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A New approach for the pseudo-quaternionic lorentzian evolute and involute curves

Pseudo-quaternionic lorentzian evolute and ınvolute eğriler için yeni bir yaklaşım

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A New Approach for the Pseudo-Quaternionic Lorentzian Evolute and Involute Curves

Highlights

- *The curve in 4-dimensional Lorentz space, which is derived from the curves obtained with Pseudo quaternions in 3D Lorentz space, is a pseudo quaternionic curve. Evolute and involute curves are defined for the obtained pseudo quaternionic timelike Lorentz curve.*
- *Some characterizations have been obtained for these curves in 3-dimensional and 4-dimensional Lorentz space.*

Graphical Abstract

The curve in 4-dimensional Lorentz space, which is derived from the curves obtained with Pseudo quaternions in 3D Lorentz space, is a pseudo quaternionic curve. Evolute and involute curves are defined for the obtained pseudo quaternionic timelike Lorentz curve. Some characterizations have been obtained for these curves in 3-dimensional and 4-dimensional Lorentz space.

Aim

Defining the evolute and involute curves for a timelike quaternion curve in quaternionic Lorentz space

Design & Methodology

Quaternion space is defined. Then, quaternionic curves in Lorentz space are defined and using this approach, evolute and involute curves are defined..

Originality

Evolute and involute curves are defined in Lorentz space for pseudo quaternionic timelike curves with the help of quaternion algebra. Previously, the evolute / involute curve pair was defined in Lorentz space, but it was re-represented with the help of Pseudo Quaternions.

Conclusion

Some characterizations are defined for Pseuudo Quaternionic evolute / involute curves obtained in Lorentz Space. Some special cases of these curves have been studied in depth. These curves; The helix can be planar or their evoluts planar or helical curves. Although evolute/involute curves have been defined in Lorentz space before, Pseudo Quaternionic timelike curves are presented in a form that has not been used before

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

A New Approach for the Pseudo-Quaternionic Lorentzian Evolute and Involute Curves

Research Article

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ABSTRACT

In this study, the main purpose is to obtain pseudo quaternionic Lorentzian evolute-involute curves on L_Q^4. In fact the pseudo quaternion is a quaternion which has 3-dimension and its temporal part is ziro. So it has only vectoral part of the quaternions. Thus, if defined a pseudo-quaternionic curve on L_Q^4, then its quaternionic part is pseudo quaternion. What is meant to do here, to write out Lorentzian evolute-involute curves of the quaternionic time-like curves in L_Q^4 using quaternions and to reveal some relationships between these curves.

Keywords: Quaternion space, Lorentzian space, binary operation, symmetric bilinear form.

1. INTRODUCTION

The involute-evolute curves in $\mathbb{R}^3$ are well known in elementary differential geometry. If a curve is differentiable at the each point of an open interval then a set of mutually orthogonal unit vectors can be constructed. These vectors are called tangent, normal and binormal unit vectors. The set of these vectors is $\{t(s), n_1(s), n_2(s)\}$ and it is named Serret-Frenet Frame. The set of frame vectors and curvatures of a curve is called Serret-Frenet Apparatus of the curve. The derivative equations of this vectors are

$$t'(s) = k_1(s)n_1(s),$$
$$n_1'(s) = -k_1(s)t(s) + k_2(s)n_2(s),$$
$$n_2'(s) = k_2(s)n_1(s).$$

Here $k_1(s)$ and $k_2(s)$ are curvatures with respect to arc length.

Let α and β be two curves on E^3 which have (I, X), (I, Y) coordinate neighborhoods, respectively. Let the Serret-Frenet vectors at the point $\alpha(s)$ of the curve α are $t(s), n_1(s), n_2(s)$ and the Serret-Frenet vectors at the point $\beta(s)$ of the curve $\beta(s)$ are $\tilde{t}(s), \tilde{n}_1(s), \tilde{n}_2(s)$. If $t(s)$ is orthogonally to $\tilde{t}(s)$ for each parameters, that is $\left\langle t(s), \tilde{t}(s) \right\rangle = 0$ then $\beta(s)$ is called involute of (s), $\alpha(s)$ is called evolute of $\beta(s)$, too [1].

William R. Hamilton was discovered the quaternions. He said that; the quaternion is appropriate is generalization one in which the real axis. Quaternions are used in many fields. Some of these are computer images and computer graphics. They can be also used in mechanics because quaternionic formulation of equation of motion in the theory of relativity.

Quaternions are constructed as a linear combination of a 3D vector with a real value. The real quaternions are coincide with $\mathbb{R}^4$. It was a four dimensional vector space over the real numbers.

Hacısalihoğlu has worked on motion geometry and quaternions [2]. Quaternion algebra is studied by Bharathi and Nagaraj [3, 4]. Later Bilici and Çalışkan have worked in many ways as time like and space like different curves [5, 6]. Bükcü and Karacan studied on the evolute and involute curves on Lorentzian space [7]. Soyfidan and Güngör also have studied this issue [8]. Kalkan et al. have studied on this issue for some special curves [9]. Altın et al studied with constant weighted curvature in Lorentzian Space [10]. Karadağ and Karadağ studied on null generalized slant helices in Lorentzian Space[11]. Later, Karadag and Sivridağ have studied many cases and gave characterizations on quaternionic curve [12,13]. In fact many famous mathematicians are working on curves of Lorentz Space. Some of those are O'Neil[14].Ozturk and Ozturk, Ilarslan, [15].Then, Almaz and Külahçı are working on Lorentzian curves [16]. Erişir and Güngör studied on quaternionic Lorentzian curve ,too[17]. Karacan and at.al studied this issue [18]. Millman and Parker studied Elements of Differential Geometry [19].

The main purpose of this study, to obtain pseudo quaternionic Lorentzian evolute-involute curves on L_Q^3. Since it is trivial task to write out Lorentzian evolute-involute curves of the quaternionic time-like curve in L_Q^4 using quaternions and to reveal some relationships between these curves. In this study L_Q^4 denotes the 4-dimensional Quaternionic Lorentzian Space.

2. PRELIMINARIES

Like a real quaternion, it is defined as four sequential numbers accompanied by four units, where $e_4 = 1$ is the real number. The other three units are

i) $e_i \times e_j = e_k = -e_j \times e_i$
ii) $e_A \times e_4 = e_4 \times e_A = e_A$
iii) $e_i \times e_i = -e_4 = -1$

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Here $A = \{1,2,3\}$ is the set of indices.

Thus a quaternion expressed as $q = q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 \in Q$, [2].

A quaternion q consists of a vector part denoted by V_q and a scalar part denoted by S_q. Thus, it is denoted as $= S_q + V_q$.

Sum of two quaternions is defined by

$$\oplus: QXQ \to Q$$

$$(q,p) \to q \oplus p = S_{q+p} + V_{q+p}, \tag{1}$$

for $\forall p, q \in Q$,

product of a scalar and a quaternion is shown in the format

$$\odot: \mathbb{R}XQ \to Q$$

$$(\lambda, q) \to \lambda \odot q = \lambda S_q + \lambda V_q\ , \tag{2}$$

$\forall \lambda \in \mathbb{R}$ and $\forall q \in Q$.

Also the product of two quaternions is defined by

$$\times: QXQ \to Q$$

$$(q,p) \to q \times p = S_qS_p + S_qV_p + S_pV_q - \langle V_q, V_p\rangle + V_q \wedge V_p, \tag{3}$$

for $\forall p, q \in Q$.

The conjugation of $q = S_q + V_q$ is defined by

$$\alpha: Q \to Q, \alpha q = S_q - V_q \tag{4}$$

The symmetric, bilinear form of the space of quaternions is defined as

$$h: QXQ \to Q$$

$$(q,p) \to h(q,p) = \frac{1}{2}[q \times \alpha p + p \times \alpha q], \tag{5}$$

for $\forall p, q \in Q$

and the norm of the quaternion is shown by

$$\|\cdot\|: Q \to \mathbb{R},\ q \to \|q\|.$$

On the other hand

$$h(q,q) = \|q\|^2 = q \times \alpha q, \tag{6}$$

for $\forall q \in Q$ [4].

In generally, for each q-quaternion is defined as

$$q = \frac{1}{2}[(q + \alpha q) + (q - \alpha q)] \tag{7}$$

Here, $\frac{1}{2}[q + \alpha q]$ is space part and $\frac{1}{2}[q - \alpha q]$ is temporal part of this quaternion, respectively [3, 4].

If the $q = q_1e_1 + q_2e_2 + q_3e_3 \in Q$ this quaternion is called a psequat (pseudo-quaternion) which is a space quaternion. Let : $I \to L_Q^3$, s $\to$ $\beta(s) = \beta_1(s)e_1 + \beta_2(s)e_2 + \beta_3(s)e_3$, be a pseudo-quaternionic curve, then every $\alpha(s) = \beta_1(s)e_1 + \beta_2(s)e_2 + \beta_3(s)e_3 + \alpha_4(s)e_4$ curve derived from β is also a pseudo-quaternionic curve on L_Q^4.

By matching the set of quaternions L_Q^3 and L_Q^4, the curves in L_Q^3 can be calculated with the help of the Serret-Frenet formulas for pseudo quaternionic curves. The matrix form of Lorentzian this formulae for a pseudo space-quaternionic Lorentzian time-like curve in L_Q^3 rederived as

$$\begin{bmatrix} \dot{t} \\ \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \end{bmatrix} \tag{8}$$

[12]. Then, by making use of (2.6), we have rederived this formulae for a quaternionic Lorentzian curve on L_Q^4 as follows

$$\begin{pmatrix} \dot{T} \\ \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}3 \end{pmatrix} = \begin{pmatrix} 0 & K & 0 & 0 \\ K & 0 & k & 0 \\ 0 & k & 0 & r-K \\ 0 & 0 & -(r-K) & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} \tag{9}$$

Definition 2.1 Let $I \subset Q$ and $\beta: I \to L_Q^4$, then

$\beta(s) = \sum_{i=1}^{4} \beta_i(s)\vec{e}_i$, $(\vec{e}_4 = +1)$ is called a quaternionic curve which is one-quaternion variable valued β transformation .

Especially, if $\beta(I) \subset Q_{\mathbb{R}^3}$ then β is called space quaternionic curve [12]. The set of space quaternions, that is $W = \{q \in Q: q + \alpha q = 0\}$ and $\mathbb{R}^3$ (3-dimensional real Euclidean space) are isomorphic. Thus, the Frenet formulas and curvatures of curves in $\mathbb{R}^3$ can be calculated with the help of space-quaternions [13].

The Serret-Frenet formulas for a time-like quaternion curve in the quaternionic Lorentz space are also calculated for a space quaternionic Lorentz curve. Using these relations, the Serret-Frenet derivative formulas for the quaternionic Lorentz curve are calculated [13].

In this study, by using this formulation, I will be able to calculate some relations for quaternionic Lorentzian evolute-involute curves.

Definition 2.2 The two bilinear forms on L_Q^4 are defined as,

$$g(x,y) = \sum_{i=1}^{3} x_iy_i - x_4y_4,\ h(x,y) = \sum_{A=1}^{4} x_Ay_A \tag{10}$$

for $\forall x, y \in L_Q^4$ [3].

In this define, for given g and h bilinear forms, like equation (3) if b any symmetric bilinear form and " $\circ$ " is a binary operation then

$$x \circ y = [x,y] + x_4S_y + y_4S_x - b(x,y)\vec{e}_4 \tag{11}$$

If $b = h$ then the quaternion is called pseudo quaternionic or $b = g$ then the quaternion is called quaternionic [3, 4].

3. MATERIAL and METHOD

Definition 3.1 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two time-like curve on L_Q^4 which have (I,X), (I,Y) coordinate neighborhoods, respectively. Let the Serret-Frenet vectors at the point of $\tilde{\alpha}(s)$ of the vurve $\tilde{\alpha}$ is $\{T(s), N_1(s), N_2(s), N_3(s)\}$ and the Serret-Frenet vectors at the point of $\tilde{\beta}(s)$ of the curve $\tilde{\beta}$ is $\{T^*(s), N_1^*(s), N_2^*(s), N_3^*(s)\}$. If $T(s)$ is g-orthogonally to $T^*(s)$ for each parameter s, that is

$g(T(s), T^{*}(s)) = 0$ (12)

then $\tilde{\beta}$ is called involute of $\tilde{\alpha}$, $\tilde{\alpha}$ is called evolute of $\tilde{\beta}$, too.

Theorem 3.1 If the α pseudo quaternionic time-like curve is evolute of the β pseudo quaternionic time-like curve then (res. it has involute equation of α) it is

$Y(s) = X(s) + (c-s)t(s),\ c = constant$ (13)

Proof. Let α be a pseudo quaternionic time-like curve. Then,

$\dot{\alpha}(s) = t(s)\ and\ \tilde{g}(t(s), t(s)) = -1$. From definition of evolute can be written;

$Y(s) = X(s) + \sigma t(s)$ (14)

If the derivative of equation (14) with respect to s, then it can be shown as;

$Y'(s) = t(s) + \sigma' t(s) + \sigma t'(s)$ (15)

Since α is evolute of β

$\tilde{g}(Y'(s), t(s)) = 0$ (16)

Thus, it can be obtained;

$\tilde{g}(t(s) + \sigma' t(s) + \sigma t'(s), t(s)) = 0$ (17)

$\tilde{g}\big(t(s), t(s)\big) + \sigma'\tilde{g}\big(t(s), t(s)\big) + \sigma\tilde{g}\big(t'(s), t(s)\big) = 0$ (18)

for a regular curve is known as;

$\tilde{g}(t'(s), t(s)) = 0$ hence, it can be written as;

$(1+\sigma')\tilde{g}(t(s), t(s)) = 0,$

and since X is a time-like curve

$\tilde{g}(t(s), t(s)) = -1$ (19)

Therefore, it can be found that;

$-(1+\sigma') = 0\ ; \sigma' = -1; \sigma = -s + \mathrm{c}$

Thus, it is obtained;

$Y(s) = X(s) + (c-s)t(s), c = constant$ (20)

The proof is completed.

As a result of this theorem, it can be found that;

$Y(s) - X(s) = (c-s)t(s)$

$\|Y(s) - X(s)\|^2 = \|(c-s)t(s)\|^2$

$= \tilde{g}\big((c-s)t(s), (c-s)t(s)\big)$

$= (c-s)^2\tilde{g}(t(s), t(s))\ d(Y(s), X(s)) = |c-s|$ (21)

Hence, the distance between of corresponding points to pair of evolute and involute curves is given as

$l = (c-s)$ (22)

Now, accounting the distance between of corresponding points to pair of evolute and involute curves on quaternionic Lorentzian space:

Let $\tilde{\beta}$ be a quaternionic curve which is derived X-curve in L_Q^4. That is,

$\tilde{\beta}(s) = x_1(s)\vec{e}_1(s) + x_2(s)\vec{e}_2(s) + x_3(s)\vec{e}_3(s) + \beta_4(s)\vec{e}_4(s)$ (23)

Then, consider that $\tilde{\beta}$ is a time-like quaternionic curve parametrized as

$\tilde{\beta}'(s) = T(s),\ g(T(s), T(s)) = -1$ (24)

On the other hand, the equation of the involute of the curve $\tilde{\alpha}(I)$ is

$\tilde{\beta}(s) = \tilde{\alpha}(s) + \rho T(s)$ (25)

then, here ρ is constant.

$\widetilde{\beta'}(s) = T(s) + \rho' T(s) + \rho T'(s)$ (26)

and it is known that $\tilde{\beta}'(s)$ is g-orthogonally to $T(s)$. That is;

$g\big(\tilde{\beta}'(s), T(s)\big) = 0$ (27)

If it can use of (26) in (27), it is defined as;

$g(T(s) + \rho' T(s) + \rho T'(s), T(s)) = 0$ (28)

$g(T(s), T(s)) + \rho' g(T(s), T(s)) + \rho g(T'(s), T(s)) = 0$ (29)

$(1+\rho')g(T(s), T(s)) = 0$ (30)

from $\tilde{\alpha}(s)$ is a time-like quaternionic Lorentzian curve, $-(1+\rho') = 0; \rho' = -1; \rho = c - s.$

Hence the equation of evolute of the curve $\tilde{\beta}(I)$ is find as

$\tilde{\beta}(s) = \tilde{\alpha}(s) + (c-s)T(s)$ (31)

So it can be written:

$\tilde{\beta}(s) - \tilde{\alpha}(s) = (c-s)T(s).$ (32)

Therefore it has

$\|\tilde{\beta}(s) - \tilde{\alpha}(s)\|^2 = g\big((c-s)T(s), (c-s)T(s)\big) = (c-s)^2 g(T(s), T(s)) = |c-s|$ (33)

That is,

$d(\tilde{\beta}(s), \tilde{\alpha}(s)) = |c-s|$ (34)

or

$$\frac{d\tilde{\beta}}{ds^*} = \frac{d\tilde{\alpha}}{ds}\frac{ds}{ds^*} \quad (35)$$

Hence, it is defined as;

$\tilde{T} = (c-s)\frac{ds}{ds^*}T' = (c-s)\frac{ds}{ds^*}KN_1$ (36)

and it is known that;

$g\big(\tilde{T}(s), T(s)\big) = 0$ (37)

so,

$(c-s)\frac{ds}{ds^*}Kg(N_1(s), T(s)) = 0$ (38)

It is derived as;

$(c-s)\frac{ds}{ds^*} = -c_1$ (39)

Then,

$g\left(\tilde{T}(s), \tilde{T}(s)\right) = g(c_1KN_1, c_1KN_1)$ (40)

$-1 = c_1^2K^2g(N_1, N_1)$ (41)

since N_1 *space-like* $g(N_1, N_1) = 1$

$-1 = c_1^2K^2$ (42)

$-1 = (c-s)\frac{ds}{ds^*}K$ (43)

and it is derived as;

$\frac{ds}{ds^*} = \frac{-1}{(c-s)K}.$ (44)

from (40) it can be written as;

$\tilde{T}(s) = c_1KN_1 \Rightarrow \tilde{T}(s) = \rho N_1(s)$ (45)

Hence,

$\tilde{T}'(s) = \frac{dN_1}{ds}\frac{ds}{ds^*} = -(KT + kN_2)\frac{1}{(c-s)K}.$ (46)

on the other hand, it has

$g\left(\widetilde{T'}(s), \widetilde{T'}(s)\right) = g(\tilde{K}\widetilde{N_1}, \tilde{K}\widetilde{N_1}) = \widetilde{K^2}g(\widetilde{N_1}, \widetilde{N_1}) = \widetilde{K^2}$ (47)

$g\big(\widetilde{N_1}, \widetilde{N_1}\big) = 1\ and\ g\big(\widetilde{N_2}, \widetilde{N_2}\big) = 1$ indirectly, it is obtained as;

$\widetilde{K^2} = g\big(\tilde{T}'(s), \tilde{T}'(s)\big)$

$$= g\left((KT + kN_2)\frac{-1}{(c-s)K}, (KT + kN_2)\frac{-1}{(c-s)K}\right)$$

$$= \left[\frac{K^2}{(c-s)^2K^2} g(T,T) + \frac{2Kk}{(c-s)^2K} g(T,N_2) + \frac{k^2}{(c-s)^2K^2} g(N_2,N_2) \right]$$

$$= \frac{-K^2}{(c-s)^2K^2} + \frac{k^2}{(c-s)^2K^2} \quad (48)$$

$$\widetilde{K} = \sqrt{\frac{k^2-K^2}{(c-s)^2K^2}} \quad (49)$$

Theorem 3.2 The evolutes of a planar time-like quaternionic Lorentzian curve are helixes.

Proof. Let us first consider the case where α is a planar time-like quaternionic Lorentz curve then its evolute is a plane curve, too. So $t^* = n_1$. Hence,

$$\alpha(s) = \beta(s) + \lambda t^*(s) \Rightarrow \beta(s) = \alpha(s) - \lambda n_1(s)$$

$$\Rightarrow t^* \frac{ds^*}{ds} = \beta'(s) = t(s) - \lambda' n_1(s) - \lambda n_1'(s),\ n_1'(s) = k(s)t(s) + r(s)n_2(s)\ \mathit{Since}\, g(t^*, t) = 0 \quad (50)$$

$$\mathit{then} - 1 + \lambda k_1(s) = 0 \Rightarrow \lambda = \frac{1}{k(s)}$$

$$\mathit{So}, \beta(s) = \alpha(s) - \frac{1}{k_1(s)} n_1(s).$$

The planar evolution of α is the geometric location of its curvature centers. Although α is a planar curve, if β is a non-planar evolute of α then

$$\beta(s) = \alpha(s) - \lambda t^*(s) \quad (51)$$

$$\beta'(s) = \alpha'(s) - \lambda' t^*(s) - \lambda t^{*\prime}(s)$$

$$= t^*(s) = t(s)\frac{ds}{ds^*} + t^*(s) - \lambda k^*(s) n_1^*(s) \quad (52)$$

$$\mathit{where}, \frac{d\lambda}{ds^*} = -1;$$

t^*, $time-like\ and\ n_1, n_2\ space-like.$

$$t(s)\frac{ds}{ds^*} - \lambda k^*(s) n_1^*(s) = 0, t(s) = \pm n_1^*(s) \quad (53)$$

$$f = \langle t^*(s), t(s) \Lambda n_1(s) \rangle \Rightarrow f' = \frac{df}{ds}$$

$$= \left\langle k^*(s) t^*(s) \frac{ds^*}{ds}, t(s)\Lambda n_1(s) \right\rangle + \langle t^*(s), k(s) n_1(s) \Lambda n_2(s) + n_1(s) \Lambda (k(s)t(s) + r(s)n_2(s)) \rangle \quad (54)$$

$f'(s) = 0;\ f(s) = constant\ and$

$\measuredangle(t^*(s), t(s)\Lambda n_1(s)) = constant.$

Therefore, the velocity vector of β is always a constant angle with the normal of planar curve α, so β is a helix. In this case, the non-planar evolutes of a α-planar curve are helical curves.

Theorem 3.3 Let $\tilde{\alpha}(s)$ be an pseudo-quaternionic Lorentzian evolute curve with s-arc parameters. If $\tilde{\beta}(s)$ is the evolute curve of $\tilde{\alpha}(s)$ then,

$$a)\ \tilde{\beta}(s) = \tilde{\alpha}(s) + \sigma n_1(s) + \rho n_2(s) \quad (55)$$

$$b)\ \sigma = -\frac{1}{k(s)}\ \ and\ \frac{\sigma' - r\rho}{\sigma} = \frac{\rho' + \sigma r}{\rho} \quad (56)$$

$$c)\ r(s) = \frac{\rho(-\frac{1}{k(s)})' - \rho'(-\frac{1}{k(s)})}{\rho^2 + (-\frac{1}{k(s)})^2} \quad (57)$$

Proof.

a) Since the vector $\tilde{\beta}(s) - \tilde{\alpha}(s)$ is perpendicular to the vector $t(s)$, it can be defines as;

$$\tilde{\beta}(s) = \tilde{\alpha}(s) = \sigma n_1(s) + \rho n_2(s) \quad (58)$$

b)

$$\frac{d\tilde{\beta}}{ds} = \tilde{\alpha}'(s) + \sigma' n_1(s) + \sigma n_1'(s) + \rho' n_2(s) + \rho n_2'(s)$$

$$= t(s) + \sigma' n_1(s) + \sigma(k(s)t(s) + r(s)n_2(s)) + \rho' n_2(s) + \rho(-r(s)n_1(s))$$

$$= (1 + \sigma k(s))t(s) + (\sigma' - \rho r(s))n_1(s) + (\rho' + \sigma r(s))n_2(s) \quad (59)$$

$$\mathit{Since} \left\langle \frac{d\tilde{\beta}}{ds}, t(s) \right\rangle = 0,$$

$$\mathit{then}\ 1 + \sigma k(s) = 0 \Rightarrow \sigma = -\frac{1}{k(s)}.$$

So,

$$\frac{d\tilde{\beta}}{ds} = (\sigma' - \rho r(s))n_1(s) + (\rho' + \sigma r(s))n_2(s) \quad (60)$$

on the other hand $\left\{\frac{d\tilde{\beta}}{ds}, \tilde{\beta}(s) - \tilde{\alpha}(s)\right\}$ is linearly dependent. Hence,

$$\tilde{\beta}(s) - \tilde{\alpha}(s) = \sigma n_1(s) + \rho n_2(s) \quad (61)$$

And

$$\frac{d\tilde{\beta}}{ds} = (\sigma' - \rho r(s))n_1(s) + (\rho' + \sigma r(s))n_2(s) \quad (62)$$

vectors in the same direction. Thus

$$\frac{(\sigma' - \rho r(s))}{\sigma} = \frac{(\rho' + \sigma r(s))}{\rho} = u \quad (63)$$

from this last equation, it is derived as;

c) $\sigma'\rho - \rho^2 r(s) = \sigma\rho' + \sigma^2 r(s)\ \mathit{and}\, \sigma = -\frac{1}{k(s)}.$

then it can be obtained

$$r(s) = \frac{\sigma'\rho - \sigma\rho'}{\sigma^2 + \rho^2} \quad (64)$$

so,

$$r(s) = \frac{\rho(-\frac{1}{k(s)})' - \rho'(-\frac{1}{k(s)})}{\rho^2 + (-\frac{1}{k(s)})^2} \quad (65)$$

6. CONCLUSION

First, the geometric properties of transformations using quaternionic and pseudo-quaternionic multiplications in 4-dimensional real vector space are given by Baharatti and Nagaraj, [3]. Later, some results were expressed by studying quaternionic curves in different spaces (eg Euclidean and Lorentzian spaces) by many mathematicians,[4,8,12].

As mentioned above, although the quaternionic curve and quaternionic Lorentzian curve has been studied by many mathematicians, this article is a different study with the quaternion terminology for a quaternionic Lorentzian curve. Because the aim of this study is how to derive evolution-inclusion curves for a psequat (pseudo-quaternionic) in Lorentz space and how to give some relations between these curves.

Therefore, this article will bring a new perspective to researchers who will work in this field.

DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Müge KARADAĞ: Completed study and wrote the manuscript.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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