x) \leq C\delta \end{aligned} \tag{6.6}$$

Since by (5.7),

$$\chi_{\{u_n<\delta\}} \chi_{\{u\neq\delta\}} \to \chi_{\{u<\delta\}} \quad \text{a.e. in } Q_T$$

Applying Fatou's Lemma in (6.6) for δ fixed, leads to

$$\int_{Q_T} f(t,x) \frac{v}{u^\gamma} \chi_{\{u<\delta\}} \psi(t,x) \leq C\delta$$

Using again Fatou's Lemma in the last inequality for $\delta \to 0$, we get

$$\int_{Q_T} f(t,x)\,\frac{v}{u^{\gamma}}\chi_{\{u=0\}}\psi(t,x) = \int_{Q_T\cap\{u=0\}} f(t,x)\,\frac{v}{u^{\gamma}}\psi(t,x) = 0 \tag{6.7}$$

this leads to

$$\int_{Q_T} f(t,x)\,\frac{v}{u^{\gamma}}\psi(t,x) = \int_{Q_T\cap\{u>0\}} f(t,x)\,\frac{v}{u^{\gamma}}\psi(t,x) \tag{6.8}$$

which are the desired identities.

(ii) In the same way with some simplifications, we prove (6.2) and (6.4). We consider a test function $\phi \in C_0^{\infty}([0,T)\times\Omega)$, $\phi \geq 0$, with $\operatorname{supp}\phi = [0,T_2]\times Y$, $T_2 < T$, $\bar{Y} \subset\subset E \subset\subset \Omega$ and $\varphi \in C_0^1(\Omega)$ with $\varphi(x) = 1$ over $\bar{Y}$, $\varphi \geq 0$ with $\operatorname{supp}\varphi = E$. By the uniform estimate (4.9), we obtain

$$\begin{aligned}
&\int_{Q_T\cap\{u_n<\delta\}} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\phi(t,x)\\
&\leq \|\phi\|_{\infty}\int_{[0,T]\times Y} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\\
&\leq \|\phi\|_{\infty}\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\varphi^2(x)\chi_{\{u_n<\delta\}}\\
&\leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\textstyle\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\phi(t,x)\\
&= \textstyle\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\chi_{\{u=\delta\}}\phi(t,x)\\
&+ \textstyle\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}}\phi(t,x)\\
&\leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases}
\end{aligned} \tag{6.9}$$

We observe that there exists at most a countable set $\bar{D}_2$ such that $\operatorname{meas}\{(t,x) : u(t,x) = \delta\} > 0$. We choose δ outside of this set $\bar{D}_2$, so that, in (6.9), the integral

$$\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\chi_{\{u=\delta\}}\phi(t,x) = 0$$

So, we have

$$\begin{aligned}
&\textstyle\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\phi(t,x)\\
&= \textstyle\int_{Q_T} |g_n(t,x)|\,\frac{v_n}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}}\phi(t,x)\\
&\leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases}
\end{aligned} \tag{6.10}$$

Since by (5.7),

$$\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}} \to \chi_{\{u<\delta\}} \quad \text{a.e. in } Q_T$$

Applying Fatou's Lemma in (6.10) for δ fixed, leads to

$$\int_{Q_T} |g_n(t,x)|\,\frac{v}{u^{\gamma}}\chi_{\{u<\delta\}}\phi(t,x) \leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases}$$

Using again Fatou's Lemma in the last inequality for $\delta \to 0$, we get

$$\int_{Q_T} |g(t,x)| \frac{v}{u^\gamma} \chi_{\{u=0\}} \phi(t,x) = \int_{Q_T \cap \{u=0\}} |g(t,x)| \frac{v}{u^\gamma} \phi(t,x) = 0 \tag{6.11}$$

this leads to

$$\int_{Q_T} |g(t,x)| \frac{v}{u^\gamma} \phi(t,x) = \int_{Q_T \cap \{u>0\}} |g(t,x)| \frac{v}{u^\gamma} \phi(t,x) \tag{6.12}$$

which also means that

$$\int_{Q_T} g(t,x) \frac{v}{u^\gamma} \phi(t,x) = \int_{Q_T \cap \{u>0\}} g(t,x) \frac{v}{u^\gamma} \phi(t,x)$$

this is the desired result □

7. Proof of the main result

Now, we give the proof of the main result of this paper. Since $u_n, v_n \geq 0$ a.e. in Q_T, thanks to (5.3) and (5.4) we obtain $u, v \geq 0$. Thanks to the convergences (5.5) and (5.6), we can now go to the limit in the parts involving the time derivatives of (3.4) and (3.5).

By (5.11) and (5.12), the sequences (∇u_n) and (∇v_n) are equi-integrable. By (5.7), (5.8), (5.13) and (5.14), thanks to Vitali's Theorem (see Theorem 1.0.16 in [19]), we obtain

$$\nabla u_n \to \nabla u \quad \text{in } L^2(Q_T) \tag{7.1}$$

$$\nabla v_n \to \nabla v \quad \text{in } L^2(Q_T) \tag{7.2}$$

We deal now with the singular lower order terms. Let be $D = [0, T_1] \times K$, $T_1 < T$, such that $K \subset\subset E \subset\subset \Omega$ and $\psi \in C_0^\infty([0,T) \times \Omega)$ with $\operatorname{supp} \psi = D$. Let φ be a function such that $\varphi(x) = 1$ on the set K, $0 \leq \varphi \leq 1$ and $\operatorname{supp}(\varphi) = E$. For any $\delta > 0$ we have

$$\begin{aligned} &\int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(t,x) \\ &= \int_{Q_T \cap \{0 \leq u_n < \delta\}} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(t,x) \\ &+ \int_{Q_T \cap \{u_n \geq \delta\}} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(t,x) \\ &= A + B \end{aligned} \tag{7.3}$$

Concerning the term A, we proceed as follows :

$$\begin{aligned} A &\leq \|\psi\|_\infty \int_{D \cap \{0 \leq u_n < \delta\}} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^2(x) \\ &\leq \|\psi\|_\infty \int_{Q_T \cap \{0 \leq u_n < \delta\}} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^2(x) \end{aligned}$$

By (4.6), we get to

$$A \leq C\delta \tag{7.4}$$

where C is a constant independent of n. For the term B, we see that

$$\begin{aligned} B &= \int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \geq \delta\}} \chi_{\{u \neq \delta\}} \psi(t,x) \\ &+ \int_{Q_T} f_n(t,x) \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \geq \delta\}} \chi_{\{u = \delta\}} \psi(t,x) \\ &= B_1 + B_2 \end{aligned}$$

We observe that there is at most a countable set $\mathcal{O}_1$ such that meas$\{(t,x) : u(t,x) = \delta\} > 0$. We choose δ outside of this set $\mathcal{O}_1$, so that the term B_2 is zero. Since (5.7) holds, for the term B_1 we have that

$$\chi_{\{u_n\geq\delta\}}\chi_{\{u\neq\delta\}} \to \chi_{\{u>\delta\}} \quad \text{a.e. in } Q_T$$
$$f_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\geq\delta\}}\chi_{\{u\neq\delta\}}\psi(x,t) \leq f(t,x)\frac{v_n}{\delta^\gamma}\psi(t,x) \in L^1(Q_T)$$

Thanks to (5.7) and (5.8), the Lebesgue Dominate Convergence Theorem guarantees that

$$\begin{aligned}&\lim_{n\to+\infty}\int_{Q_T} f_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\geq\delta\}}\chi_{\{u\neq\delta\}}\psi(t,x)\\ =&\int_{Q_T} f(t,x)\frac{v}{u^\gamma}\chi_{\{u>\delta\}}\psi(t,x)\end{aligned}$$

Then

$$\lim_{n\to+\infty} B = \int_{Q_T} f(t,x)\frac{v}{u^\gamma}\chi_{\{u>\delta\}}\psi(t,x) \tag{7.5}$$

By (7.3), (7.4), (7.5) and (6.8), we can deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_T} f_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x) &= \lim_{\substack{\delta\to0\\ n\to\infty}}\int_{Q_T} f_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x)\chi_{\{u_n>\delta\}}\\ &= \int_{Q_T\cap\{u>0\}} f(t,x)\frac{v}{u^\gamma}\psi(t,x)\\ &= \int_{Q_T} f(t,x)\frac{v}{u^\gamma}\psi(t,x)\end{aligned}$$

In the same way, we have for any $\delta > 0$

$$\begin{aligned}&\textstyle\int_{Q_T} g_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x)\\ &= \textstyle\int_{Q_T\cap\{0\leq u_n<\delta\}} g_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x)\\ &+ \textstyle\int_{Q_T\cap\{u_n\geq\delta\}} g_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x)\\ &= \bar{A}+\bar{B}\end{aligned} \tag{7.6}$$

Concerning the term $\bar{A}$, we have

$$\left|\bar{A}\right| \leq \|\psi\|_\infty \int_{Q_T\cap\{0\leq u_n<\delta\}} |g_n(t,x)|\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\varphi^2(x)$$

By (4.9), we get to

$$\left|\bar{A}\right| \leq \|\psi\|_\infty.\begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases} \tag{7.7}$$

which implies $\lim_{\delta\to0}\left|\bar{A}\right| = 0$, where C is a constant independent of n. For the term $\bar{B}$, we see that

$$\begin{aligned}\bar{B} &= \int_{Q_T} g_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\geq\delta\}}\chi_{\{u\neq\delta\}}\psi(t,x)\\ &+ \int_{Q_T} g_n(t,x)\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\geq\delta\}}\chi_{\{u=\delta\}}\psi(t,x)\\ &= \bar{B}_1+\bar{B}_2\end{aligned}$$

We observe that there is at most a countable set $\mathcal{O}_2$ such that $\text{meas}\{(t,x) : u(t,x) = \delta\} > 0$. We choose δ outside of this set $\mathcal{O}_2$, so that the term $\bar{B}_2$ is zero. Since (5.7) holds, for the term $\bar{B}_1$ we have that

$$\chi_{\{u_n \geq \delta\}}\chi_{\{u\neq\delta\}} \to \chi_{\{u>\delta\}} \quad \text{a.e. in } Q_T$$
$$|g_n(t,x)| \frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\geq\delta\}}\chi_{\{u\neq\delta\}}\psi(x,t) \leq |g(t,x)| \frac{v_n}{\delta^\gamma}\psi(t,x) \in L^1(Q_T)$$

Thanks to (5.7) and (5.8), the Lebesgue Dominate Convergence Theorem guarantees that

$$\lim_{n\to\infty} \bar{B} = \int_{Q_T} g(t,x) \frac{v}{u^\gamma}\chi_{\{u>\delta\}}\psi(t,x) \tag{7.8}$$

By (7.6), (7.7), (7.8) and (6.8), we can deduce

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_T} g_n(t,x) \frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x) &= \lim_{\substack{\delta\to 0\\ n\to\infty}}\int_{Q_T} g_n(t,x) \frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi(t,x)\chi_{\{u_n>\delta\}}\\ &= \int_{Q_T\cap\{u>0\}} g(t,x)\frac{v}{u^\gamma}\psi(t,x)\\ &= \int_{Q_T} g(t,x)\frac{v}{u^\gamma}\psi(t,x)\end{aligned}$$

Repeating the same argument for u_n to deal with the case of v_n, but this time we use (4.7) and (4.8), which ends the proof of our main result Theorem 2.2.

8. Concluding remarks and perspectives

This work has mainly focused on the question of the existence of weak solutions for a class of singular reaction diffusion systems. Many important results have been obtained with additional assumptions that can be applied to extinction models and other models in biology, ecology, physics, and others as appropriate.

We have developed original methods to overcome certain difficulties, and despite the complexity of the model studied, we have succeeded in obtaining an existence result.

In addition to this work, we can address the following interesting questions :

- Question of uniqueness, by considering the notion of entropic solutions.
- Mathematical analysis of anisotropic system, which consists in adding diffusion coefficients to the studied system depending on (t,x) or more generally depending on $(t,x,u,\nabla u)$.
- Asymptotic behaviour of solutions.
- Numerical simulation.

This list of loose themes corresponds to work in progress or prospective. Some are a continuation of the work already done, and some are new research projects.

This not only makes it possible to be closer to the reality and concerns of the current industrial world, but also goes beyond the theoretical framework by developing models and tools that can be used and transferred to various industries.

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