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PÓLYA-AEPPLI PROCESS OF ORDER k OF THE SECOND TYPE WITH AN APPLICATION

Stefanka Chukova
School of Mathematics and Statistics,
Victoria University of Wellington of Wellington,
Wellington, NZ

Meglena Lazarova
Faculty of Applied Mathematics and Informatics,
Technical University of Sofia,
Sofia, Bulgaria

Leda Minkova*
Faculty of Mathematics and Informatics,
Sofia University,
Sofia, Bulgaria

Abstract: In this paper we propose and study the so called Pólya-Aeppli process of order k of the second type. Firstly, the process is defined using probability generating function, followed by its definition as a birth process. The distribution of the related counting process is presented by recursion formulae. The Pólya-Aeppli process of order k of the second type is considered within the framework of the risk process and corresponding probability of ruin is studied. Using simulation, some interesting results for the probability of ruin are obtained. Also, a comparison between the Pólya-Aeppli process of order k and Pólya-Aeppli process of order k of the second type is discussed.

Key words: Pólya-Aeppli distribution, Distributions of order k, Compound distributions, Ruin probability.

1. Introduction

Our motivation is based on the risk process, $\{X(t), t \geq 0\}$, and its use as a main tool in modeling of the surplus of an insurance company. In details, the risk process is given by

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i, \tag{1.1}$$

where c is a premium income per unit time, $N(t)$ is a counting process, $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent identically distributed, positive random variables, independent of $N(t)$, with Z_i representing the size of the ith claim. We assume that the individual claim amount has a continuous distribution with distribution function F, $F(0) = 0$, and mean value $\mu = EZ_1 < \infty$. In the classical risk model the process $N(t)$ is assumed to be a homogeneous Poisson process.

Let us consider the following stochastic process $N(t) = X_1 + \ldots + X_{N_1(t)}$, where $X_1, X_2, \ldots$ are mutually independent random variables and also independent of the process $N_1(t)$.

It is well known that if the compounding random variable X has a discrete distribution with a finite support and truncated at 0, the random variable $N(t)$ has a distribution of order k, see for example [1], [3], [8] and [2]

Pólya-Aeppli distribution of order k was introduced by [10], and applied as a counting distribution in the risk model considered in [4]. There, the random variable $N_1(t)$ is Poisson distributed

* Corresponding author. E-mail address:leda@fmi.uni-sofia.bg

with parameter λ and X_i are truncated geometrically distributed with probability mass function (PMF) and probability generating function (PGF) given by

$$P(X=i)=\frac{1-\rho}{1-\rho^k}\rho^{i-1}, \quad i=1,2,\dots k, \tag{1.2}$$

and

$$\psi_X(s)=\frac{(1-\rho)s}{1-\rho^k}\frac{1-\rho^k s^k}{1-\rho s},$$

where $k \geq 1$ is a fixed integer number and $\rho \in [0,1)$. As a result, the above process $N(t)$ is called Pólya-Aeppli process of order k, denoted by $PA_k(\lambda,\rho)$.

In this paper we introduce another Pólya-Aeppli process of order k and call it Pólya-Aeppli process of order k of the second type, and denote it by $PA_{kII}(\lambda,\rho)$. The two Pólya-Aeppli processes of order k are different due to the difference of the compounding distributions included in their definitions. In the truncated geometric distribution in (1.2) the mass from $k+1$ to infinity is uniformly distributed over the points $1,2,\dots,k$. Here, we consider the case when the mass from $k+1$ to infinity is clumped at point k.

So, what is the motivation for this new model $PA_{kII}(\lambda,\rho)$, and what is the difference between the current model and the model $PA_k(\lambda,\rho)$ in [4]? As mentioned above, our motivation for both modeling approaches comes from the risk process presented in (1.1). For the first model (the model in [4]) we consider (1.1) embedded in an usual operational environment for the insurance company, i.e., an environment without any major natural disasters or calamities, say storms, hurricanes, floods, earthquakes and so on. The only restriction we impose in this model is a limitation on the maximum possible number of simultaneous claims at any time, say k, which reasonably represents the reality faced by the insurance company in its everyday operations. Also, an environment with no major natural disasters, suggests no preference on any of the allowed integer numbers within $[1,k]$ of simultaneous claims, which means that the tail probability should be uniformly distributed over the domain $[1,k]$, leading to the truncated distribution considered in [4]. So, what is the motivation for the current model? The modeling in this study is for (1.1) embedded in an operational environment for the insurance company under the occurrence of a major natural disaster. As before, we preserve the limitation on the maximum possible number of simultaneous claims k at any time, but due to the external disastrous conditions we expect to have high number of simultaneous claims, so we place the tail probability at the maximum possible number of simultaneous claims to reflect the severity of the disaster, which leads to our model in this study.

The paper is organize as follows. In Section 2, we introduce the Pólya-Aeppli process of order k of second type. In Section 3, we define this process as a birth process. Some applications of this process to the risk model are given in Section 4. In Section 5, we present and discuss some simulation results related to Pólya-Aeppli process of order k of second type and Section 6 concludes this study.

2. Pólya-Aeppli process of order k of the second type

In this section we introduce the distribution of the Pólya-Aeppli process of order k of the second type as a compound Poisson distribution. The distribution of the compounding random variables X_i is given by the following PMF, which clumps the right tail of the distribution at point k:

$$P(X=i)=\begin{cases} (1-\rho)\rho^{i-1}, & i=1,2,\dots k-1 \\ \rho^{i-1}, & i=k. \end{cases} \tag{2.1}$$

The corresponding PGF is given by

$$\psi_X(s)=\frac{(1-\rho)s+(1-s)(\rho s)^k}{1-\rho s}. \tag{2.2}$$

Definition 1. The distribution defined by (2.1) or (2.2) is called a clumped geometric distribution with parameters k and $1-\rho$, and it is denoted by $CGe(k, 1-\rho)$.
In this case, the PGF of the $N(t)$ is given by

$$\psi_{N(t)}(s) = e^{-\lambda t\left(1-\frac{(1-\rho)s+(1-s)(\rho s)^k}{1-\rho s}\right)}. \tag{2.3}$$

Definition 2. The process defined by the PGF in (2.3) is called a Pólya-Aeppli process of order k of the second type with parameters $\lambda > 0$ and $\rho \in [0, 1)$, and denoted by $PA_{kII}(\lambda, \rho)$.
If $k \to \infty$, the clumped geometric distribution approaches the usual geometric distribution with parameter $1-\rho$.
If $k \to \infty$, the Pólya-Aeppli process of order k of second type, approaches the usual Pólya-Aeppli process, see [9] and [5]. If $\rho = 0$, it is the usual homogeneous Poisson process.
The mean and the variance functions of the $PA_{kII}(\lambda, \rho)$ are given by

$$EN(t) = \lambda t \frac{1-\rho^k}{1-\rho}$$

and

$$Var(N(t)) = \frac{\lambda t}{(1-\rho)^2}[1+\rho-(2k+1)\rho^k+(2k-1)\rho^{k+1}].$$

For the Fisher index, we obtain

$$FI(N(t)) = \frac{Var(N(t))}{E(N(t))} = \frac{1+\rho}{1-\rho} - 2k\frac{\rho^k}{1-\rho^k}.$$

The Fisher index of the distribution of the Pólya-Aeppli process is equal to $\frac{1+\rho}{1-\rho}$, see [5]. Hence, the distribution of the counting process $PA_{kII}(\lambda, \rho)$ is underdispersed with respect to the distribution of the Pólya-Aeppli process.

Let us denote by $P_n(t) = P(N(t) = n),\ n = 0, 1, \ldots$ The following proposition gives an extension of the Panjer recursion formulas, see [11].

Proposition 1. *The PMF of the $N(t) \sim PA_{kII}(\lambda, \rho)$ satisfies the following recursion formulae:*

$$\begin{aligned} P_1(t) &= \lambda t(1-\rho)P_0(t), \\ P_n(t) &= (2\rho + \tfrac{\lambda t(1-\rho)-2\rho}{n})P_{n-1}(t) - (1-\tfrac{2}{n})\rho^2 P_{n-2}(t), \quad n = 2, 3, \ldots k-1 \\ P_n(t) &= (2\rho + \tfrac{\lambda t(1-\rho)-2\rho}{n})P_{n-1}(t) - (1-\tfrac{2}{n})\rho^2 P_{n-2}(t) + \lambda t\rho^k \tfrac{k}{n} P_{n-k}(t) \\ &\quad -\lambda t\rho^k[\tfrac{k+1}{n} + \tfrac{k-1}{n}\rho]P_{n-k-1}(t) + \lambda t\rho^{k+1}\tfrac{k}{n}P_{n-k-2}(t), \quad n = k, k+1, k+2, \ldots \end{aligned}$$

and $P_{-1}(t) = P_{-2}(t) = 0$.

Proof. Differentiation in (2.3) leads to

$$(1-\rho s)^2 \frac{\partial}{\partial s}\psi_{N(t)}(s) = \lambda t[1-\rho+k\rho^k s^{k-1} - \rho^k((k+1)+(k-1)\rho)s^k + k\rho^{k+1}s^{k+1}]\psi_{N(t)}(s), \tag{2.4}$$

where $\psi_{N(t)}(s) = \sum_{n=0}^{\infty} P_n(t)s^n$ and $\frac{\partial}{\partial s}\psi_{N(t)}(s) = \sum_{n=0}^{\infty}(n+1)P_{n+1}(t)s^n$. The recursions are obtained by equating the coefficients of s^n on both sides of (2.4) for fixed $n = 0, 1, 2, \ldots$.

□

3. Pólya-Aeppli process of order k of the second type as a birth process

Suppose that $N(t) \sim PA_{kII}(\lambda, \rho)$. The properties of this process are specified by the following assumptions: For any small $h > 0$

$$P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - \lambda h + o(h), & n = m, \\ (1-\rho)\rho^{i-1}\lambda h + o(h), & n = m+i, \\ & i = 1, 2, \ldots, k-1, \\ \rho^{k-1}\lambda h + o(h), & n = m+k, \end{cases} \tag{3.1}$$

for every $m = 0, 1, \ldots,$ where $o(h) \to 0$ as $h \to 0$. Note that the assumptions imply that for $i = k+1, k+2, \ldots,$ $P(N(t+h) = m+i \mid N(t) = m) = o(h)$.

The above assumptions yield the following Kolmogorov forward equations:

$$\left| \begin{array}{l} P_0'(t) = -\lambda P_0(t), \\ P_n'(t) = -\lambda P_n(t) + (1-\rho)\lambda \sum_{j=1}^{n} \rho^{j-1} P_{n-j}(t), \quad n = 1, 2, \ldots, k-1, \\ P_n'(t) = -\lambda P_n(t) + (1-\rho)\lambda \sum_{j=1}^{k-1} \rho^{j-1} P_{n-j}(t) + \lambda \rho^{k-1} P_{n-k}(t), \quad n = k, k+1, \ldots, \end{array} \right. \tag{3.2}$$

with the conditions

$$P_0(0) = 1 \quad \text{and} \quad P_n(0) = 0, \quad n = 1, 2, \ldots. \tag{3.3}$$

Multiplying the nth equation of (3.2) by s^n and summing for all $n = 0, 1, 2, \ldots$ we get the following differential equation

$$\frac{\partial \Psi_{N(t)}(s)}{\partial t} = -\lambda[1 - \psi_X(s)]\Psi_{N(t)}(s). \tag{3.4}$$

The solution of (3.4) with the initial condition

$$\Psi_{N(1)}(s) = 1$$

is given by (2.3), which is the PGF of the distribution of $PA_{kII}(\lambda, \rho)$. This leads to the following definition for the Pólya-Aeppli process of order k of second type, namely:

Definitin 3. The process defined by (3.2) and (3.3) is the Pólya-Aeppli process of order k of second type.

4. Application to risk model

We consider the risk model (1.1), where $N(t) \sim PA_{kII}(\lambda, \rho)$. We call this model a Pólya-Aeppli of order k of second type risk model. In this case the relative safety loading θ is defined by

$$\theta = \frac{EX(t)}{E\sum_{i=1}^{N(t)} Z_i} = \frac{c(1-\rho)}{\lambda\mu(1-\rho^k)} - 1.$$

To ensure that $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \frac{\lambda\mu(1-\rho^k)}{1-\rho}.$$

Denote by $\tau = \inf\{t : X(t) < -u\}$ the time to ruin of an insurance company having initial capital $u \geq 0$, and by

$$\Psi(u) = P(\tau < \infty) \tag{4.1}$$

the related ruin probability. Let $G(u,y)$ be the probability of the following event: $\{$ruin occurs with initial capital u and deficit, immediately after ruin occurs, is at most $y\}$ with $u \geq 0$ and $y \geq 0$. Hence

$$G(u,y) = P(\tau < \infty, D \leq y), \tag{4.2}$$

where $D = |u + X(\tau)|$ is the deficit immediately after ruin occurs. Therefore

$$\lim_{y \longrightarrow \infty} G(u,y) = \Psi(u). \tag{4.3}$$

Using the assumptions in (3.1), and for any small $h > 0$, we have

$$\begin{aligned} G(u,y) &= (1-\lambda h)\, G(u+ch,y) + \\ &+ (1-\rho)\lambda h \sum_{i=1}^{k-1} \rho^{i-1} \left[\int_0^{u+ch} G(u+ch-x,y) dF^{*i}(x) + \left(F^{*i}(u+ch+y) - F^{*i}(u+ch) \right) \right] + \\ &+ \rho^{k-1}\lambda h \left[\int_0^{u+ch} G(u+ch-x,y) dF^{\star k}(x) + \left(F^{\star k}(u+ch+y) - F^{\star k}(u+ch) \right) \right] + o(h), \end{aligned} \tag{4.4}$$

where $F^{\star i}(x), \quad i = 1, 2, \ldots$ is the distribution function of $Z_1 + Z_2 + \ldots + Z_i$.

Let us denote by

$$H(x) = (1-\rho) \sum_{i=1}^{k-1} \rho^{i-1} F^{*i}(x) + \rho^{k-1} F^{*k}(x) \tag{4.5}$$

the non defective probability distribution function of the claims with

$$H(0) = 0, \qquad H(\infty) = 1.$$

Rearranging the terms in (4.4) and letting $h \to 0$ we obtain the following differential equation

$$\frac{\partial G(u,y)}{\partial u} = \frac{\lambda}{c} \left[G(u,y) - \int_0^u G(u-x,y) dH(x) - [H(u+y) - H(u)] \right]. \tag{4.6}$$

In terms of the safety loading the equation has the form

$$\frac{\partial G(u,y)}{\partial u} = \frac{1-\rho}{\mu(1-\rho^k)} \frac{1}{1+\theta} \left[G(u,y) - \int_0^u G(u-x,y) dH(x) - [H(u+y) - H(u)] \right]. \tag{4.7}$$

4.1. Ruin probability

THEOREM 1. *The probability of ruin $\Psi(u)$ satisfies the equation*

$$\frac{d\Psi(u)}{du} = \frac{\lambda}{c} \left[\Psi(u) - \int_0^u \Psi(u-x) dH(x) - [1 - H(u)] \right], \quad u \geq 0. \tag{4.8}$$

PROOF. The result follows from (4.6) and (4.3).

□

Similarly to [4], we obtain the function $G(0,y)$ given by

$$G(0,y) = \frac{\lambda}{c} \int_0^y [1 - H(u)] du, \tag{4.9}$$

and for the ruin probability with no initial capital we obtain

$$\Psi(0) = \frac{\lambda\mu}{(1-\rho)c} (1-\rho^k). \tag{4.10}$$

4.2. Exponentially distributed claims

Let us consider the case of exponentially distributed claim sizes with mean μ, i.e. $F(x) = 1 - e^{-\frac{x}{\mu}}, \quad x \geq 0, \quad \mu > 0$. In this case, the function

$$F^{*i}(x) = 1 - \sum_{j=0}^{i-1} \frac{\left(\frac{x}{\mu}\right)^j}{j!} e^{-\frac{x}{\mu}}, \; x \geq 0$$

is an Erlang distribution function. Then, the distribution function $H(x)$ in (4.5) is given by

$$H(x) = 1 - \sum_{i=0}^{k-1} \frac{\left(\frac{\rho x}{\mu}\right)^i}{i!} e^{-\frac{x}{\mu}}.$$

The density function $h(x)$ has the form

$$h(x) = \frac{1}{\mu}\left[(1-\rho)\sum_{i=0}^{k-2} \frac{\left(\frac{\rho x}{\mu}\right)^i}{i!} + \frac{\left(\frac{\rho x}{\mu}\right)^{k-1}}{(k-1)!}\right] e^{-\frac{x}{\mu}}.$$

So, the initial condition (4.9) in the case of exponential distribution is

$$G(0,y) = \frac{\lambda\mu}{c} \sum_{i=0}^{k-1} \frac{\rho^i}{i!} \gamma(i+1, y/\mu),$$

where $\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$ is the incomplete Gamma function.

5. Simulation

In what follows, we apply the simulation approach for calculating the probability of ruin suggested in [6] for the case of exponentially distributed claims with initial capital $u = 0$. We confirm the validity of our simulated results by matching them with the value of the ruin probability computed analytically using (4.10). Then, using our simulator, we provide results for the case of non-zero initial capital not only for exponentially distributed claims but also for claims with gamma and Weibull distributions. For a summary of the simulation approach for calculating the probability of ruin see [4]. All of our simulation results are based on 3 000 000 runs. Next, we provide some results regarding the probability of ruin for different scenarios of the claim distribution as well as the value of the initial capital.

5.1. Results

We consider the case of exponentially distributed claims and no initial capital $u = 0$. We verify the correctness of our simulator by comparing the results for the probability of ruin for fixed model parameters, produced in two different ways : (i) by the simulator, given in column "simulated", and (ii) computed using (4.10) given in column "analytical". These are given in Table 1.

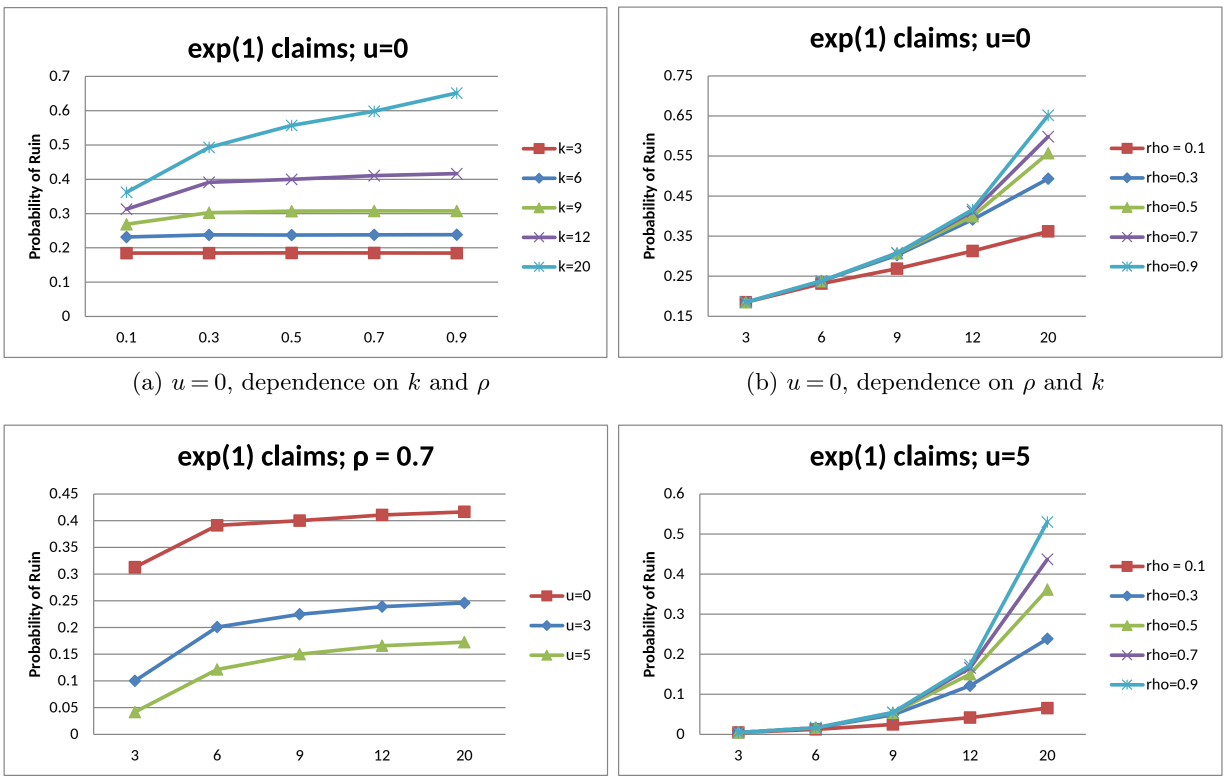
λ	k	ρ	simulated Exp(1)	analytical Exp(1)
1.0	15	0.6	0.208117	0.208235
1.5	4	0.8	0.316531	0.316286
2.0	10	0.4	0.256365	0.256383
2.5	3	0.9	0.423526	0.423437
3.0	6	0.2	0.288426	0.288443

Table 1: Simulated and analytical Exp(1)

As it is easy to see, the "analytical" and "simulated" results are very close. So, we use our simulator, written in MATHEMATICA, to compute a reasonable approximation of the probability of ruin for non-exponentially distributed claims and non-zero initial capital ($u \neq 0$) and a summary of our results is given in subsection 5.1.1.

5.1.1. Case 1: Exponentially distributed claims

Here, we present some simulation results for the case of exponentially distributed claims with non-zero initial capital.



(a) $u=0$, dependence on k and ρ

(b) $u=0$, dependence on ρ and k

(c) $\rho=0.7$, dependence on u and k

(d) $u=5$, dependence on ρ and k

FIGURE 1. Probability of ruin: exponentially distributed claims

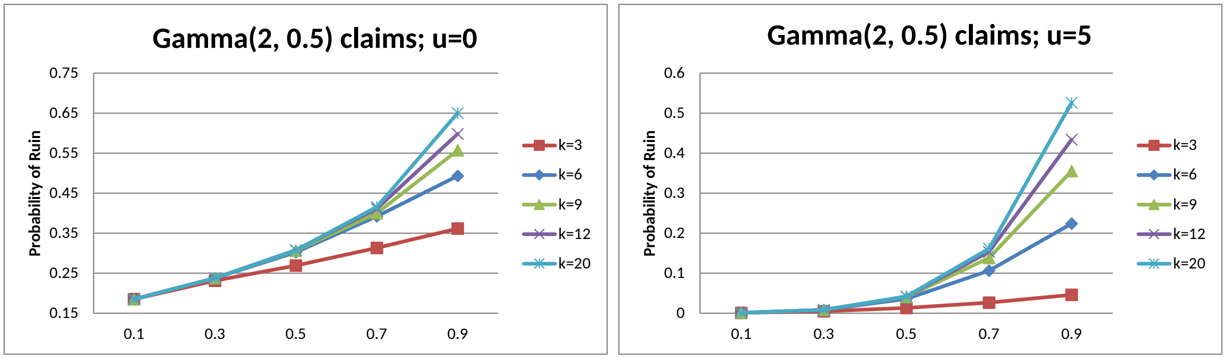
Comparing part(b) and part(d) of Figure 1, both with x-label k, it is easy to see that the probability of ruin is shifted downwards as the initial capital increases. If the initial capital is $u=0$, the smallest values for the probability of ruin is just above 0.35 for $\rho=0.1$, whereas the analogous value for $u=5$ is just below 0.1. The depicted overall dependence on ρ, regardless of the value of the initial capital, is as expected, the probability of ruin increases as ρ increases. The overall trends depicted in part(a), with x-label ρ, and part(c), with x-label k, of Figure 1 also agree with our intuition. Namely, for a fixed value of ρ, the probability of ruin is higher for low values of the initial capital and it increases on k. It is worth to point out the sharp increase of the probability of ruin for large values of ρ and large k, as shown in part(a) of Figure 1.

5.1.2. Case 2: Gamma distributed claims

Next, we consider gamma distributed claims with parameters α and β, i.e., the density function of the claim sizes is

$$f(x) = \frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{x}{\beta}}, \quad x \geq 0,$$

where $\Gamma(\alpha)$ is the Gamma function. Suppose that $\alpha = 2$ and $\beta = 0.5$. In this case the mean values of the claims are $EZ_i = \alpha\beta = 1$. We present results for different values of the model parameters u, k and ρ.



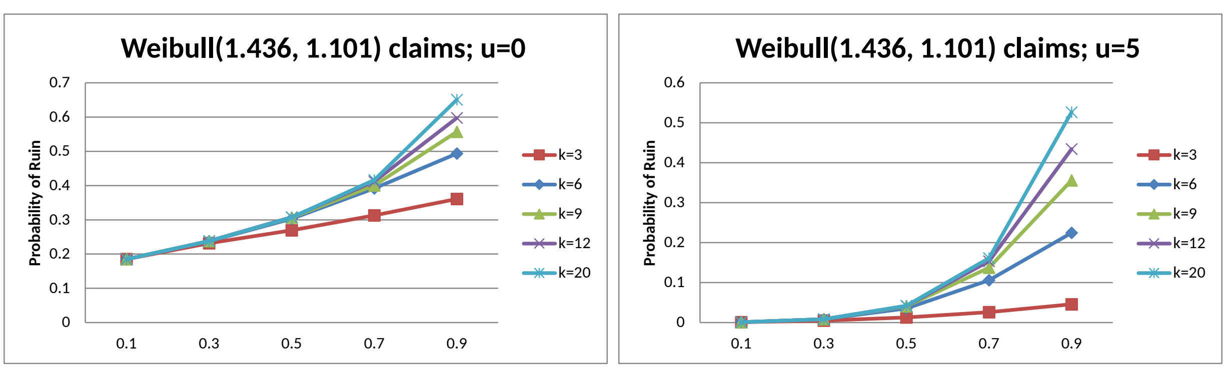
(a) $u = 0$, dependence on k and ρ (b) $u = 5$, dependence on k and ρ

FIGURE 2. Probability of ruin: gamma distributed claims

The trends observed for the gamma distributed claims are similar to the one we have presented and discussed for the case of exponentially distributed claims in subsection 5.1.1. Here, in Figure 2, with x-label ρ, we depict the dependence of the probability of ruin from u, for similar ρ and k. Overall the probability of ruin for lower value of the capital u is higher, similar to what we have observed in the exponential case. In addition we see that for high values of u and ρ, k have a strong impact on the probability of ruin, e.g., see for $u = 0$ and $\rho = 0.9$, the range of the probability of ruin is approximately $(0.35, 0.65)$, whereas for $u = 5$ this range is much larger, approximately $(0.05, 0.53)$.

5.1.3. Case 3: Weibull distributed claims

Next, we consider the Weibull distribution with parameters $\alpha = 1.43552259$ and $\beta = 1.1013206$ distributed claims. Here α is the shape parameter and β is the scale parameter. The parameters of the Weibull and gamma distributions were selected so that the three claim size distributions considered in sections 5.1.1, 5.1.2 and 5.1.3 have the same expectation $\mu = 1$ and the Weibull and gamma claim sizes have the same variances.



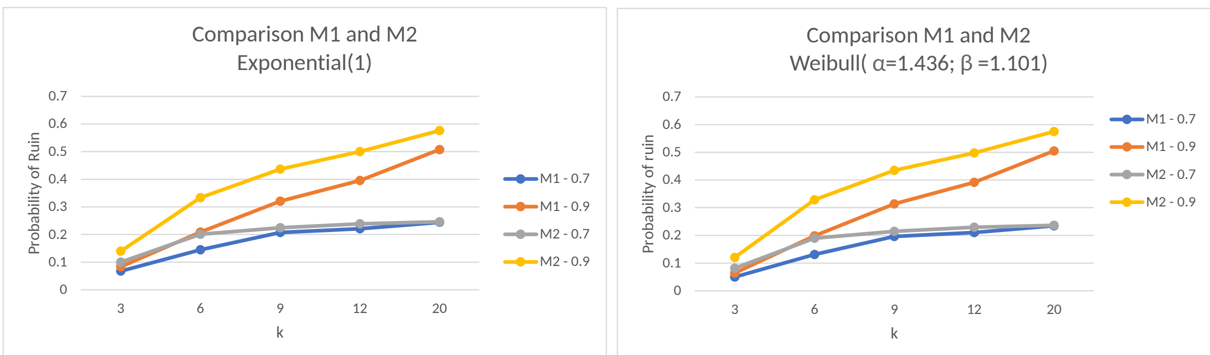
(a) $u = 0$, dependence on k and ρ (b) $u = 5$, dependence on k and ρ

FIGURE 3. Probability of ruin: Weibull distributed claims

We were quite surprised to see that the behavior of the probability of ruin under Weibull distributed claims, part(a) and part(b) in Figure 3, with x-label ρ, mimics quite closely the behavior of this probability for gamma distributed claims. So, then the natural question is: under a risk model based on the Pólya-Aeppli process of order k, are the mean value and the variance of the claim distribution what determines the probability of ruin, i.e., the actual form of the claim size distribution does not have an effect on the probability of ruin. Interestingly, similar observations were made in [4]. Again, observing these results is a good motivation for future research because at this point we are not able to answer this question.

5.2. Comparison between M1 and M2

For brevity we will refer to the current model as M2 and to $PA_k(\lambda,\rho)$ from [4] as M1. Here we provide a brief comparison between the probabilities of ruin for the two models.



(a) $\rho = 0.7, 0.9,\ u = 3$ (b) $\rho = 0.7, 0.9,\ u = 3$

FIGURE 4. Probability of ruin: comparison between M1 and M2

In part(a) and part(b) in Figure 4, we fix the value of the parameter $u = 3$, and illustrate the dependence of the probability of ruin for M1 and M2 for two different values of $\rho = 0.7, 0.9$. Again, the probability of ruin for M1 and M2 is similar for the selected exponential and Weibull claim size distributions. The probability of ruin is an increasing function of k and its value is shifted upwards for higher values of parameter ρ. As expected, the probability of ruin for M2 is higher than for M1 and this is exactly what we expect to observe as an outcome for the insurance company at the time of severe natural disaster. Having model $PA_{kII}(\lambda,\rho)$ in place provides a reasonable theoretical background for the company to plan accordingly for natural calamities. From the observations above a natural question arises: are there any condition on the mean and the variance of the claim size distribution that will guaranty the satisfaction of some inequalities on the related ruin probabilities. These inequalities will be very useful in the sense that, even at the time of calamity, the probability of ruin would not exceed a known value. Again, further numerical and theoretical studies are needed to gain some insight on this question.

6. Conclusions

In the present study we have defined and studied the Pólya-Aeppli process of order k of second type as a compound Poisson process with clumped geometric compounding distribution with success probability equal to $1-\rho > 0$. We have discussed some possible application of this process in risk theory. We have studied the probability of ruin for the related risk model and have derived an exact expression for the ruin probability in the particular case of zero initial capital. Also, we have adopted a simulation approach, given in [6] for our particular model. Using this simulation approach we have provided results for general cases of the model, such as non-exponential claim distribution and non-zero initial capital. The simulation results have opened for discussion several very interesting questions related to the probability of ruin for Pólya-Aeppli of order k second type risk model. Also, a motivation for $PA_k(\lambda, \rho)$ studied in [4] and $PA_{kII}(\lambda, \rho)$ is outlined and a comparison between these two models is discussed.

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