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Research Article

Estimation after selection from bivariate normal population with application to poultry feeds data

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Abstract

In many practical situations, it is often desired to select a population (treatment, product, technology, etc.) from a choice of several populations on the basis of a particular characteristic that associated with each population, and then estimate the characteristic associated with the selected population. The present paper is focused on estimating a characteristic of the selected bivariate normal population, using a LINEX loss function. A natural selection rule is used for achieving the aim of selecting the best bivariate normal population. Some natural-type estimators and Bayes estimator (using a conjugate prior) of a parameter of the selected population are presented. An admissible subclass of equivariant estimators, using the LINEX loss function, is obtained. Further, a sufficient condition for improving the competing estimators is derived. Using this sufficient condition, several estimators improving upon the proposed natural estimators are obtained. Further, an application of the derived results is provided by considering the poultry feeds data. Finally, a comparative study on the competing estimators of a parameter of the selected population is carried-out using simulation.

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1. Introduction

The estimation of a characteristic after selection has been recognized as an important practical problem for many years. The problem arises naturally in multiple applications where one wishes to select a population from the available $k\,(\geq 2)$ populations and then estimate some characteristics (or parametric functions) associated with the population selected by a fixed selection rule. For example, in modeling economic phenomenons, often the economist is faced with the problem of choosing an economic model from $k\,(\geq 2)$ different models that returns a minimum loss to the capital economic. After the selection

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of the desired economic model, using a pre-specified selection procedure, the economist may like to have an estimate of the return losses from the selected model. In clinical research, after the selection of the most effective treatment from a choice of k available treatments, a doctor may wishes to have an estimate of the effectiveness of the selected treatment. The aforementioned problems are continuation of the general formulation of the Ranking and Selection problems. Several inferential methods for statistical selection and estimation related to these problems have been developed by many authors, see [1–8, 10, 12, 15–21, 29–31].

The majority of prior studies on selection and estimation following selection problems have exclusively focused on a selected univariate population, and very few papers have appeared for a selected bivariate/multivariate population. Some of the works devoted to the bivariate/multivariate case are due to [1] and [23]. In particular, Mohammadi and Towhidi [23] considered the estimation of a characteristic after selection from bivariate normal population, using a squared error loss function. The authors used this loss function and derived a Bayes estimator of a characteristic of the bivariate normal population selected by a natural selection rule. The authors also provided some admissibility and inadmissibility results. This paper continues the study of [23] by considering the following loss function

$$L(\delta,\theta) = e^{a(\delta-\theta)} - a(\delta-\theta) - 1. \quad \delta \in \mathbb{D},\ \theta \in \Theta, \tag{1.1}$$

where δ is an estimator of the unknown parameter θ, a is a location parameter of the loss function given in Equation (1.1), Θ denotes the parametric space, and $\mathbb{D}$ represents a class of estimators of θ. The loss function in Equation (1.1) is generally called an asymmetric linear exponential (LINEX) loss and is useful in situations where positive bias (overestimation) is assumed to be more preferable than negative bias (underestimation) or vice versa. Many researchers have used the above loss function, see among others [3, 14, 24, 32].

The normal distribution is the most important and used probability model in many natural phenomena. For instance, variables such as psychological, educational, blood pressure, and heights, etc., follow normal distribution. One generalization of the univariate normal distribution is the bivariate normal distribution. Consider two independent populations π_1 and π_2. Let $\boldsymbol{Z}_i = (X_i, Y_i)^\intercal$ be a random vector associated with the bivariate normal population $\pi_i \equiv N(\boldsymbol{\theta}^{(i)}, \boldsymbol{\Sigma})$, where $\boldsymbol{\theta^{(i)}} = \left(\theta_x^{(i)}, \theta_y^{(i)}\right)^\intercal$ denotes the 2-dimensional unknown mean vector $(i = 1, 2)$, and $\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix}$ denotes the common known positive-definite variance-covariance matrix. Suppose that a population is selected on the basis of their X-variate which is a characteristic that is easy to observed or can be measured at the time of selection, and Y-variate is an associated characteristic that is of main interest but can not be measured at the time of selection or can be observed later. Then, based on available information of the X-variate, we wish to draw some inferences about the corresponding Y-variate. For example, an experiment is conducted to compare the effect of organic and inorganic feeds in poultry. The aim of the study is to produce eggs with more weights and less cholesterol levels. Here X represents weights of eggs and Y represents cholesterol levels. A comprehensive details of this study is provided in Section 5. One more example is that, X may be the grade of an applicant on a particular test and Y is a grade on a future test. Then, based on the X-grade we want to see the behavior of the corresponding Y-grade. Let $X_{(1)}$ and $X_{(2)}$ be the order statistics from X_1 and X_2. Then, the Y-variates induced by the order statistic $X_{(i)}$ is called the concomitant of $X_{(i)}$ and is denoted by $Y_{[i]}$ $(i = 1, 2)$. Assume that the bivariate population associated with $\max\left\{\theta_x^{(1)}, \theta_x^{(2)}\right\}$ is referred as the better population. For selecting the better population, a natural selection rule $\boldsymbol{\psi} = (\psi_1, \psi_2)$ selects the population associated with $X_{(2)} = \max(X_1, X_2)$, so that, the natural selection rule $\boldsymbol{\psi} = (\psi_1, \psi_2)$ can be expressed as

$$\psi_1(\boldsymbol{x}) = \begin{cases} 1, & \text{if} \ \ X_1 > X_2 \\ 0, & \text{if} \ \ X_1 \leq X_2, \end{cases} \tag{1.2}$$

and $\psi_2(\boldsymbol{x}) = 1 - \psi_1(\boldsymbol{x})$. After a bivariate normal population is selected using the selection rule $\boldsymbol{\psi}$, given in Equation (1.2), we are interested in the estimation of the second component of the mean vector associated with the selected population, which can be expressed as

$$\begin{aligned} \theta_{\mathrm{y}}^{S}(\boldsymbol{x}) &= \theta_y^{(1)}\psi_1(\boldsymbol{x}) + \theta_y^{(2)}\psi_2(\boldsymbol{x}) \\ &= \begin{cases} \theta_y^{(1)}, & \text{if} \ \ X_1 > X_2 \\ \theta_y^{(2)}, & \text{if} \ \ X_1 \leq X_2. \end{cases} \end{aligned}$$

Note that θ_{y}^{S} depends on the variables X_1 and X_2, i.e., θ_{y}^{S} is a random parametric function of $\theta_y^{(1)}$, $\theta_y^{(2)}$, X_1 and X_2. Our goal is to estimate θ_{y}^{S} using the loss function given in Equation (1.1).

Putter and Rubinstein [27] have shown that an unbiased estimator of the mean after selection from univariate normal population does not exist. Dahiya [11] continued the study of [27] by proposing several different estimators of mean and investigated their corresponding bias and mean squared error. Later, Parsian and Farsipour [26] considered two univariate normal populations having same known variance but unknown means, using the loss function given in Equation (1.1). They suggested seven different estimators for the mean and investigated their respective biases and risk functions. Misra and van der Muelen [22] continued the study of [26] by deriving some admissibility and inadmissibility results for estimators of the mean of the univariate normal population selected by a natural selection rule. As a consequence, they obtained some estimators better than those suggested by [26]. Recently, Mohammadi and Towhidi [23] extended the study of [11] by considering a bivariate normal population. The authors derived Bayes and minimax estimators and an admissible subclass of natural estimators were also obtained. Further, they provided some improved estimators of the mean of the selected bivariate normal population. This article continues the investigation of [23] by deriving various competing estimators and decision theoretic results under the LINEX loss function.

Note that, using the loss function given in Equation (1.1) for estimating θ_{y}^{S}, the estimation problem under consideration is location invariant with regard to a group of permutation and a location group of transformations. Moreover, its appropriate to use permutation and location invariant estimators satisfying $\delta\left(\boldsymbol{Z}_1, \boldsymbol{Z}_2\right) = \delta\left(\boldsymbol{Z}_2, \boldsymbol{Z}_1\right)$ and $\delta\left(\boldsymbol{Z}_1 + \boldsymbol{c}, \boldsymbol{Z}_2 + \boldsymbol{c}\right) = \delta\left(\boldsymbol{Z}_1, \boldsymbol{Z}_2\right) + c_2, \ \forall \ \boldsymbol{c} = (c_1, c_2)^{\intercal} \in \mathbb{R}^2$, where $\mathbb{R}^2$ denotes the 2-dimensional Euclidean space. Therefore, any location equivariant estimator of θ_{y}^{S} will be of the form

$$\delta_{\varphi}\left(\boldsymbol{Z}_1, \boldsymbol{Z}_2\right) = Y_{[2]} + \varphi\left(X_{(1)} - X_{(2)}, Y_{[1]} - Y_{[2]}\right), \tag{1.3}$$

where $\varphi(\cdot)$ is a function of $X_{(1)} - X_{(2)}$ and $Y_{[1]} - Y_{[2]}$. Let $\mathcal{Q}_c$ represents the class of all equivariant estimators of the form (1.3). For notational simplicity, the following notations will be adapted throughout the paper; $\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2)$, $\theta_x = \max\left(\theta_x^{(1)}, \theta_x^{(2)}\right) - \min\left(\theta_x^{(1)}, \theta_x^{(2)}\right)$, $\theta_y = \max\left(\theta_y^{(1)}, \theta_y^{(2)}\right) - \min\left(\theta_y^{(1)}, \theta_y^{(2)}\right)$, $\boldsymbol{\theta}^* = (\theta_x, \theta_y)^{\intercal} \in \mathbb{R}_+^2$, where $\mathbb{R}_+^2$ denotes the positive part of the two dimensional Euclidean space $\mathbb{R}^2$, and $\phi(\cdot)$ and $\Phi(\cdot)$ denote the usual pdf and cdf of $N(0, 1)$.

We presented some natural estimators and Bayes estimator, under the loss function given in a location parameter of the loss function given in Equation (1.1), of θ_{y}^{S} in Section 2. In Section 3, an admissible subclass of natural type estimators is obtained. Further, a result of improved estimators is derived in Section 4. In Section 5, an application of the derived results is provided by considering the poultry feeds data. Finally, in Section

6, using the LINEX loss function, risk comparison of the estimators of θ_{y}^{S} is carried-out using a simulation study.

2. Estimators of $\theta_{\mathbf{y}}^{S}$

In this section, we present various estimators of θ_{y}^{S} of the selected population. First, based on the maximum likelihood estimator (MLE), an estimator of θ_{y}^{S} is given by

$$\delta_{N,1}(\boldsymbol{Z}) = Y_{[2]}.$$

Similarly, based on the minimum risk equivariant estimator (MREE), an estimator of θ_{y}^{S} is given by

$$\delta_{N,2}(\boldsymbol{Z}) = Y_{[2]} - \frac{1}{2} a\sigma_{yy}.$$

The third estimator of θ_{y}^{S} that we propose is given by

$$\delta_{N,3}(\boldsymbol{Z}) = Y_{[2]} + \frac{1}{a} \ln \left[1 + \left(e^{a\left(Y_{[1]} - Y_{[2]}\right)} - 1 \right) \Phi \left(\frac{X_{(1)} - X_{(2)}}{\sqrt{2\sigma_{xx}}} \right) \right].$$

Note that the estimator $\delta_{N,3}$ is based on the MLE of $\frac{1}{a} \ln \left[E\left(e^{a\theta_{\mathrm{y}}^{S}} \right) \right]$, where $E\left(e^{a\theta_{\mathrm{y}}^{S}} \right) = e^{a\theta_{y}^{(2)}} \left[1 + \left(e^{a\left(\theta_{y}^{(1)} - \theta_{y}^{(2)} \right)} - 1 \right) \Phi \left(\frac{\theta_{x}^{(1)} - \theta_{x}^{(2)}}{\sqrt{2\sigma_{xx}}} \right) \right]$.

Another natural estimator of θ_{y}^{S}, which is similar to the estimator studied by [11], is given by

$$\delta_{N,4}(\boldsymbol{Z}) = \begin{cases} \frac{Y_{[1]} + Y_{[2]}}{2}, & \text{if } X_{(1)} - X_{(2)} > -c\sqrt{2\sigma_{xx}} \\ Y_{[2]}, & \text{if } X_{(1)} - X_{(2)} \le -c\sqrt{2\sigma_{xx}}, \end{cases}$$

where $c > 0$ is a constant. The estimator $\delta_{N,4}$ is called hybrid estimator and is same as the estimator $\delta_{N,1}$ for $c = 0$.

Theorem 2.1. *Under the conjugate prior $\Pi^{m}\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}\right) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\vartheta})$ and the loss function given in Equation (1.1), the Bayes estimator of θ_{y}^{S} is given by*

$$\delta_{\Pi^m}(\boldsymbol{Z}) = \frac{\mu_2(|\Sigma| + m\sigma_{yy}) + mY_{[2]}(m + \sigma_{xx}) + m\sigma_{xy}(\mu_1 - X_{(2)})}{m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|} - \frac{a}{2} \frac{m^2\sigma_{yy} + m|\Sigma|}{(m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|)}.$$

Proof. Suppose that $\boldsymbol{\theta}^{(i)}$ has a conjugate bivariate normal prior $\Pi^{m}\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}\right) = \prod_{i=1}^{2} \Pi_{(i)}^{m}\left(\boldsymbol{\theta}^{(i)}\right) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\vartheta}), i = 1, 2$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $\boldsymbol{\vartheta} = mI$, and I denotes an identity matrix of order 2 and m is a positive real number. Then, the posterior distribution of $\boldsymbol{\theta}^{(i)}$, given $\boldsymbol{Z}_i = \boldsymbol{z}_i$, is

$$\boldsymbol{\theta}^{(i)} | \boldsymbol{z}_i \sim N_2 \left(\boldsymbol{K}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{z}_i + \boldsymbol{\vartheta}^{-1}\boldsymbol{\mu} \right), \boldsymbol{K} \right), \quad i = 1, 2, \tag{2.1}$$

where $\boldsymbol{K} = \left(\boldsymbol{\Sigma}^{-1} + \boldsymbol{\vartheta}^{-1} \right)^{-1}$.

The posterior risk of an estimator δ_i of $\theta_{y}^{(i)}$ under the loss function given in Equation (1.1) is

$$EL\left(\delta_i(\boldsymbol{Z}_i), \theta_{y}^{(i)}\right) = e^{a\delta_i(\boldsymbol{Z}_i)} E\left[e^{-a\theta_{y}^{(i)}} \middle| \boldsymbol{Z}_i = \boldsymbol{z}_i \right] - a\left(\delta_i(\boldsymbol{Z}_i) - E\left(\theta_{y}^{(i)} | \boldsymbol{Z}_i = \boldsymbol{z}_i \right) \right) - 1, \tag{2.2}$$

$i = 1, 2$. It is not difficult to check that the Bayes estimator $\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i)$ of $\theta_y^{(i)}$, which minimizes the posterior risk in Equation (2.2), is given by

$$\begin{aligned}\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i) &= -\frac{1}{a}\ln\left[E\left[e^{-a\theta_y^{(i)}}\Big|\boldsymbol{Z}_i=\boldsymbol{z}_i\right]\right]\\ &= -\frac{1}{a}\ln\left[M_{\theta_y^{(i)}|\boldsymbol{z}_i}(-a)\right],\quad i=1,2,\end{aligned} \tag{2.3}$$

where $M_{\theta_y^{(i)}|\boldsymbol{z}_i}(\cdot)$ denotes the moment generating function (MGF) of $\theta_y^{(i)}|\boldsymbol{z}_i$. It follows from Equation (2.1) that $\theta_y^{(i)}|\boldsymbol{z}_i$ has univariate normal distribution $N(p_i^*, q_i^*)$, where

$$p_i^* = \frac{\mu_2(|\Sigma| + m\sigma_{yy}) + mY_i(m+\sigma_{xx}) + m\sigma_{xy}(\mu_1 - X_i)}{m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|},$$

and

$$q_i^* = \frac{m^2\sigma_{yy} + m|\Sigma|}{(m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|)},\quad i=1,2.$$

Therefore,

$$M_{\boldsymbol{\theta}^{(i)}|\boldsymbol{z}_i}(-a) = e^{-ap_i^* + \frac{1}{2}a^2q_i^*},\quad i=1,2. \tag{2.4}$$

Combining Equations (2.3) and (2.4), we get

$$\begin{aligned}&\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i) =\\ &\frac{\mu_2(|\Sigma| + m\sigma_{yy}) + mY_i(m+\sigma_{xx}) + m\sigma_{xy}(\mu_1 - X_i)}{m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|}\\ &-\frac{a}{2}\frac{m^2\sigma_{yy} + m|\Sigma|}{(m^2 + m\sigma_{xx} + m\sigma_{yy} + |\Sigma|)},\quad i=1,2.\end{aligned}$$

It can be verified that the posterior risk of the Bayes estimator $\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i)$ of $\theta_y^{(i)}$, is given by

$$r(\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i)) = \frac{a^2}{2}\frac{(m^2\sigma_{yy} + |\Sigma|m)}{(|\Sigma| + m^2 + m\sigma_{yy} + m\sigma_{xx})}. \tag{2.5}$$

Since the posterior risk in Equation (2.5) does not depend on $\boldsymbol{Z}_i$, $i = 1, 2$, it follows form Theorem 3.1 of Sackrowitz and Samuel-Cahn [28] that the posterior risk $r\left(\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i)\right)$, given in Equation (2.5), is also the Bayes risk of $\delta_{\Pi^m_{(i)}}(\boldsymbol{Z}_i)$. Now an application of Lemma 3.2 of [28] leads to the result. □

Remark 2.2. It can be easily checked that the estimator $\delta_{N,2}$ is a limit of the Bayes estimators $\delta_{\Pi^m}(\boldsymbol{Z})$ as $m \to \infty$.

Remark 2.3. Following the procedures in the proof of Theorem 2.1, it can be verified that, the estimator $\delta_{N,2}$ is also a generalized Bayes estimator of θ_{y}^S, using the loss function given in Equation (1.1) and the improper prior $\Pi\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}\right) = 1,\ \forall\ \boldsymbol{\theta}^{(i)} \in \mathbb{R}^2,\ i = 1, 2.$

3. Some admissibility results

An admissible subclass of equivariant estimators within the class $\mathcal{Q}_d$ is obtained, using the loss function given in Equation (1.1), where

$$\mathcal{Q}_d = \left\{\delta_d : \delta_d(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = Y_{[2]} + d,\ \forall\ d \in \mathbb{R}\right\},$$

here $\mathbb{R}$ denotes the real line. For obtaining the admissibility of the estimators within the above class we require the following lemma.

Lemma 3.1. *Let* $W = Y_{[2]} - \theta_{\rm y}^S$ *and* $\rho = \frac{\sigma_{xy}}{\sqrt{\sigma_{xx}\sigma_{yy}}}$. *Then,* W *has the pdf*

$$f_W(w|\boldsymbol{\theta}^*) = \frac{1}{\sqrt{\sigma_{yy}}}\phi\left(\frac{w}{\sqrt{\sigma_{yy}}}\right)\left\{\Phi\left(\frac{\frac{\rho w}{\sqrt{\sigma_{yy}}}+\frac{\theta_x}{\sqrt{\sigma_{xx}}}}{\sqrt{2-\rho^2}}\right)+\Phi\left(\frac{\frac{\rho w}{\sqrt{\sigma_{yy}}}-\frac{\theta_x}{\sqrt{\sigma_{xx}}}}{\sqrt{2-\rho^2}}\right)\right\}, \quad w \in \mathbb{R}.$$

Proof. For fixed $\boldsymbol{\theta}^* \in \mathbb{R}_+^2$, the cdf of W is given by

$$\begin{aligned}
F_W(w) =& P\left(Y_{[2]} - \theta_{\rm y}^S \leq w\right)\\
=& P\left(Y_2 - \theta_y^{(2)} \leq w, X_1 \leq X_2\right) + P\left(Y_1 - \theta_y^{(1)} \leq w, X_1 > X_2\right)\\
=& P\left(V_2\sqrt{\sigma_{yy}} \leq w, U_1 \leq U_2 + \frac{\theta_x}{\sqrt{\sigma_{xx}}}\right) + P\left(V_1\sqrt{\sigma_{yy}} \leq w, U_1 > U_2 + \frac{\theta_x}{\sqrt{\sigma_{xx}}}\right),
\end{aligned}$$

where $U_1 = \frac{X_1-\theta_x^{(1)}}{\sqrt{\sigma_{xx}}}$, $U_2 = \frac{X_2-\theta_x^{(2)}}{\sqrt{\sigma_{xx}}}$, $V_1 = \frac{Y_1-\theta_y^{(1)}}{\sqrt{\sigma_{yy}}}$, and $V_2 = \frac{Y_2-\theta_y^{(2)}}{\sqrt{\sigma_{yy}}}$. Clearly, (U_1, V_1) and (U_2, V_2) have bivariate normal distribution $N_2\left((0,0), \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$. It follows that

$$\begin{aligned}
F_W(w) =& \int_{-\infty}^{\infty}\left[\int_{u_1-\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\int_{-\infty}^{\frac{w}{\sqrt{\sigma_{yy}}}}\phi_2(u_2,v_2)\,dv_2du_2\right]\phi(u_1)du_1\\
&+\int_{-\infty}^{\infty}\left[\int_{u_2+\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\int_{-\infty}^{\frac{w}{\sqrt{\sigma_{yy}}}}\phi_2(u_1,v_1)\,dv_1du_1\right]\phi(u_2)du_2,
\end{aligned}$$

where $\phi_2(\cdot,\cdot)$ is the pdf of bivariate normal distribution $N_2\left((0,0), \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$, and $\phi(\cdot)$ is the pdf of univariate standard normal distribution. Now, differentiating with respect to w, we get

$$\begin{aligned}
f_W(w) =& \int_{-\infty}^{\infty}\left[\int_{u_1-\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\phi_2\left(u_2,\frac{w}{\sqrt{\sigma_{yy}}}\right)du_2\right]\phi(u_1)du_1\\
&+\int_{-\infty}^{\infty}\left[\int_{u_2+\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\phi_2\left(u_1,\frac{w}{\sqrt{\sigma_{yy}}}\right)du_1\right]\phi(u_2)du_2\\
=& \frac{1}{\sqrt{\sigma_{yy}}\sqrt{2\pi}}e^{-\frac{w^2}{2\sigma_{yy}}}\int_{-\infty}^{\infty}\left[\int_{u_1-\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(u_2-\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)^2}du_2\right]\phi(u_1)du_1\\
&+\frac{1}{\sqrt{\sigma_{yy}}\sqrt{2\pi}}e^{-\frac{w^2}{2\sigma_{yy}}}\int_{-\infty}^{\infty}\left[\int_{u_2+\frac{\theta_x}{\sqrt{\sigma_{xx}}}}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(u_1-\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)^2}du_1\right]\phi(u_2)du_2\\
=& \frac{1}{\sqrt{\sigma_{yy}}\sqrt{2\pi}}e^{-\frac{w^2}{2\sigma_{yy}}}\left[\int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{u_1-\left(\frac{\theta_x}{\sqrt{\sigma_{xx}}}+\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)}{\sqrt{1-\rho^2}}\right)\right)\phi(u_1)du_1\right.\\
&\left.+\int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{u_2+\left(\frac{\theta_x}{\sqrt{\sigma_{xx}}}-\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)}{\sqrt{1-\rho^2}}\right)\right)\phi(u_2)du_2\right]\\
=& \frac{1}{\sqrt{\sigma_{yy}}\sqrt{2\pi}}e^{-\frac{w^2}{2\sigma_{yy}}}\left[\left(1-\int_{-\infty}^{\infty}\phi(u_1)\Phi\left(\frac{u_1-\left(\frac{\theta_x}{\sqrt{\sigma_{xx}}}+\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)}{\sqrt{1-\rho^2}}\right)du_1\right)\right.\\
&\left.+\left(1-\int_{-\infty}^{\infty}\phi(u_2)\Phi\left(\frac{u_2+\left(\frac{\theta_x}{\sqrt{\sigma_{xx}}}-\frac{\rho w}{\sqrt{\sigma_{yy}}}\right)}{\sqrt{1-\rho^2}}\right)du_2\right)\right].
\end{aligned}$$

Using the identity (see [11]) $\int_{-\infty}^{\infty}\Phi\left(\frac{u+a}{\sqrt{b}}\right)\phi(u)du=\Phi\left(\frac{a}{\sqrt{1+b}}\right)$, we have

$$f_W(w)=\frac{1}{\sqrt{\sigma_{yy}}\sqrt{2\pi}}e^{-\frac{w^2}{2\sigma_{yy}}}\left[\Phi\left(\frac{\frac{\rho w}{\sqrt{\sigma_{yy}}}+\frac{\theta_x}{\sqrt{\sigma_{xx}}}}{\sqrt{2-\rho^2}}\right)+\Phi\left(\frac{\frac{\rho w}{\sqrt{\sigma_{yy}}}-\frac{\theta_x}{\sqrt{\sigma_{xx}}}}{\sqrt{2-\rho^2}}\right)\right].$$

Hence the result follows. □

The following theorem establishes the admissibility of the estimators δ_d within the class $\mathcal{Q}_d$.

Theorem 3.2. *Let*

$$d_0=\begin{cases}-\frac{a\sigma_{yy}}{2}-\frac{1}{a}\left[\ln 2+\ln\left\{\Phi\left(\frac{a\sigma_{xy}}{\sqrt{2\sigma_{xx}}}\right)\right\}\right], & \text{if } \sigma_{xy}>0\\ -\frac{a\sigma_{yy}}{2}, & \text{if } \sigma_{xy}\le 0,\end{cases}$$

and

$$d_1=\begin{cases}-\frac{a\sigma_{yy}}{2}, & \text{if } \sigma_{xy}\ge 0\\ -\frac{a\sigma_{yy}}{2}-\frac{1}{a}\left[\ln 2+\ln\left\{\Phi\left(\frac{a\sigma_{xy}}{\sqrt{2\sigma_{xx}}}\right)\right\}\right], & \text{if } \sigma_{xy}<0.\end{cases}$$

Let $\delta_d\in\mathcal{Q}_d$ be given estimators of θ_{y}^S. Then,
(i) Within the class $\mathcal{Q}_d$, the equivariant estimators δ_d are admissible for $d_0\le d\le d_1$, under the loss function given in Equation (1.1),
(ii) The equivariant estimators δ_d for $d\in(-\infty,d_0)\cup(d_1,\infty)$ are inadmissible even within the class $\mathcal{Q}_d$.

Proof. For a fixed $\boldsymbol{\theta}^*=(\theta_x,\theta_y)^\intercal\in\mathbb{R}_+^2$, define $\Psi(\boldsymbol{\theta}^*)=-\frac{1}{a}\ln\left[E_{\theta^*}\left(e^{aW}\right)\right]$, where $W=Y_{[2]}-\theta_{\mathrm{y}}^S$. Then, for fixed $\boldsymbol{\theta}^*\in\mathbb{R}_+^2$, the risk function of the estimators δ_d is given by

$$R(\delta_d,\boldsymbol{\theta}^*)=E_{\boldsymbol{\theta}^*}\left[e^{a\left(Y_{[2]}+d-\theta_{\mathrm{y}}^S\right)}-a\left(Y_{[2]}+d-\theta_{\mathrm{y}}^S\right)-1\right]$$

It is easy to verify that $R(\delta_d,\boldsymbol{\theta}^*)$ is minimized at $d=\Psi(\boldsymbol{\theta}^*)=-\frac{1}{a}\ln\left[E_{\boldsymbol{\theta}^*}\left(e^{aW}\right)\right]$. Using Lemma 3.1, we have

$$\Psi(\boldsymbol{\theta}^*)=-\frac{a\sigma_{yy}}{2}-\frac{1}{a}\ln\left[H_a(\theta_x)\right],$$

where for $a\neq 0$, $H_a(\theta_x)=\Phi\left(\frac{a\sigma_{xy}+\theta_x}{\sqrt{2\sigma_{xx}}}\right)+\Phi\left(\frac{a\sigma_{xy}-\theta_x}{\sqrt{2\sigma_{xx}}}\right)$. Clearly, the behavior of $H_a(\theta_x)$ depends on $\theta_x\in(0,\infty)$. To see the behavior of $H_a(\theta_x)$, we will differentiate $H_a(\theta_x)$ w.r.t θ_x. We have

$$\begin{aligned}H_a'(\theta_x)=&\frac{1}{\sqrt{2\sigma_{xx}}}\phi\left(\frac{a\sigma_{xy}+\theta_x}{\sqrt{2\sigma_{xx}}}\right)-\frac{1}{\sqrt{2\sigma_{xx}}}\phi\left(\frac{a\sigma_{xy}-\theta_x}{\sqrt{2\sigma_{xx}}}\right)>0\\ \Leftrightarrow\ &\phi\left(\frac{a\sigma_{xy}+\theta_x}{\sqrt{2\sigma_{xx}}}\right)>\phi\left(\frac{a\sigma_{xy}-\theta_x}{\sqrt{2\sigma_{xx}}}\right)\\ \Leftrightarrow\ &e^{-\frac{1}{2}\left(\frac{a\sigma_{xy}+\theta_x}{\sqrt{2\sigma_{xx}}}\right)^2}>e^{-\frac{1}{2}\left(\frac{a\sigma_{xy}-\theta_x}{\sqrt{2\sigma_{xx}}}\right)^2}\\ \Leftrightarrow\ &e^{-\frac{1}{2}\left(\frac{a^2\sigma_{xy}^2+\theta_x^2+2a\sigma_{xy}\theta_x}{2\sigma_{xx}}\right)}>e^{-\frac{1}{2}\left(\frac{a^2\sigma_{xy}^2+\theta_x^2-2a\sigma_{xy}\theta_x}{2\sigma_{xx}}\right)}\\ \Leftrightarrow\ &e^{-\frac{a\sigma_{xy}\theta_x}{2\sigma_{xx}}}>e^{\frac{a\sigma_{xy}\theta_x}{2\sigma_{xx}}}\\ \Leftrightarrow\ &a\sigma_{xy}<0.\end{aligned}$$

Therefore, for $a\sigma_{xy}>0$ $(a\sigma_{xy}<0)$, $H_a(\theta_x)$ is a decreasing (an increasing) function of $\theta_x\in(0,\infty)$. Using the monotonicity of $H_a(\theta_x)$, we conclude that for $\sigma_{xy}>0$ $(\sigma_{xy}<0)$,

$\Psi(\boldsymbol{\theta}^*)$ is an increasing (a decreasing) function of θ_x. Therefore, for $\sigma_{xy} > 0$

$$\inf_{\boldsymbol{\theta}^* \in \mathbb{R}_+^2} \Psi(\boldsymbol{\theta}^*) = d_0 \quad \text{and} \quad \sup_{\boldsymbol{\theta}^* \in \mathbb{R}_+^2} \Psi(\boldsymbol{\theta}^*) = \lim_{\theta_x \to \infty} \Psi(\boldsymbol{\theta}^*) = d_1, \tag{3.1}$$

and for $\sigma_{xy} < 0$

$$\inf_{\boldsymbol{\theta}^* \in \mathbb{R}_+^2} \Psi(\boldsymbol{\theta}^*) = \lim_{\theta_x \to \infty} \Psi(\boldsymbol{\theta}^*) = d_0 \quad \text{and} \quad \sup_{\boldsymbol{\theta}^* \in \mathbb{R}_+^2} \Psi(\boldsymbol{\theta}^*) = d_1. \tag{3.2}$$

(i) Since $\Psi(\boldsymbol{\theta}^*)$ is a continuous function of $\boldsymbol{\theta}^*$, it follows from Equations (3.1) and (3.2) that any value of d in the interval (d_0, d_1) minimizes the risk function $R(\delta_d, \boldsymbol{\theta}^*)$ for some $\boldsymbol{\theta}^* \in \mathbb{R}_+^2$. Consequently, the estimators δ_d, for any value of $d \in (d_0, d_1)$ are admissible within the subclass $\mathcal{Q}_d$. The admissibility of the estimators δ_{d_0} and δ_{d_1}, within the class $\mathcal{Q}_d$, follows form continuity of $R(\delta_d, \boldsymbol{\theta}^*)$.
(ii) For a fixed $\boldsymbol{\theta}^* \in \mathbb{R}_+^2$, the risk function $R(\delta_d, \boldsymbol{\theta}^*)$ is a decreasing (an increasing) function of d for $d < \Psi(\boldsymbol{\theta}^*)$ $(d > \Psi(\boldsymbol{\theta}^*))$. Since $d_0 \le \Psi(\boldsymbol{\theta}^*) \le d_1, \forall\ \boldsymbol{\theta}^* \in \mathbb{R}_+^2$, it follows that the equivariant estimators δ_d are dominated by δ_{d_0} for $d < d_0$ and δ_{d_1} for $d > d_1$. ☐

Remark 3.3. The estimator $\delta_{N,2}$ is a member of the class $\mathcal{Q}_d$ for $d = -\frac{1}{2}a\sigma_{yy}$. Then, using Theorem 3.2, the estimator $\delta_{N,2}$ is admissible within the class $\mathcal{Q}_d$.

4. Some Results of Improved Estimators

In this section, using the loss function given in Equation (1.1), a sufficient condition for improving equivariant estimators of θ_{y}^S in the general class $\mathcal{Q}_c$ is derived. The following lemmas are needed for establishing the result.

Lemma 4.1. *Let* $T_1 = X_{(1)} - X_{(2)}$, $T_2 = Y_{[1]} - Y_{[2]}$, $T_3 = Y_{[2]} - \theta_{\mathrm{y}}^S$, $\rho = \frac{\sigma_{xy}}{\sqrt{\sigma_{xx}\sigma_{yy}}}$, *and* $\boldsymbol{\theta}^* = (\theta_x, \theta_y)^\intercal \in \mathbb{R}_+^2$ *For* $t_1 \le 0$, $t_2 \in \mathbb{R}$, *the conditional pdf of* T_3 *given* $T_1 = t_1, T_2 = t_2$ *is given by*

$$f_{T_3|T_1,T_2}(T_3|T_1,T_2)$$
$$= \sqrt{\frac{2}{\sigma_{yy}}}\left[\frac{\phi\left(\sqrt{\frac{2}{\sigma_{yy}}}\left(t_3 + \frac{t_2-\theta_y}{2}\right)\right) D_1(t_1,t_2,\boldsymbol{\theta}^*) + \phi\left(\sqrt{\frac{2}{\sigma_{yy}}}\left(t_3 + \frac{t_2+\theta_y}{2}\right)\right) D_2(t_1,t_2,\boldsymbol{\theta}^*)}{D_1(t_1,t_2,\boldsymbol{\theta}^*) + D_2(t_1,t_2,\boldsymbol{\theta}^*)}\right],$$

where

$$D_1(t_1,t_2,\theta^*) = \phi\left(\frac{t_2-\theta_y}{\sqrt{2\sigma_{yy}}}\right)\phi\left(\frac{\rho\left(\frac{t_2-\theta_y}{\sqrt{\sigma_{yy}}}\right) - \left(\frac{t_1-\theta_x}{\sqrt{\sigma_{xx}}}\right)}{\sqrt{2(1-\rho^2)}}\right),$$

and

$$D_2(t_1,t_2,\theta^*) = \phi\left(\frac{t_2+\theta_y}{\sqrt{2\sigma_{yy}}}\right)\phi\left(\frac{\rho\left(\frac{t_2+\theta_y}{\sqrt{\sigma_{yy}}}\right) - \left(\frac{t_1+\theta_x}{\sqrt{2\sigma_{xx}}}\right)}{\sqrt{2(1-\rho^2)}}\right).$$

(ii) *For* $t_1 \le 0$ *and* $t_2 \in \mathbb{R}$,

$$E\left(e^{aT_3}|T_1 = t_1, T_2 = t_2\right) = e^{\frac{a^2\sigma_{yy}}{4} - \frac{at_2}{2}}\left[\Delta(t_1,t_2,\boldsymbol{\theta}^*)\right],$$

where for $t_1 \le 0$ *and* $t_2 \in \mathbb{R}$,

$$\Delta(t_1,t_2,\boldsymbol{\theta}^*) = \frac{D_1(t_1,t_2,\boldsymbol{\theta}^*)\, e^{\frac{a\boldsymbol{\theta}_y}{2}} + D_2(t_1,t_2,\boldsymbol{\theta}^*)\, e^{\frac{-a\boldsymbol{\theta}_y}{2}}}{D_1(t_1,t_2,\boldsymbol{\theta}^*) + D_2(t_1,t_2,\boldsymbol{\theta}^*)}, \tag{4.1}$$

$\forall\ \boldsymbol{\theta}^* \in \mathbb{R}_+^2$

Lemma 4.2. *For $t_1 \leq 0$ and $t_2 \in \mathbb{R}$, define*

$$\varphi\left(t_1, t_2, \boldsymbol{\theta}^*\right) = -\frac{1}{a} \ln \left[E\left(e^{aT_3} | T_1 = t_1, T_2 = t_2\right)\right]$$
$$= \frac{t_2}{2} - \frac{a\sigma_{yy}}{4} - \frac{1}{a} \ln \left[\Delta\left(t_1, t_2, \boldsymbol{\theta}^*\right)\right] \quad \textit{(Using Lemma 4.1 (ii))},$$

where $\Delta(\cdot)$ is given by (4.1). Then, for $t_1 \leq 0$ and $t_2 \in \mathbb{R}$,

$$\varphi_I\left(t_1, t_2\right) \leq \varphi\left(t_1, t_2, \boldsymbol{\theta}^*\right) \leq \varphi_S\left(t_1, t_2\right), \quad \forall \boldsymbol{\theta}^* \in \mathbb{R}_+^2,$$

where

$$\varphi_I\left(t_1, t_2\right) = \begin{cases} \frac{t_2}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \ t_1\xi - \rho t_2 < 0 \textit{ and } t_2 - \xi\rho t_1 < -a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$\varphi_S\left(t_1, t_2\right) = \begin{cases} \frac{t_2}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \ t_1\xi - \rho t_2 > 0 \textit{ and } t_2 - \xi\rho t_1 > -a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \infty, & \text{otherwise,} \end{cases}$$

where $\xi = \sqrt{\frac{\sigma_{yy}}{\sigma_{xx}}}$.

Now, we exploit the approach of [9] to obtain a sufficient condition for improving the equivariant estimators of the form $\delta_\varphi\left(\boldsymbol{Z}\right) = Y_{[2]} + \varphi\left(T_1, T_2\right)$, where $T_1 = X_{(1)} - X_{(2)}$ and $T_2 = Y_{[1]} - Y_{[2]}$.

Theorem 4.3. *Consider an equivariant estimator $\delta_\varphi\left(\boldsymbol{Z}\right) = Y_{[2]} + \varphi\left(T_1, T_2\right)$ of θ_{y}^S, where $\varphi(\cdot)$ denotes a function of T_1 and T_2. Suppose that*

$$P\left(\{\varphi(T_1, T_2) \leq \varphi_I(T_1, T_2)\} \cup \{\varphi(T_1, T_2) \geq \varphi_S(T_1, T_2)\}\right) > 0,$$

where $\varphi_I(\cdot)$ and $\varphi_S(\cdot)$ are as given in Lemma 4.2. Then, using the loss function given in Equation (1.1), the estimator $\delta_\varphi(\cdot)$ is improved by $\delta_\varphi^*(\boldsymbol{Z}) = Y_{[2]} + \varphi^*(T_1, T_2)$, where

$$\varphi^*(T_1, T_2) = \begin{cases} \varphi_I(T_1, T_2), & \textit{if } \ \varphi(T_1, T_2) \leq \varphi_I(T_1, T_2) \\ \varphi(T_1, T_2), & \textit{if } \ \varphi_I(T_1, T_2) < \varphi(T_1, T_2) < \varphi_S(T_1, T_2) \\ \varphi_S(T_1, T_2), & \textit{if } \ \varphi(T_1, T_2) \geq \varphi_S(T_1, T_2). \end{cases}$$

Proof. (i) Consider the risk difference of the estimators δ_φ and δ_φ^* and

$$R(\boldsymbol{\theta}^*, \delta_\varphi) - R(\boldsymbol{\theta}^*, \delta_\varphi^*) = E\left[K_{\boldsymbol{\theta}^*}(T_1, T_2)\right],$$

$$\begin{aligned}
K_{\boldsymbol{\theta}^*}(t_1, t_2) &= E\left[e^{a\left(\delta_\varphi(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S\right)} - a(\delta_\varphi(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S) - 1 \middle| T_1 = t_1, T_2 = t_2\right] \\
&\qquad - E\left[e^{a\left(\delta_\varphi^*(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S\right)} - a\left(\delta_\varphi^*(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S\right) - 1 \middle| T_1 = t_1, T_2 = t_2\right] \\
&= E\left[e^{a\left(\delta_\varphi(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S\right)} - e^{a\left(\delta_\varphi^*(\boldsymbol{Z}) - \theta_{\mathrm{y}}^S\right)} \middle| T_1 = t_1, T_2 = t_2\right] \\
&\qquad - aE\left[\delta_\varphi(\boldsymbol{Z}) - \delta_\varphi^*(\boldsymbol{Z}) \middle| T_1 = t_1, T_2 = t_2\right] \\
&= E\left[e^{a\left(Y_{[2]} + \varphi(t_1, t_2) - \theta_{\mathrm{y}}^S\right)} - e^{a\left(Y_{[2]} + \varphi^*(t_1, t_2) - \theta_{\mathrm{y}}^S\right)} | T_1 = t_1, T_2 = t_2\right] - a\left[\varphi(t_1, t_2) - \varphi^*(t_1, t_2)\right] \\
&= \left[e^{a\varphi(t_1, t_2)} - e^{a\varphi^*(t_1, t_2)}\right] E\left(e^{a\left(Y_{[2]} - \theta_{\mathrm{y}}^S\right)} | T_1 = t_1, T_2 = t_2\right) - a\left[\varphi(t_1, t_2) - \varphi^*(t_1, t_2)\right] \\
&= \left[e^{a\varphi(t_1, t_2)} - e^{a\varphi^*(t_1, t_2)}\right] e^{-a\varphi(t_1, t_2, \boldsymbol{\theta}^*)} - a\left[\varphi\left(t_1, t_2\right) - \varphi^*\left(t_1, t_2\right)\right].
\end{aligned}$$

The last line of the above expression follows from Lemma 4.1 and Lemma 4.2. Now suppose that $\varphi(t_1,t_2) \le \varphi_I(t_1,t_2)$ (so that $\varphi^*(t_1,t_2) = \varphi_I(t_1,t_2)$), then

$$\begin{aligned} K_{\boldsymbol{\theta}^*}(t_1,t_2) &= \left[e^{a\varphi(t_1,t_2)} - e^{a\varphi_I(t_1,t_2)}\right] e^{-a\varphi(t_1,t_2,\boldsymbol{\theta}^*)} - a\left(\varphi(t_1,t_2) - \varphi_I(t_1,t_2)\right) \\ &\ge \left[e^{a\varphi(t_1,t_2)} - e^{a\varphi_I(t_1,t_2)}\right] e^{-a\varphi_I(t_1,t_2)} - a\left(\varphi(t_1,t_2) - \varphi_I(t_1,t_2)\right) \\ &= \left[e^{a\{\varphi(t_1,t_2)-\varphi_I(t_1,t_2)\}} - 1\right] - a\left[\varphi(t_1,t_2) - \varphi_I(t_1,t_2)\right]. \end{aligned}$$

Using the property $e^x > 1+x, \forall\, x \ne 0$, we have $K_{\boldsymbol{\theta}^*}(t_1,t_2) \ge 0, \forall\, a \ne 0$. If $\varphi_I(t_1,t_2) < \varphi(t_1,t_2) < \varphi_S(t_1,t_2)$ (so that $\varphi^*(t_1,t_2) = \varphi(t_1,t_2)$), then, $K_{\boldsymbol{\theta}^*}(t_1,t_2) = 0$. If $\varphi(t_1,t_2) \ge \varphi_S(t_1,t_2)$ (so that $\varphi^*(t_1,t_2) = \varphi_S(t_1,t_2)$), then,

$$\begin{aligned} K_{\boldsymbol{\theta}^*}(t_1,t_2) &= \left[e^{a\varphi(t_1,t_2)} - e^{a\varphi_S(t_1,t_2)}\right] e^{-a\varphi(t_1,t_2,\boldsymbol{\theta}^*)} - a\left(\varphi(t_1,t_2) - \varphi_S(t_1,t_2)\right) \\ &\ge \left[e^{a\varphi(t_1,t_2)} - e^{a\varphi_S(t_1,t_2)}\right] e^{-a\varphi_S(t_1,t_2)} - a\left(\varphi(t_1,t_2) - \varphi_S(t_1,t_2)\right) \\ &= \left[e^{a\{\varphi(t_1,t_2)-\varphi_S(t_1,t_2)\}} - 1\right] - a\left[\varphi(t_1,t_2) - \varphi_S(t_1,t_2)\right]. \end{aligned}$$

Again using the property $e^x > 1+x, \forall\, x \ne 0$, we have $K_{\boldsymbol{\theta}^*}(t_1,t_2) \ge 0, \forall\, a \ne 0$. Now, since $P\left(\{\varphi(T_1,T_2) \le \varphi_I(T_1,T_2)\} \cup \{\varphi(T_1,T_2) \ge \varphi_S(T_1,T_2)\}\right) > 0$, we conclude that

$$R(\boldsymbol{\theta}^*, \delta_\varphi) - R(\boldsymbol{\theta}^*, \delta_\varphi^*) \ge 0, \quad \forall\, \boldsymbol{\theta}^* \in \mathbb{R}_+^2,$$

and the strict inequality holds for some $\boldsymbol{\theta}^* \in \mathbb{R}_+^2$. Hence the result follows. □

Improved estimators

Here, we provide some improved estimators of θ_{y}^S by using the result of Theorem 4.3.

Improved estimator 1: The estimator $\delta_{N,1} = Y_{[2]}$ is a member of the class $\mathcal{Q}_c$ (δ_φ with $\varphi = 0$). It follows from Theorem 4.1 that, the estimator $\delta_{N,1}$ is improved by $\delta_{N,1}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = Y_{[2]} + \varphi_{N,1}(T_1,T_2)$, where

$$\varphi_{N,1}(T_1,T_2) = \begin{cases} \varphi_I(T_1,T_2), & \text{if } \ 0 \le \varphi_I(T_1,T_2) \\ 0, & \text{if } \ \varphi_I(T_1,T_2) < 0 < \varphi_S(T_1,T_2) \\ \varphi_S(T_1,T_2), & \text{if } \ 0 \ge \varphi_S(T_1,T_2), \end{cases}$$

and $\varphi_I(T_1,T_2)$ and $\varphi_S(T_1,T_2)$ are given in Lemma 4.2. For $a > 0$ and $0 < \rho \le 1$, the estimator $\delta_{N,1}$ is improved by

$$\delta_{N,1}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \ T_1 > \frac{\rho T_2}{\xi} \text{ and } \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) < T_2 < \frac{a\sigma_{yy}}{2} \\ \delta_{N,1}, & \text{otherwise.} \end{cases}$$

Improved estimator 2: For $a < 0$ and $-1 \le \rho < 0$, the estimator $\delta_{N,1}$ is improved by

$$\delta_{N,1}^{I2}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \ T_1 < \frac{\rho T_2}{\xi} \text{ and } \frac{a\sigma_{yy}}{2} \le T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,1}, & \text{otherwise.} \end{cases}$$

Improved estimator 3: For $a > 0$ ($a < 0$) and $-1 \le \rho < 0$ ($0 < \rho \le 1$), the estimator $\delta_{N,1}$ is improved by

$$\delta_{N,1}^{I3}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \ T_1 < \frac{\rho T_2}{\xi} \text{ and } \frac{a\sigma_{yy}}{2} \le T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ & \text{or } \ T_1 > \frac{\rho T_2}{\xi} \text{ and } \frac{a\sigma_{yy}}{2} \ge T_2 > \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,1}, & \text{otherwise.} \end{cases}$$

Improved estimator 4: For $a < 0$ and $\rho = 0$, the estimator $\delta_{N,1}$ is improved by

$$\delta_{N,1}^{I4}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \frac{a\sigma_{yy}}{2} \le T_2 < -\frac{a\sigma_{yy}}{2} \\ \delta_{N,1}, & \text{otherwise.} \end{cases}$$

For $a > 0$ and $\rho = 0$, Theorem 4.3 fails to provide an improved estimator upon the estimator $\delta_{N,1}$.

Improved estimator 5: The estimator $\delta_{N,2} = Y_{[2]} - \frac{a\sigma_{yy}}{2}$ is a member of the class Ω_c (δ_φ with $\varphi = -\frac{a\sigma_{yy}}{2}$). It follows from Theorem 4.1 that, the estimator $\delta_{N,2}$ is improved by $\delta_{N,2}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = Y_{[2]} + \varphi_{N,2}(T_1, T_2)$, where

$$\varphi_{N,2}(T_1, T_2) = \begin{cases} \varphi_I(T_1, T_2), & \text{if } -\frac{a\sigma_{yy}}{2} \le \varphi_I(T_1, T_2) \\ -\frac{a\sigma_{yy}}{2}, & \text{if } \varphi_I(T_1, T_2) < -\frac{a\sigma_{yy}}{2} < \varphi_S(T_1, T_2) \\ \varphi_S(T_1, T_2), & \text{if } -\frac{a\sigma_{yy}}{2} \ge \varphi_S(T_1, T_2). \end{cases}$$

For $a > 0$ ($a < 0$) and $0 < \rho \le 1$ ($-1 \le \rho < 0$), the estimator $\delta_{N,2}$ is improved by

$$\delta_{N,2}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \frac{\rho T_2}{\xi} \text{ and } -\frac{a\sigma_{yy}}{2} \le T_2 < \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ & \text{or } T_1 > \frac{\rho T_2}{\xi} \text{ and } -\frac{a\sigma_{yy}}{2} \ge T_2 > \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,2}, & \text{otherwise.} \end{cases}$$

Improved estimator 6: For $a > 0$ and $-1 \le \rho < 0$, the estimator $\delta_{N,2}$ is improved by

$$\delta_{N,2}^{I2}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \frac{\rho T_2}{\xi} \text{ and } -\frac{a\sigma_{yy}}{2} \le T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,2}, & \text{otherwise.} \end{cases}$$

For $a < 0$ ($a \neq 0$) and $0 < \rho \le 1$ ($\rho = 0$), Theorem 4.3 fails to provide an improved estimator upon the estimator $\delta_{N,2}$.

Improved estimator 7: For $a > 0$, $0 < \rho \le 1$, and $\varphi_3 \le \varphi_I$ or $\varphi_3 \ge \varphi_S$, where $\varphi_3 = \frac{1}{a}\ln\left[1 + \left(e^{aT_2} - 1\right)\ \Phi\left(\frac{T_1}{\sqrt{2\sigma_{xx}}}\right)\right]$, and φ_I and φ_S are as given in Lemma 4.2, the estimator $\delta_{N,3}$ is improved by

$$\delta_{N,3}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \frac{\rho T_2}{\xi} \text{ and } T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ & \text{or } T_1 > \frac{\rho T_2}{\xi} \text{ and } T_2 > \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,3}, & \text{otherwise.} \end{cases}$$

Improved estimator 8: For $a < 0$ and $0 < \rho \le 1$ and $\varphi_3 \le \varphi_I$, the estimator $\delta_{N,3}$ is improved by

$$\delta_{N,3}^{I2}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \frac{\rho T_2}{\xi} \text{ and } T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,3}, & \text{otherwise.} \end{cases}$$

Improved estimator 9: For $a \neq 0$, $-1 \le \rho < 0$ and $\varphi_3 \le \varphi_I$ or $\varphi_3 \ge \varphi_I$, the estimator $\delta_{N,3}$ is improved by

$$\delta_{N,3}^{I3}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_2 < \min\left\{\frac{\xi T_1}{\rho}, \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2)\right\} \\ & \text{or } \max\left\{\frac{\xi T_1}{\rho}, \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2)\right\} < T_2 \\ \delta_{N,3}, & \text{otherwise.} \end{cases}$$

Improved estimator 10: For $a \neq 0$, $\rho = 0$ and $\varphi_3 \leq \varphi_I$, the estimator $\delta_{N,3}$ is improved by

$$\delta_{N,3}^{I4}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4} & \text{if } T_1 < 0 \text{ and } T_2 < -\frac{a\sigma_{yy}}{2} \\ \delta_{N,3}, & \text{otherwise.} \end{cases}$$

Improved estimator 11: For $a > 0$ and $0 < \rho \leq 1$, the estimator $\delta_{N,4}$ is improved by

$$\delta_{N,4}^{I1}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 > \max\left\{-c\sqrt{2\sigma_{xx}}, \frac{\rho T_2}{\xi}\right\} \text{ and } T_2 > \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } \frac{\rho T_2}{\xi} < T_1 \leq -c\sqrt{2\sigma_{xx}} \text{ and } \frac{a\sigma_{yy}}{2} \geq T_2 > \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,4}, & \text{otherwise.} \end{cases}$$

Improved estimator 12: For $a > 0$ and $-1 \leq \rho < 0$, the estimator $\delta_{N,4}$ is improved by

$$\delta_{N,4}^{I2}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 > \max\left\{-c\sqrt{2\sigma_{xx}}, \frac{\rho T_2}{\xi}\right\} \text{ and } T_2 > \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \min\left\{-c\sqrt{2\sigma_{xx}}, \frac{\rho T_2}{\xi}\right\} \text{ and } \frac{a\sigma_{yy}}{2} \leq T_2 < \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ & \text{or } \frac{\rho T_2}{\xi} < T_1 \leq -c\sqrt{2\sigma_{xx}} \text{ and } \frac{a\sigma_{yy}}{2} \geq T_2 > \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,4}, & \text{otherwise.} \end{cases}$$

Improved estimator 13: For $a < 0$ and $0 < \rho \leq 1$, the estimator $\delta_{N,4}$ is improved by

$$\delta_{N,4}^{I3}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } -c\sqrt{2\sigma_{xx}} < T_1 < \frac{\rho T_2}{\xi} \text{ and } T_2 < \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \min\left\{-c\sqrt{2\sigma_{xx}}, \frac{\rho T_2}{\xi}\right\} \text{ and } \frac{a\sigma_{yy}}{2} \leq T_2 < \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ & \text{or } \frac{\rho T_2}{\xi} < T_1 \leq -c\sqrt{2\sigma_{xx}} \text{ and } \frac{a\sigma_{yy}}{2} \geq T_2 > \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,4}, & \text{otherwise.} \end{cases}$$

Improved estimator 14: For $a < 0$ and $-1 \leq \rho < 0$, the estimator $\delta_{N,4}$ is improved by

$$\delta_{N,4}^{I4}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } -c\sqrt{2\sigma_{xx}} < T_1 < \frac{\rho T_2}{\xi} \text{ and } T_2 < \xi\rho T_1 - \frac{a\sigma_{yy}}{2}(1-\rho^2) \\ \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 < \min\left\{-c\sqrt{2\sigma_{xx}}, \frac{\rho T_2}{\xi}\right\} \text{ and } \frac{a\sigma_{yy}}{2} \leq T_2 < \xi\rho T_1 - a\frac{\sigma_{yy}}{2}(1-\rho^2) \\ \delta_{N,4}, & \text{otherwise.} \end{cases}$$

Improved estimator 15: For $a < 0$ and $\rho = 0$, the estimator $\delta_{N,4}$ is improved by

$$\delta_{N,4}^{I5}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \begin{cases} \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 > -c\sqrt{2\sigma_{xx}} \text{ and } T_2 < -\frac{a\sigma_{yy}}{2} \\ \frac{Y_{[1]}+Y_{[2]}}{2} - \frac{a\sigma_{yy}}{4}, & \text{if } T_1 \leq -c\sqrt{2\sigma_{xx}} \text{ and } \frac{a\sigma_{yy}}{2} \leq T_2 < -\frac{a\sigma_{yy}}{2} \\ \delta_{N,4}, & \text{otherwise.} \end{cases}$$

For $a > 0$ and $\rho = 0$, Theorem 4.3 fails to provide an improved estimator upon the estimator $\delta_{N,4}$.

5. An application to Poultry feeds data

In this section, a data analysis is presented using a real data set (reported in [25]) to demonstrate the computation of various estimates of θ_{y}^{S}. Olosunde [25] conducted a study to compare the effect of two different copper-salt combinations on eggs produced by chicken in poultry feeds. An equal number of chickens were randomly assigned to be fed with each of the two combinations. A sample of 96 chickens were randomly selected from the poultry and were divided into two groups, of 48 chickens each. One group was given an organic copper-salt combination and an inorganic copper-salt combination was given to the another group. After a period of time, the weight and the cholesterol level of the eggs produced by the two groups were measured. The observed data from the organic and the inorganic Copper-Salt combinations are reported in [25] and presented in Table 1. The eggs with more weights and less cholesterol is preferable.

Table 1. Organic and Inorganic copper-salt combinations observed data.

Organic Copper-Salt				Inorganic Copper-Salt			
Weight		Cholesterol		Weight		Cholesterol	
56.08	56.34	60.73	66.03	52.67	53.17	164.23	167.42
56.61	56.87	71.33	76.63	53.67	54.17	170.60	173.78
57.13	57.39	81.86	81.93	54.67	55.17	176.96	180.14
57.65	57.92	81.93	87.16	55.67	56.17	183.32	186.51
58.18	58.44	92.46	92.52	56.67	57.17	189.69	192.87
58.70	58.96	97.76	97.82	57.67	58.17	196.05	199.24
59.23	59.45	103.06	103.11	58.67	59.17	202.42	205.60
59.75	60.01	108.36	108.41	59.67	60.17	208.78	211.96
60.27	60.54	113.66	113.70	60.67	61.17	215.14	218.33
60.80	61.06	118.96	119.00	61.67	62.17	221.52	224.69
61.32	61.58	124.26	124.30	62.67	63.17	224.85	227.88
61.85	62.34	129.56	129.60	63.43	65.15	228.03	231.06
62.11	61.85	134.86	134.89	65.67	63.43	228.01	224.83
61.58	61.32	140.16	140.19	62.93	62.43	221.65	218.46
61.06	60.80	1745.46	145.48	61.93	61.43	215.28	212.10
60.54	60.27	150.76	150.78	60.93	60.43	208.92	205.74
60.01	59.75	156.06	156.08	59.93	59.43	202.56	199.37
59.49	59.23	161.36	161.37	58.93	58.43	196.19	193.01
59.00	58.70	166.66	166.67	57.93	57.43	189.83	186.65
58.44	58.18	171.96	171.97	56.93	56.43	183.46	180.28
57.92	57.65	177.26	177.26	55.93	55.43	177.10	173.72
57.39	57.13	182.56	182.56	54.93	54.43	170.74	167.55
56.87	56.61	182.56	187.86	53.93	53.43	164.37	161.19
56.34	56.08	187.86	193.16	52.93	52.43	158.01	154.83

Let π_1 and π_2 represent the populations given an organic copper-salt combination and an inorganic copper-salt combination, respectively. Let (X_i, Y_i) be a pair of observations from the population π_i, $i = 1, 2$, where the X-variate denotes the average weights of eggs and the Y-variate denotes the corresponding average cholesterol levels. A number of 48 observations corresponding to each measurement is available from the data obtained by [25]. Since the sample sizes of the two populations are same, the pooled variance-covariance matrix is used. The obtained data are assumed to have a bivariate normal distribution with different means and common known variance-covariance matrix. To check the validity of the bivariate normality assumption for the available data set, we apply the Royston's normality test, given in the R-software package "MVN" that provided by [13]. Royston's test combines the Shapiro-Wilk (S-W) test statistics for univariate normality and obtain one test statistic for bivariate/multivariate normality. The Royston's and Shapiro-Wilk tests statistic with corresponding p-values are presented in Table 2.

From Table 2, we may conclude that the data set satisfy the bivariate normality assumption at 0.05 level of significance. The estimated parameters of the bivariate normal model (based on ML) are presented in Table 3.

Table 2. Normality test, p-values, kurtosis and skewness.

Test	Measure	Statistic	p-value	kurtosis	Skewness	Normality
Royston	π_1	5.878109	0.0529			Yes
S-W	π_1-weight	0.9569	0.0758	-1.256476	0.01487668	Yes
S-W	π_1-cholesterol	0.9598	0.0988	-1.213823	-0.09288089	Yes
Royston	π_2	2.867	0.1051			Yes
S-W	π_2-weight	0.9679	0.2089	-1.110509	0.1015675	Yes
S-W	π_2-cholesterol	0.9543	0.0592	-1.263555	-0.0816612	Yes

Table 3. Estimated parameters of the bivariate normal distribution.

Population	Measure	Mean	Variance	Covariance
π_1	weight	59.0997	8.1645	40.0655
	cholesterol	131.4569	952.9425	
π_2	weight	58.3516	8.1645	40.0655
	cholesterol	195.7275	952.9425	

Recall that, the quality of a population is determined with regard to their X-variate, while the corresponding Y-variate is of main interest. We say that the population $\pi_1 \equiv N\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\Sigma}\right)$ is better than the population $\pi_2 \equiv N\left(\boldsymbol{\theta}^{(2)}, \boldsymbol{\Sigma}\right)$ if $\theta_x^{(1)} > \theta_x^{(2)}$ and the population π_2 is considered better than the population π_1 if $\theta_x^{(1)} \leq \theta_x^{(2)}$, where $\boldsymbol{\theta}^{(1)} = \left(\theta_x^{(1)}, \theta_y^{(1)}\right)^\intercal$ and $\boldsymbol{\theta}^{(2)} = \left(\theta_x^{(2)}, \theta_y^{(2)}\right)^\intercal$ are the mean vectors of the populations π_1 and π_2 respectively. From the data we have $\hat{\boldsymbol{\theta}}^{(1)} = (59.0998, 131.4569)^\intercal$, $\hat{\boldsymbol{\theta}}^{(2)} = (58.3517, 195.7275)^\intercal$, and $\boldsymbol{\Sigma} = \begin{bmatrix} 8.1645 & 40.0655 \\ 40.0655 & 952.9425 \end{bmatrix}$. It can be observed that the average weight of eggs from chicken fed with an organic copper-salt combination is larger than the one with an inorganic copper-salt combination. Therefore, using the natural selection rule $\boldsymbol{\psi}$ given in Equation (1.2), we may conclude that the population π_1 is preferable over the population π_2. Also, the average cholesterol level for the population π_1 is less than that for the population π_2. Hence, based on the above observations, the organic copper-salt combination is recommended. This result was also obtained by [25]. The various estimates of θ_{y}^S of the selected bivariate normal population are presented in Table 4.

Table 4. The various estimates of θ_{y}^S for different values of a.

Estimators	a=0.1	a=-0.1	a=0.01	a=-0.01	0.001	-0.001
$\delta_{N,1}$	131.4569	131.4569	131.4569	131.4569	131.4569	131.4569
$\delta_{N,2}$	83.80977	179.104	126.6922	136.2216	130.9804	131.9334
$\delta_{N,3}$	188.1252	137.7294	166.5778	137.7294	161.9665	160.9386
$\delta_{N,4}(c)$	131.4569 (0.1)	131.4569 (0.15)	163.5922 (0.5)	163.5922 (1)	163.59 ($\sqrt{2}$)	163.59 (2)
$\delta_{N,1}^{I1}$	131.4569	-	131.4569	-	131.4569	-
$\delta_{N,1}^{I3}$	-	131.4569	-	131.4569	-	131.4569
$\delta_{N,2}^{I2}$	83.80977	179.104	126.6922	136.2216	130.9804	131.9334
$\delta_{N,3}^{I1}$	188.1252	137.7294	166.5778	137.7294	161.9665	160.9386
$\delta_{N,3}^{I2}$	-	137.7294	-	137.7294	-	160.9386
$\delta_{N,4}^{I1}(c)$	131.4569 (0.1)	131.4569 (0.15)	163.5922 (0.5)	163.5922 (1)	163.59 ($\sqrt{2}$)	163.59 (2)
$\delta_{N,4}^{I3}(c)$	131.4569 (0.1)	131.4569 (0.15)	163.5922 (0.5)	163.5922 (1)	163.59 ($\sqrt{2}$)	163.59 (2)

6. Risk comparisons of estimators

In this section, we compare the risk performance of the proposed estimators of θ_{y}^S, using the loss function given in Equation (1.1). For this purpose, a simulation study is performed using MATLAB software to compute the values of risk of the various estimators. 20,000 simulation runs with different configurations of parameters are used to obtain the risk values. Note that the estimator with the least average risk values is preferable. Further, the natural selection rule ψ presented in Equation (1.2) is used for achieving the aim of selecting the best bivariate normal population. It is easy to see that, the risk of the proposed estimators of θ_{y}^S depend on the parameters σ_{xx}, σ_{yy}, ρ, a and $\theta^{(1)} = \left(\theta_x^{(1)}, \theta_y^{(1)}\right)$, $\theta^{(2)} = \left(\theta_x^{(2)}, \theta_y^{(2)}\right)$ (only through θ_x and θ_y). So that, the risk functions are vary for different combinations of these parameters. The computed values of risks of the various estimators of θ_{y}^S are presented in Tables 5-10, for different combinations of $\theta^{(1)}$, $\theta^{(2)}$, and for $\sigma_{xx} = \sigma_{yy} = 2$, $\rho \in \{-1, 0, 1\}$, and $a \in \{-1, 1\}$. Note that the computation of risk values was carried-out for other values of a and ρ but these values were omitted from the tables because the same results were obtained. The risk values of the hybrid estimator $\delta_{N,4}$ were calculated for $c = 1$. In view of the risk values in Tables 5-10, we present the following assessment of the estimators of θ_{y}^S.

(1) For $a > 0$ and $0 < \rho \leq 1$, the improved estimators $\delta_{N,1}^{I1}$ and $\delta_{N,2}^{I2}$ provide a considerable improvement upon the estimators $\delta_{N,1}$ and $\delta_{N,2}$, respectively. The improved estimators $\delta_{N,3}^{I1}$ and $\delta_{N,4}^{I1}$ have the same performance with the estimators $\delta_{N,3}$ and $\delta_{N,4}$, respectively, hence their risk values were omitted form Table 5. The improved estimator $\delta_{N,2}^{I2}$ dominate all other estimators and has the least values of risk among other estimators.

(2) For $a > 0$ and $-1 \leq \rho < 0$, the improved estimators $\delta_{N,1}^{I3}$, $\delta_{N,2}^{I1}$, $\delta_{N,3}^{I3}$ and $\delta_{N,4}^{I2}$ perform better than their respective natural estimators. However, among all these estimators the improved estimator $\delta_{N,1}^{I3}$ has the best performance.

(3) For $a > 0$ and $\rho = 0$, the improved estimator $\delta_{N,3}^{I4}$ provides a significant improvement upon the estimator $\delta_{N,3}$. Also, the estimator $\delta_{N,3}^{I4}$ has better performance than the estimators $\delta_{N,2}$ and $\delta_{N,4}$ when $\theta_y \leq 0.2$. But, when $\theta_y > 0.2$ the estimator $\delta_{N,2}$ performs better than $\delta_{N,3}^{I4}$. Further, the estimator $\delta_{N,2}$ dominates the three estimators $\delta_{N,1}$, $\delta_{N,3}$ and $\delta_{N,4}$.

(4) For $a < 0$ and $0 < \rho \leq 1$, the estimator $\delta_{N,4}$ dominates the estimators $\delta_{N,2}$ and $\delta_{N,3}$, but, when θ_x and θ_y are very close to zero, $\delta_{N,3}$ dominates $\delta_{N,4}$. The estimator $\delta_{N,1}$ dominates all the estimators of θ_{y}^S. The improved estimators $\delta_{N,1}^{I3}$, $\delta_{N,3}^{I2}$ and $\delta_{N,4}^{I3}$ have the same values of risk with the estimators $\delta_{N,1}$ $\delta_{N,3}$ and $\delta_{N,4}$, respectively, hence their risk values were omitted form Table 8.

(5) For $a < 0$ and $-1 \leq \rho < 0$, the improved estimators $\delta_{N,1}^{I2}$, $\delta_{N,2}^{I2}$, $\delta_{N,3}^{I3}$ and $\delta_{N,4}^{I4}$ provide considerable improvement upon their respective natural estimators. However, the improved estimator $\delta_{N,2}^{I2}$ has the least risk values among all these estimators.

(6) For $a < 0$ and $\rho = 0$, the improved estimators $\delta_{N,1}^{I4}$, $\delta_{N,3}^{I4}$ and $\delta_{N,4}^{I5}$ provide only marginal improvement upon the estimators $\delta_{N,1}$, $\delta_{N,3}$ and $\delta_{N,4}$, respectively. The estimator $\delta_{N,4}^{I5}$ dominates the other estimators when θ_x and θ_y are very close to zero, but when θ_x and θ_y are not close to zero the estimator $\delta_{N,2}$ dominates $\delta_{N,4}^{I5}$.

Based on the above observations, we conclude that, for $a > 0$ and $0 \leq \rho \leq 1$ the performance of the estimator $\delta_{N,2}^{I2}$ is satisfactory, hence is recommended for practical purposes. For $a > 0$ and $-1 \leq \rho < 0$, the estimator $\delta_{N,1}^{I3}$ is recommended. For $a > 0$ and $\rho = 0$, the estimator $\delta_{N,3}^{I4}$ is recommended when $\theta_y \leq 0.2$ and the estimator $\delta_{N,2}$ is recommended for other values of θ_x and θ_y. For $a < 0$, the use of the natural estimator $\delta_{N,1}$ is recommended for $0 < \rho \le 1$ and the estimator $\delta^{I2}_{N,2}$ is recommended for $-1 \le \rho < 0$. Also, for $a < 0$ and $\rho = 0$, the estimator $\delta^{I5}_{N,4}$ is recommended when θ_x and θ_y are very close to zero, and the estimator $\delta_{N,2}$ is recommended when θ_x and θ_y are not close to zero.

Table 5. Risk values of the various estimators of θ^S_{y} for $a = 1$, $\sigma_{xx} = \sigma_{yy} = 2$, $\rho = 1$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta^{I1}_{N,1}$	$\delta_{N,2}$	$\delta^{I2}_{N,2}$	$\delta_{N,3}$	$\delta_{N,4}$
(0.2,2)	(2,0.2)	2.6462	1.7312	0.9743	0.8315	3.7656	2.5723
(0.4,1.8)	(1.8,0.4)	2.6458	1.6916	1.0069	0.8184	3.4462	2.7156
(0.6,1.6)	(1.6,0.6)	2.6915	1.6833	1.0001	0.7655	2.9903	2.3291
(0.8,1.4)	(1.4,0.8)	2.9213	1.7103	1.0110	0.7371	2.4982	2.2103
(1,1.2)	(1.2,1)	2.7271	1.5697	1.0090	0.7050	2.5228	2.2934
(0,0)	(0,0)	2.6445	2.6445	1.0463	1.0463	2.4251	2.3460
(1.2,1)	(1,1.2)	2.8050	1.5987	0.9524	0.7978	2.6302	2.3102
(1.4,0.8)	(0.8,1.4)	3.0249	1.7512	0.9608	0.8304	2.5218	2.3739
(1.6,0.6)	(0.6,1.6)	2.7631	1.6928	0.9847	0.8421	2.5866	2.4471
(1.8,0.4)	(0.4,1.8)	2.5929	1.6722	0.9783	0.8634	2.8642	2.5125
(2,0.2)	(0.2,2)	2.5158	1.7084	0.9673	0.8592	3.4061	2.6093

Table 6. Risk values of the various estimators of θ^S_{y} for $a = 1$, $\sigma_{xx} = \sigma_{yy} = 2$, $\rho = 0$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta_{N,2}$	$\delta_{N,3}$	$\delta^{I4}_{N,3}$	$\delta_{N,4}$
(0.2,2)	(2,0.2)	1.8102	1.0294	2.8037	0.2382	1.7059
(0.4,1.8)	(1.8,0.4)	1.7510	1.0135	2.2707	0.2913	1.4270
(0.6,1.6)	(1.6,0.6)	1.7064	0.9899	1.7890	0.3291	1.2453
(0.8,1.4)	(1.4,0.8)	1.7238	1.0044	1.5421	0.5055	1.0612
(1,1.2)	(1.2,1)	1.6948	0.9882	1.4484	0.6394	1.0201
(0,0)	(0,0)	1.7815	0.9668	1.4718	0.7055	1.0048
(1.2,1)	(1,1.2)	1.7621	0.9876	1.4629	0.8513	1.0175
(1.4,0.8)	(0.8,1.4)	1.7868	1.0149	1.6371	1.1382	1.1297
(1.6,0.6)	(0.6,1.6)	1.8065	1.0233	1.8261	1.4093	1.2745
(1.8,0.4)	(0.4,1.8)	1.7162	1.0005	2.1955	1.8950	1.3780
(2,0.2)	(0.2,2)	1.7324	1.00108	2.7087	2.5046	1.6951

Table 7. Risk values of the various estimators of θ^S_{y} for $a = -1$, $\sigma_{xx} = \sigma_{yy} = 4$, $\rho = 1$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta_{N,2}$	$\delta_{N,3}$	$\delta_{N,4}$
(0.2,2)	(2,0.2)	0.8470	1.0127	5.7682	0.8711
(0.4,1.8)	(1.8,0.4)	0.7248	1.0865	2.5465	0.8804
(0.6,1.6)	(1.6,0.6)	0.7089	1.0710	1.3354	0.8717
(0.8,1.4)	(1.4,0.8)	0.7276	1.0884	0.8350	0.7409
(1,1.2)	(1.2,1)	0.6622	1.0792	0.6940	0.7321
(0,0)	(0,0)	0.7230	1.0702	0.8046	0.8953
(1.2,1)	(1,1.2)	0.6552	1.0754	0.6857	0.7314
(1.4,0.8)	(0.8,1.4)	0.7326	1.0777	0.8444	0.7707
(1.6,0.6)	(0.6,1.6)	0.7397	1.0828	1.2542	0.7920
(1.8,0.4)	(0.4,1.8)	0.7428	1.0912	2.5213	0.9157
(2,0.2)	(0.2,2)	0.7429	1.0854	5.7609	0.9237

Table 8. Risk values of the various estimators of θ_{y}^{S} for $a=-1$, $\sigma_{xx}=\sigma_{yy}=4$, $\rho=-1$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta_{N,1}^{I2}$	$\delta_{N,2}$	$\delta_{N,2}^{I2}$	$\delta_{N,3}$	$\delta_{N,3}^{I3}$	$\delta_{N,4}$	$\delta_{N,4}^{I4}$
(0.2,2)	(2,0.2)	2.5510	1.5050	0.9719	0.8279	5.4113	2.6828	2.2766	1.4729
(0.4,1.8)	(1.8,0.4)	2.7091	1.5983	0.9564	0.7988	5.8670	2.7114	2.3859	1.4868
(0.6,1.6)	(1.6,0.6)	2.6500	1.3048	0.8987	0.6791	5.9427	2.1990	2.3006	1.1539
(0.8,1.4)	(1.4,0.8)	2.9947	1.5611	0.9747	0.6968	6.7696	3.0248	2.5833	1.4237
(1,1.2)	(1.2,1)	2.6574	1.3893	0.8401	0.6219	6.1065	2.6089	2.2276	1.2886
(0,0)	(0,0)	2.5950	0.9487	0.8343	0.5545	6.0156	1.3589	2.1910	0.8812
(1.2,1)	(1,1.2)	2.7163	1.3609	0.8578	0.6191	6.2608	2.5868	2.2753	1.2710
(1.4,0.8)	(0.8,1.4)	2.7172	1.4031	0.8875	0.6452	6.2037	2.7325	2.3345	1.2680
(1.6,0.6)	(0.6,1.6)	2.6410	1.2674	0.8919	0.6566	5.8921	2.1696	2.2734	1.1366
(1.8,0.4)	(0.4,1.8)	2.6376	1.5342	0.9428	0.7901	5.7970	2.6678	2.3202	1.4445
(2,0.2)	(0.2,2)	2.5747	1.4634	0.9800	0.8183	5.5067	2.5940	2.2982	1.4377

Table 9. Risk values of the various estimators of θ_{y}^{S} for $a=1$, $\sigma_{xx}=\sigma_{yy}=2$, $\rho=-1$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta_{N,1}^{I3}$	$\delta_{N,2}$	$\delta_{N,2}^{I1}$	$\delta_{N,3}$	$\delta_{N,3}^{I3}$	$\delta_{N,4}$	$\delta_{N,4}^{I2}$
(0.2,2)	(2,0.2)	0.8444	0.8311	1.0076	0.9003	1.1567	1.0309	1.1740	1.0267
(0.4,1.8)	(1.8,0.4)	0.8144	0.7603	1.0551	0.8648	1.0014	0.8922	1.0819	0.9385
(0.6,1.6)	(1.6,0.6)	0.7200	0.6401	1.0794	0.7991	0.7685	0.7001	0.8720	0.7522
(0.8,1.4)	(1.4,0.8)	0.6935	0.6061	1.1007	0.8165	0.6789	0.6317	0.8052	0.7123
(1,1.2)	(1.2,1)	0.6668	0.5711	1.1303	0.7972	0.6010	0.5749	0.7359	0.6551
(0,0)	(0,0)	0.6636	0.5244	1.1273	0.6705	0.5974	0.5728	0.7312	0.6167
(1.2,1)	(1,1.2)	0.6759	0.5883	1.1219	0.8072	0.6331	0.6077	0.7545	0.6796
(1.4,0.8)	(0.8,1.4)	0.6741	0.5921	1.0815	0.8039	0.6631	0.6163	0.7860	0.6968
(1.6,0.6)	(0.6,1.6)	0.7262	0.6411	1.0883	0.7959	0.7793	0.7183	0.8866	0.7703
(1.8,0.4)	(0.4,1.8)	0.8091	0.7601	1.0564	0.8658	0.9857	0.8826	1.0557	0.9217
(2,0.2)	(0.2,2)	0.8739	0.8523	1.0286	0.9121	1.1834	1.0477	1.2171	1.0514

Table 10. Risk values of the various estimators of θ_{y}^{S} for $a=-1$, $\sigma_{xx}=\sigma_{yy}=4$, $\rho=0$.

$\theta^{(1)}$	$\theta^{(2)}$	$\delta_{N,1}$	$\delta_{N,1}^{I4}$	$\delta_{N,2}$	$\delta_{N,3}$	$\delta_{N,4}$	$\delta_{N,4}^{I5}$
(0.2,2)	(2,0.2)	1.6912	1.6477	0.9909	80.5726	1.3579	1.2440
(0.4,1.8)	(1.8,0.4)	1.7057	1.6595	0.9975	63.5385	1.2313	1.1164
(0.6,1.6)	(1.6,0.6)	1.7053	1.6616	0.9974	92.9430	1.1331	1.0304
(0.8,1.4)	(1.4,0.8)	1.7269	1.6796	1.0031	39.0709	1.0501	0.9585
(1,1.2)	(1.2,1)	1.6670	1.6183	0.9756	24.7978	1.0183	0.9209
(0,0)	(0,0)	1.6932	1.6436	0.9835	24.7124	0.9684	0.8822
(1.2,1)	(1,1.2)	1.7682	1.7364	1.0191	47.2484	0.9901	0.8853
(1.4,0.8)	(0.8,1.4)	1.7416	1.7215	0.9995	34.1616	1.0370	0.9302
(1.6,0.6)	(0.6,1.6)	1.6809	1.6567	1.0217	53.2835	1.1860	1.0729
(1.8,0.4)	(0.4,1.8)	1.6833	1.6786	0.9869	136.5534	1.2239	1.0576
(2,0.2)	(0.2,2)	1.6913	1.6913	1.0117	76.2712	1.4199	1.2616

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