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Research Article

Multicomponent stress-strength reliability estimation for the standard two-sided power distribution

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Abstract

A system of k components that functions as long as at least s components survive is termed as s-out-of-k:G system, where G refers to "good". In this study, we consider the s-out-of-k:G system when $X_1, X_2, \cdots, X_k$ are independent and identically distributed strength components and each component is exposed to common random stress Y when the underlying distributions all belong to the standard two-sided power distribution. The system is regarded as surviving only if at least s out of k $(1 < s < k)$ strengths exceed the stress. The reliability of such a system is the surviving probability and is estimated by using the maximum likelihood and Bayesian approaches. Parametric and nonparametric bootstrap confidence intervals for the maximum likelihood estimates and the highest posterior density confidence intervals for Bayes estimates by using the Markov Chain Monte Carlo technique are obtained. A real data set is also analyzed to illustrate the performances of the estimators.

Mathematics Subject Classification (2020). 62N05, 62F10, 62F15

Keywords. Bayesian estimation, maximum likelihood estimation, reliability, stress-strength model, two-sided power distribution

1. Introduction

The standard two-sided power distribution, denoted by STSP, is introduced by [26] and proposed as a peaked alternative to the beta distribution by [17]. It has the following probability density function (pdf) and the cumulative distribution function (cdf), respectively,

$$f(x|\alpha,\beta) = \begin{cases} \alpha(\frac{x}{\beta})^{\alpha-1} & ,0 < x \leq \beta \\ \alpha(\frac{1-x}{1-\beta})^{\alpha-1} & ,\beta \leq x < 1 \end{cases} \tag{1.1}$$

$$F(x|\alpha,\beta) = \begin{cases} \beta(\frac{x}{\beta})^{\alpha} & ,0 \leq x \leq \beta \\ 1-(1-\beta)(\frac{1-x}{1-\beta})^{\alpha} & ,\beta \leq x \leq 1 \end{cases} \tag{1.2}$$

where $\alpha > 0$ is the shape and $0 < \beta < 1$ is the reflection/threshold parameter. Since it is defined on a bounded support and has a similar flexibility, the STSP distribution is

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a beta-like distribution. The parameters in the distribution determine the shapes of the distribution. For example, for $0 < \beta < 1$ and $\alpha > 0$, the distribution is unimodal; for $0 < \beta < 1$ and $0 < \alpha < 1$, the distribution is U shaped; for $\alpha = 1$ the distribution is uniform on $(0, 1)$; for $\alpha = 2$, the distribution is triangular. When $\beta = 0.5$, the distribution is symmetric (see Figure 1). The left-skewed and right-skewed distributions are obtained when $\beta > 0.5$ and $\beta < 0.5$, respectively, for $\alpha > 1$. The STSP distribution is clearly more flexible than the power function distribution which is obtained for the case $\beta = 1$. When the support of the distribution is extended to a finite interval (a, b), it is called the two-sided power (TSP) distribution. In literature, the STPS distribution is commonly used for modeling financial data with excess kurtosis. Many authors studied the STSP distribution and its extensions in detail. Recently, Akther et al. [2] handled new explicit expressions for the moments of order statistics of the STSP distribution. Kharazmi et al. [13] proposed a general change-point family of distributions as an extension of the STSP distribution. A matrix variate two-sided power distribution is proposed by [27].

The STSP distribution has also applications in risk analysis such as project evaluation and review technique (PERT) [22]. The STSP distribution can be used in reliability and life testing experiments on $[0, 1]$. Particularly, when these types of lifetime data have any threshold point, they are convenient for modeling by a two-sided distribution. Mance et al. [21] studied some features of the TSP which is an extension of the STSP distribution in reliability analysis, firstly. They introduced the reliability and hazard functions of the TSP distribution and presented their plots with usefulness in engineering. Using an analytical estimation procedure, they obtained the TSP parameters and compared the distribution with the Weibull distribution. Recently, Çetinkaya and Genç [7] studied the STSP distribution for the stress-strength reliability where a component operates if its strength exceeds the stress imposed on it, respectively. The aim of this paper is to study the reliability of a multicomponent stress-strength model under the STSP distribution.

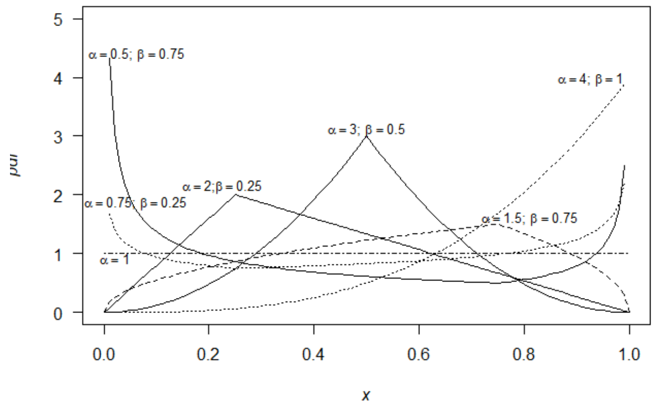

Figure 1. Plots of probability denstiy function of the STSP distribution for various choices of its parameters

A multicomponent system has more than one component and occurs when a component under consideration of k independent components with the strengths $X_1, X_2, \cdots, X_k$ and each component of the system is subjected to common stress Y [18]. A system belonging to this class can be one of the two types:

- The system functions when at least s $(1 \leq s \leq k)$, components are working; denoted by s-out-of-k:G system, where G refers to "good",
- The system fails when the failure of the kth component $(1 \leq s \leq k)$; denoted by s-out-of-k:F system, where F refers to "fail".

Many examples of s-out-of-k system can be considered. For example; A panel consisting of k identical solar cells maintains an adequate power output if at least s of the cells are active during the duration of the mission. The external force interfering with the operation of the cells may be extreme temperatures and the strength of a cell, in this context, may be taken as its capacity to withstand the external temperatures [12]. As another example, we consider a suspension bridge consisting of n vertical cable pairs. One vertical cable pair consists of two cables on both sides of a deck. The bridge will only survive if a minimum of s vertical cable pairs through the deck are not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion, and so on. These types of systems also appear in different areas such as industrial and military applications.

Suppose a system with the strengths of the components, $X_1, X_2, \cdots, X_k$, are independent and identically distributed random variables with cdf F_X and subjected to the common stress Y having cdf F_Y, then the reliability, probability of successful operation, of the s-out-of-k system defined by [4] is given by

$$\begin{aligned} R_{s,k} &= P[\text{at least s of the } (X_1, X_2, \cdots, X_k) \text{ exceed } Y] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y). \end{aligned} \tag{1.3}$$

When the system operates whenever at least one of k components survives, that is $s = 1$, it is named as a parallel system. However, if the system operates when all s components survive, that is $s = k$, it is termed as a series system.

Reliability estimation of a s-out-of-k system has been discussed by several authors in the literature. In recent years, there has been growing interest for the s-out-of-k and related systems under different probability distributions. This type of multicomponent stress-strength reliability estimation studies is considered for log-logistic distribution by [24], generalized exponential by [23], PoissonWeibull models by [10], Weibull distribution by [16], Kumaraswamy distribution by [8], the general class of inverse exponential distributions by [14], Topp-Leone distribution by [1]. Furthermore, Kzlaslan [15] considered a multicomponent system when the underlying distributions belonging to the proportional reversed hazard rate model. Barbiero [3] defined a general discretization procedure for both stress and strength are defined as complex functions of continuous random variables.

Unlike all previous studies based on multicomponent stress-strength reliability, we studied a distribution with bounded support and two-sided. With these aspects, the STSP distribution provides more flexibility for the lifetimes which have a threshold point. Further, it has a bathtub failure curve which makes it useful for modeling early life, useful life, and wear out processes of a component lifetime with only a single model for $\alpha < 1$ values of its shape parameter. Also, it belongs to the increasing failure rate (IFR) class of distributions for $\alpha > 1$ and has a better chance of surviving any shorter period and a worse chance of surviving any larger period (see Figure 2).

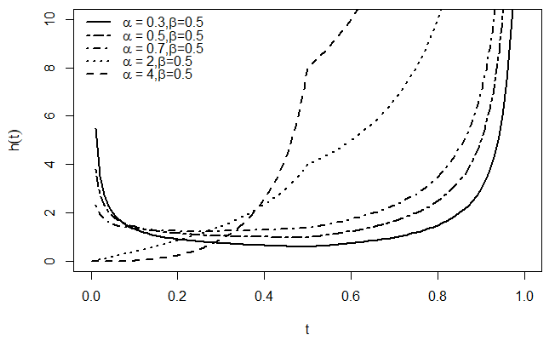

Figure 2. Hazard function plots of the symmetrical $STSP$ distribution ($\beta = 0.5$) for different shape parameters.

2. Maximum likelihood estimation of $R_{s,k}$

Let us assume that $X_1, X_2, \cdots, X_k$ are independently and identically distributed (iid) random strengths from $STSP(\alpha_1, \beta_1)$ and Y is a common stress from $STSP(\alpha_2, \beta_2)$. We will first derive an expression for $R_{s,k}$ and then construct the likelihood function. To compute $R_{s,k}$, we can assume, without loss of generality, that $\beta_2 < \beta_1$. Then, the reliability of a multicomponent stress-strength model for the STSP distribution using Equation(1.1) and Equation (1.2) can be obtained by

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \\ &= \sum_{i=s}^{k} \binom{k}{i} \Big[\int_0^{\beta_2} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \\ &+ \int_{\beta_2}^{\beta_1} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) + \int_{\beta_1}^{1} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \Big]. \end{aligned} \tag{2.1}$$

The three integrals from the left to right in Equation(2.1) are denoted by $I_i, i = 1, 2, 3$ and can be evaluated as in the following. The first one is

$$\begin{aligned} I_1 &= \int_0^{\beta_2} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \\ &= \int_0^{\beta_2} \left[1 - \beta_1 \left(\frac{y}{\beta_1}\right)^{\alpha_1}\right]^i \left[\beta_1 \left(\frac{y}{\beta_1}\right)^{\alpha_1}\right]^{k-i} \alpha_2 \left(\frac{y}{\beta_2}\right)^{\alpha_2 - 1} dy. \end{aligned}$$

by change of variable $u = \beta_1 \left(\frac{y}{\beta_1}\right)^{\alpha_1}$, we obtain I_1 as

$$I_1 = \rho \frac{\beta_1^{\rho(\alpha_1 - 1)}}{\beta_2^{(\alpha_2 - 1)}} B(\beta_1 \delta^{\alpha_1}; k - i + \rho, i + 1).$$

where $B(x; a, b)$ is the incomplete beta function defined by

$$B(x; a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} dt.$$

$\rho = \alpha_2/\alpha_1$ and $\delta = \beta_2/\beta_1$. The second integral

$$\begin{aligned} I_2 &= \int_{\beta_2}^{\beta_1} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \\ &= \int_{\beta_2}^{\beta_1} \left[1 - \beta_1 \left(\frac{y}{\beta_1}\right)^{\alpha_1}\right]^i \left[\beta_1 \left(\frac{y}{\beta_1}\right)^{\alpha_1}\right]^{k-i} \alpha_2 \left(\frac{1 - y}{1 - \beta_2}\right)^{\alpha_2 - 1} dy. \end{aligned}$$

By expanding the term with power i by binomial theorem, we obtain I_2 as

$$\begin{aligned} I_2 = &\sum_{j=0}^{i} \binom{i}{j} (-1)^j \beta_1^{(1-\alpha_1)(j+k-i)} (1 - \beta_2)^{(1-\alpha_2)} \\ &\times \alpha_2 \big[B(\beta_1; \alpha_1(k + j - i) + 1, \alpha_2) - B(\beta_2; \alpha_1(k + j - i) + 1, \alpha_2) \big]. \end{aligned}$$

And for the third integral

$$\begin{aligned} I_3 &= \int_{\beta_1}^{1} [1 - F_X(y)]^i [F_X(y)]^{k-i} dF_Y(y) \\ &= \int_{\beta_1}^{1} \left[(1 - \beta_1) \left(\frac{1 - y}{1 - \beta_1}\right)^{\alpha_1}\right]^i \left[1 - (1 - \beta_1) \left(\frac{1 - y}{1 - \beta_1}\right)^{\alpha_1}\right]^{k-i} \alpha_2 \left(\frac{1 - y}{1 - \beta_2}\right)^{\alpha_2 - 1} dy. \end{aligned}$$

by change of variable $u = (1-\beta_1)\left(\frac{1-y}{1-\beta_1}\right)^{\alpha_1}$, we obtain I_3 as

$$I_3 = \rho\frac{(1-\beta_1)^{\rho(\alpha_1-1)}}{(1-\beta_2)^{(\alpha_2-1)}}B(1-\beta_1; i+\rho, k-i+1).$$

Combining the integral expressions above in Equation (2.1), we get $R_{s,k}$ as

$$\begin{aligned}
R_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\rho\Big[&\frac{\beta_1^{\rho(\alpha_1-1)}}{\beta_2^{(\alpha_2-1)}}B(\beta_1\delta^{\alpha_1}; k-i+\rho, i+1)\\
&+(1-\beta_2)^{(1-\alpha_2)}\Big(\alpha_1\sum_{j=0}^{i}\binom{i}{j}(-1)^j\beta_1^{(1-\alpha_1)(j+k-i)}\big[B(\beta_1;\alpha_1(k+j-i)+1,\alpha_2) \qquad (2.2)\\
&-B(\beta_2;\alpha_1(k+j-i)+1,\alpha_2)\big]+(1-\beta_1)^{\rho(\alpha_1-1)}B(1-\beta_1; i+\rho, k-i+1)\Big)\Big].
\end{aligned}$$

The the likelihood function of the observed samples $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ are given as

$$\begin{aligned}
L(\alpha_1,\alpha_2,\beta_1,\beta_2|\mathbf{x},\mathbf{y}) = \alpha_1^n\alpha_2^m&\left\{\frac{\prod_{i=1}^{r}x_{(i)}\prod_{i=r+1}^{n}(1-x_{(i)})}{\beta_1^r(1-\beta_1)^{n-r}}\right\}^{\alpha_1-1}\\
&\times\left\{\frac{\prod_{i=1}^{r'}y_{(i)}\prod_{i=r'+1}^{m}(1-y_{(i)})}{\beta_2^{r'}(1-\beta_2)^{m-r'}}\right\}^{\alpha_2-1}.
\end{aligned}$$

where $x_{(r)} \le \beta_1 < x_{(r+1)}$ and $y_{(r')} \le \beta_2 < y_{(r'+1)}$ with $x_{(0)} \equiv 0$, $y_{(0)} \equiv 0$, $x_{(n+1)} \equiv 1$ and $y_{(m+1)} \equiv 1$.

The maximum likelihood estimates (MLEs) of the parameters based on a sample of size n from the $STSP(\alpha,\beta)$ distribution are obtained by [26], and they are given by

$$\hat{\beta} = X_{(\hat{r})} \qquad \text{and} \qquad \hat{\alpha} = -\frac{n}{\log M(\hat{r})}.$$

where $\hat{r} = arg\ max_{\{r\in 1,2,\cdots,n\}}M(r)$ and

$$M(r) = \prod_{i=1}^{r-1}\frac{X_{(i)}}{X_{(r)}}\prod_{i=r+1}^{n}\frac{1-X_{(i)}}{1-X_{(r)}}.$$

Hence, the MLE of $R_{s,k}$ is obtained from Equation (2.2) by using the invariance property of MLEs. That is, MLE of $R_{s,k}$ can be obtained by replacing the parameters in Equation (2.2) with their estimates. Also, bootstrap methods can be used to find approximate confidence intervals since neither exact nor approximate sampling distribution $\hat{R}_{s,k}$ is not available.

2.1. Bootstrap confidence intervals for $R_{s,k}$

In this subsection, we propose parametric and nonparametric bootstrap methods for obtaining an approximate confidence interval for $R_{s,k}$. Çetinkaya and Genç [7] mentioned that the STSP distribution does not belong to a regular family of distributions since the support of the distribution depends on its threshold parameter β. Therefore, asymptotic variance of parameters cannot be determined by using the standard theory of the Fisher information matrix. Thus, bootstrap percentile method (boot-p) will be used in place of bootstrap-t confidence interval that require a regular family in its construction. We propose to use the following algorithm to generate parametric and non-parametric bootstrap samples, as suggested by [9].

- Step 1: Generate random samples $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ from $STSP(\alpha_1, \beta_1)$ and $STSP(\alpha_2, \beta_2)$, respectively.
  For obtaining a non-parametric bootstrap sample
- Step 2: By sampling with replacement, generate bootstrap samples $x_1^*, x_2^*, \cdots, x_n^*$ and $y_1^*, y_2^*, \cdots, y_m^*$ by using the random samples $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$.
  For obtaining a parametric bootstrap sample
- Step 2: After computing the MLEs of all parameters $\hat{\alpha}_1$,$\hat{\alpha}_2$,$\hat{\beta}_1$ and $\hat{\beta}_2$, generate independent bootstrap samples $x_1^*, x_2^*, \cdots, x_n^*$ and $y_1^*, y_2^*, \cdots, y_m^*$ from $STSP(\hat{\alpha}_1, \hat{\beta}_1)$ and $STSP(\hat{\alpha}_2, \hat{\beta}_2)$, respectively. Then, compute the MLEs of all parameters based on the bootstrap samples, denoted by $\hat{\alpha}_1^*$,$\hat{\alpha}_2^*$,$\hat{\beta}_1^*$ and $\hat{\beta}_2^*$.
- Step 3: Compute the bootstrap estimate of $R_{s,k}$ by replacing the parameters in Equation (2.2) with their bootstrap estimates and denote by $\hat{R}^*_{s,k}$.
- Step 4: Repeat Step 2 B times and obtain a set of bootstrap estimates of $R_{s,k}$, say $\hat{R}^*_{(s,k)i} : i = 1, 2, \cdots, B$.

By using the bootstrap samples which are obtained above, compute $(\hat{R}^{*(\alpha/2)}_{s,k}, \hat{R}^{*(1-\alpha/2)}_{s,k})$ where $\hat{R}^{*(\gamma)}_{s,k}$ is the γ-percentile of $\hat{R}^*_{(s,k)i} : i = 1, 2, \cdots, B$, that is a number such that

$$\frac{1}{B}\sum_{i=1}^{B} I(\hat{R}^*_{(s,k)i} \leq \hat{R}^{*(\gamma)}_{s,k}), \quad 0 < \gamma < 1.$$

and $I(.)$ is the indicator function.

3. Bayes estimation of $R_{s,k}$

In this section, the Bayes estimate of $R_{s,k}$, denoted by $R^B_{s,k}$, is obtained by the following the findings of [7]. They noted that the form of the pdf of the STSP distribution given in Equation 1.1 is not compatible with developing classical Bayesian models. A similar problem was handled for the triangular distribution by [11], also. Hence, we used the hierarchical model proposed by [7] to obtain $R^B_{s,k}$. For this purpose, we first consider all the parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 are unknown random parameters. Then, we need to obtain posterior densities of the parameters, denoted by $\pi(\alpha_1, \alpha_2, \beta_1, \beta_2|\mathbf{x},\mathbf{y})$, by considering the independent prior distributions for parameters, denoted by $\pi(\alpha_1)$, $\pi(\alpha_2)$, $\pi(\beta_1)$ and $\pi(\beta_2)$. Then, based on the given assumptions, we have the likelihood function as

$$L(\mathbf{x},\mathbf{y}|\alpha_1, \alpha_2, \beta_1, \beta_2) = \alpha_1^n \alpha_2^m \left\{ \frac{\prod_{i=1}^{r} x_{(i)} \prod_{i=r+1}^{n} (1 - x_{(i)})}{\beta_1^r (1-\beta_1)^{n-r}} \right\}^{\alpha_1 - 1} \times \left\{ \frac{\prod_{i=1}^{r'} y_{(i)} \prod_{i=r'+1}^{m} (1 - y_{(i)})}{\beta_2^{r'} (1-\beta_2)^{m-r'}} \right\}^{\alpha_2 - 1}.$$

Following, the joint density of the data and the parameters, α_1, α_2, β_1 and β_2, becomes

$$\pi(\alpha_1, \alpha_2, \beta_1, \beta_2, x, y) = L(\mathbf{x},\mathbf{y}|\alpha_1, \alpha_2, \beta_1, \beta_2)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta_1)\pi(\beta_2)$$

Then, the joint posterior density of the parameters is obtained as

$$\pi(\alpha_1, \alpha_2, \beta_1, \beta_2|\mathbf{x},\mathbf{y}) = \frac{\pi(\alpha_1, \alpha_2, \beta_1, \beta_2, data)}{\int_0^1 \int_0^1 \int_0^\infty \int_0^\infty L(\mathbf{x},\mathbf{y}, \alpha_1, \alpha_2, \beta_1, \beta_2) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2}.$$

Thus, under squared error loss function, the Bayes estimate of $R_{s,k}$, say $\hat{R}^B_{s,k}$ is defined as

$$\hat{R}^B_{s,k} = \int_0^1 \int_0^1 \int_0^\infty \int_0^\infty R_{s,k}\ \pi(\alpha_1, \alpha_2, \beta_1, \beta_2|\mathbf{x},\mathbf{y}) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2. \tag{3.1}$$

The squared error loss function (SELF) is the most commonly used loss function because it is symmetrical and it provides equal distance to the losses through overestimation and underestimation. The performances of the Bayes estimations under the different loss functions are handled by [6] and it is observed that there are no significant differences between them. Therefore, we propose to use SELF as a symmetrical loss function.

Unfortunately, it is not possible to obtain an explicit expression for this posterior mean of $R_{s,k}$. Çetinkaya and Genç [7] mentioned that the form of the $STSP$ distribution given in Equation (1.1) is not compatible for developing Bayesian models. Since the support of the distribution depends on the reflection parameter, posterior distributions of α and β cannot be obtained. This fact was previously pointed out for the triangular distribution which is special form of the STSP distribution ($\alpha = 2$ case) by [11]. To overcome this adversity and obtain a Bayesian inference for the STSP distribution, Çetinkaya and Genç [7] proposed a hierarchical model construction. This model provides conditional distributions of parameters to build a MCMC algorithm using a Gibbs sampler as given in the following.

Theorem 3.1 ([7]). *Let V be a random variable with parameter $\alpha > 1$. Suppose that V has the pdf*

$$f_V(v;\alpha) = \alpha\big[1-(1-v)^{1/(\alpha-1)}\big], \quad 0 < v < 1.$$

Further, let the conditional distribution of X given $V = v$ be the uniform distribution represented by

$$U\big[\beta(1-v)^{1/(\alpha-1)}, 1-(1-\beta)(1-v)^{1/(\alpha-1)}\big].$$

Then the marginal distribution of X has the STSP distribution with pdf given in Equation (1.1). Thus, this hierarchical model will simplify the computational procedures for Bayesian calculations. In order to implement a Gibbs sampler, Çetinkaya and Genç [7] obtained the conditional distributions of α, β and v as in the following

$$\begin{aligned} f(v|\alpha,\beta,x) &\propto f(v|\alpha)f(x|\alpha,\beta,v) \\ &\propto I\Bigg(\max\bigg\{1-\Big(\frac{x}{\beta}\Big)^{\alpha-1}, 1-\Big(\frac{1-x}{1-\beta}\Big)^{\alpha-1}\bigg\} < v < 1\Bigg), \end{aligned} \tag{3.2}$$

$$\begin{aligned} f(\beta|\alpha,v,x) &\propto \pi(\beta)f(x|\beta,v,\alpha) \\ &\propto \pi(\beta)I\Bigg(1-\frac{1-x}{(1-v)^{1/(\alpha-1)}} < \beta < \frac{x}{(1-v)^{1/(\alpha-1)}}\Bigg), \end{aligned}$$

$$\begin{aligned} f(\alpha|v,\beta,x) &\propto \pi(\alpha)f(v|\alpha)f(x|\beta,v,\alpha) \\ &\propto \alpha\pi(\alpha)I\Bigg(1 < \alpha < \min\bigg\{\frac{\ln(1-v)}{\ln(\frac{x^{<}}{\beta})}+1, \frac{\ln(1-v)}{\ln(\frac{1-x^{>}}{1-\beta})}+1\bigg\}\Bigg), \end{aligned}$$

where $I(.)$ denotes indicator function, $x^{<}$ denotes observations below β and $x^{>}$ observations above β, $\pi(\alpha)$ and $\pi(\beta)$ denotes prior distributions for the parameters.
In the following, similarly to Gibbs sampler algorithm which is given by [7], we give an algorithm for simulation from the STSP distributions with $\alpha_1 > 1$ and $\alpha_2 > 1$.

Step 1: Choose n, m, chain size M and initial $\alpha_1^{(0)}$ and $\beta_1^{(0)}$ values for α_1 and β_1, similarly for α_2 and β_2.
Step 2: Set $t = 1$.
Step 3: Generate $\{x_1, x_2, \cdots, x_n\}$ from the $STSP(\alpha_1^{(t-1)}, \beta_1^{(t-1)})$ distribution and $\{y_1, y_2, \cdots, y_m\}$ from the $STSP(\alpha_2^{(t-1)}, \beta_2^{(t-1)})$ distribution.
Step 4: Given $\alpha_1^{(t-1)}$ and $\beta_1^{(t-1)}$ and $\{x_1, x_2, \cdots, x_n\}$ generate $\{v_1, v_2, \cdots, v_n\}$ using Equation(3.2), similarly generate $\{v_1^{'}, v_2^{'}, \cdots, v_m^{'}\}$ given $\alpha_2^{(t-1)}$ and $\beta_2^{(t-1)}$ and $\{y_1, y_2, \cdots, y_m\}$.
Step 5: Considering uniform prior on $[0, 1]$ for β_1, given $\alpha_1^{(t-1)}$, $\{x_1, x_2, \cdots, x_n\}$ and

$\{v_1, v_2, \cdots, v_n\}$, generate $\beta_1^{(t)}$ using

$$I\Big(\max\Big\{1 - \frac{1-x_i}{(1-v_i)^{1/(\alpha^{(t-1)}-1)}}, 0\Big\} < \beta_1 < \min\Big\{\frac{x_i}{(1-v_i)^{1/(\alpha^{(t-1)}-1)}}, 1\Big\}\Big).$$

Similarly, generate for $\beta_2^{(t)}$.

Step 6: Considering uniform prior on $[1, c]$ for α_1 and choosing $c = 100$ generate $\alpha_1^{(t)}$ from the pdf $[(n+1)/(b^{n+1}-1)]\alpha^n$ using inverse transformation method, where

$$b = \min\Big\{1 + \frac{\ln(1-v_i)}{\ln(\frac{x_i^<}{\beta^{(t)}})}, 1 + \frac{\ln(1-v_i)}{\ln(\frac{1-x_i^>}{1-\beta^{(t)}})}, c\Big\}$$

Similarly, generate for $\alpha_2^{(t)}$.

Step 7: Using Equation (2.2), compute $R_{s,k}^{B(t)}$ at $(\alpha_1^{(t)}, \beta_1^{(t)}, \alpha_2^{(t)}, \beta_2^{(t)})$.

Step 8: Set $t = t + 1$.

Step 9: Repeat steps $2 - 8$, M times and obtain a posterior sample $(R_{s,k}^{B(t)} : t = 1, 2, \cdots, M)$.

Finally, the posterior mean under mean sqaured error, say $\hat{R}_{s,k}^B$,can be obtained as follows;

$$\hat{R}_{s,k}^B = \frac{1}{M}\sum_{t=1}^{M} R_{s,k}^{B(t)}$$

Using the method proposed by [5], we construct the $100(1-\gamma)\%$ highest posterior density (HPD) credible interval as

$$\Big(\hat{R}^B_{s,k[\frac{\gamma}{2}M]}, \hat{R}^B_{s,k[(1-\frac{\gamma}{2})M]}\Big)$$

where $[\frac{\gamma}{2}M]$ and $[(1-\frac{\gamma}{2})M]$ are the smallest integers less than or equal to $\frac{\gamma}{2}M$ and $(1-\frac{\gamma}{2})M$, respectively.

On Step 5 and Step 6, we used uniform priors which is useful in non-informative cases. Alternatively, a beta prior for β and truncated gamma for α may also choose as informative priors. Following, by using these conjugate priors, we obtained truncated beta and truncated gamma posterior distributions for parameters, respectively. However, truncated gamma prior does not give any numerical results for large samples size n. Çetinkaya and Genç [7] implied that this difficulty may be due to the special function existing in the cdf of the gamma distribution. Additionally, generating random variable from beta distribution in truncated cases is highly sensitive on near the borders. For these reasons, we used only non-informative priors as given in the algorithm above.

4. Simulation studies

In this section, we present some numerical results to compare the performances of the MLEs and Bayes estimates of the $R_{s,k}$ with their bootstrap and HPD credible intervals, respectively. The stress and the strength populations are generated as $(3, 2, 0.7, 0.4)$ and $(2, 2, 0.85, 0.15)$ for $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ with different sample sizes $(n, m) = (5, 10), (10, 20), (20, 30), (50, 50)$ and $(100, 100)$. The true values of reliability in multicomponent stress-strength with the given combinations for $(s, k) = (1, 3)$ are 0.865535 and 0.933949; and for $(s, k) = (2, 4)$ are 0.793819 and 0.876481. We generate 2000 samples for each sample size combinations in two cases. We compute the MLEs and Bayes estimates with their mean squared errors (MSE). We run the Gibbs sampler to generate a Markov chain with 3500 observations using the algorithm given in Section 3. The first 500 values is discarded as burn-in and we take every third variate as an independent and identically distributed observation in thinning procedure. Adjusted for autocorrelation in the chain by burning-in and thinning the chain, a sample of 1000 resulted which is used to calculate the posterior estimates. Then, the simulation is performed by Markov Chain Monte Carlo (MCMC) which is run for 2000 replicates. For the MLEs, we also obtain 95% confidence intervals based on parametric and nonparametric boot-p methods which are given in Section 2.1. We report the average bootstrap confidence intervals, confidence lengths based on 2000 replications. We used 200 bootstrap intervals for each replication. On the other hand, we obtained the 95% HPD credible intervals for the Bayes estimates. We use R program [25] to perform these simulation studies. The corresponding R codes are presented for readers in the Appendix. Then, all the results of this simulation scenarios appear in Table 4 and Table 5.

We observe that both MLEs and Bayes estimates approach the actual values of $R_{s,k}$ for large sample sizes and their MSE's decrease as the sample size increases, as expected. The MLEs overestimates and Bayes estimate underestimates the actual values. According to MSE values, for the large values of $R_{s,k}$, MLEs give smaller MSE for large samples and Bayes estimates for the small and moderate sample sizes. On the other hand, for the smaller values of $R_{s,k}$, Bayes estimates give smallest MSE values in all sample sizes. The HPD credible intervals are better in large samples than bootstrap confidence intervals. In the case of small and moderate sample sizes, nonparametric boot-p gives the shortest confidence intervals. Unexpectedly, HPD credible intervals do not perform better than boot-p confidence intervals unlike the claims in the literature. The main reason of this may be using a hierarchical model construction by using a latent or auxiliary variable. Unlike the known inference studies, the original form of the pdf cannot be used in this method. Difference due to using a latent or auxiliary variable may lead these results.

4.1. Real data example

In this subsection, the considered estimation methods are illustrated with two real data sets. The original data are from [20], frosted flakes data, and represent the sugar concentration for 25 g samples of a cereal as measured by two methods; high performance liquid chromatography (Data Set I) and a quick method using the infra-analyzer 400 (Data Set II). Whereas liquid chromatography is a slow accurate lab method, infra-analyzer 400 method provides quick measurements. In such a case, experimenters may consider that at least s out of k infra-analyzer 400 method measurements exceed measurements based on the liquid chromatography. Such a reliability problem concern about the probability of that the quick measurement method do not cause less measurement than accurate and slow method. Thus, let X denote the strength random variable which given in Data Set II and Y denote the stress random variable which given in Data Set I. The corresponding sample sizes are $n = m = 100$ for both. We applied min-max scaling method $(\frac{x-a}{b-a})$ on data sets. For both data sets, a denotes the floor integer of the range and b denotes ceiling integer of the range. We choose $a = 31, b = 44$ for the Data Set I and $a = 30, b = 46$ for the Data Set II, respectively. Thus, the points in the data sets all lie in the interval $(0, 1)$ and we present the transformed data sets in Table 1 and Table 2 for convenience.

We fit the STSP distributions to these datasets and use both MLE and Bayes estimation procedure. The MLEs are $\hat{\alpha}_1 = 2.493, \hat{\beta}_1 = 0.563, \hat{\alpha}_2 = 2.257$ and $\hat{\beta}_2 = 0.554$. Corresponding Kolmogorov-Simirnov goodness of fit test statistics and associated p-values are obtained as 0.09 and 0.8127 for X, 0.08 and 0.9062 for Y. The Bayes estimates are $\hat{\alpha}_1 = 2.487, \hat{\beta}_1 = 0.552, \hat{\alpha}_2 = 2.262$ and $\hat{\beta}_2 = 0.532$. Corresponding Kolmogorov-Simirnov goodness of fit test statistics and associated p-values are obtained as 0.1 and 0.6994 for X, 0.08 and 0.9062 for Y. Thus, we cannot reject the null hypotheses that these data sets come from the STSP distributions. Also, the Q-Q plots (Figs. 3 and 4), hazard plots (Figs. 5 and 6) and histograms with density curves (Figs. 7 and 8) support these observations. Consequently, the STSP models fit reasonably well to the transformed data sets.

Table 1. Transformed measurements using high performance liquid chromatography, ($m = 100$).

0.4077	0.1692	0.6154	0.4846	0.7462	0.5692	0.3692	0.3846	0.5308
0.6923	0.9615	0.4308	0.2077	0.7308	0.5923	0.4000	0.8154	0.5231
0.6077	0.3846	0.7769	0.6923	0.3462	0.2538	0.1538	0.4538	0.4077
0.3154	0.5923	0.2308	0.8692	0.8000	0.5538	0.5615	0.4923	0.5000
0.2923	0.5462	0.7692	0.3385	0.3077	0.5308	0.6231	0.1769	0.7231
0.1923	0.3692	0.4308	0.5615	0.1615	0.4923	0.6846	0.5692	0.3923
0.3154	0.4923	0.5538	0.6154	0.0462	0.3923	0.4308	0.4154	0.6846
0.6231	0.2846	0.4846	0.6231	0.6000	0.5923	0.4231	0.4846	0.6846
0.6462	0.4154	0.2923	0.7769	0.2154	0.8000	0.6231	0.5615	0.5154
0.7615	0.4231	0.7308	0.6692	0.7385	0.6077	0.6385	0.4692	0.3538
0.8923	0.7308	0.5769	0.4231	0.3000	0.2000	0.6000	0.2308	0.5692
0.4462								

Table 2. Transformed measurements based on the infra-analyzer 400, ($n = 100$).

0.3188	0.3687	0.6313	0.3438	0.4937	0.5938	0.5312	0.4937	0.7000
0.5187	0.7687	0.5625	0.0500	0.4562	0.5938	0.6437	0.7500	0.4312
0.5812	0.3500	0.6812	0.4750	0.3438	0.3438	0.1500	0.3813	0.3813
0.5312	0.6250	0.3375	0.6812	0.5875	0.5375	0.6000	0.5750	0.4000
0.5063	0.6000	0.6750	0.4625	0.4750	0.3750	0.4500	0.1875	0.7437
0.5063	0.4625	0.6313	0.6688	0.3375	0.7125	0.6375	0.5688	0.2938
0.4188	0.3937	0.5500	0.8125	0.2437	0.3500	0.3000	0.3625	0.7125
0.4312	0.2313	0.3813	0.6375	0.6187	0.4125	0.3625	0.5375	0.4688
0.6187	0.3438	0.3062	0.6688	0.0813	0.5938	0.4937	0.5438	0.3813
0.7938	0.6812	0.6250	0.7000	0.7188	0.7500	0.7875	0.6750	0.6375
0.9813	0.7000	0.5625	0.3813	0.493	0.2437	0.5750	0.2313	0.8000
0.6625								

Thus, the MLEs and Bayes estimates of $R_{s,k}$ for different choices of (s, k) are obtained with corresponding bootstrap and HPD confidence intervals and reported in Table 3. We calculate the % 95 bootstrap confidence intervals for R_{sk} based on 5000 replications for each of which 500 bootstrap intervals. Also, we perform the simulation algorithm given in Section 3 with the iteration number $B = 100,000$. We see that the parametric bootstrap method gives the shortest average length result. On the contrary, nonparametric bootstrap method gives the longest average length result.

Table 3. MLEs and Bayes estimates of $R_{s,k}$ based on the given real data, the average 95% bootstrap intervals and highest posterior density credible intervals (HPD CI) with their lengths for different choices of (s, k).

(s, k)	$(1, 3)$	$(2, 4)$
$\hat{R}_{s,k}$	0.747901	0.605892
$\hat{R}^B_{s,k}$	0.753345	0.613226
Parametric BCI	0.667421, 0.816491 (0.149070)	0.506669, 0.699124 (0.192455)
Non-Parametric BCI	0.657196, 0.835423 (0.178227)	0.489451, 0.730182. (0.240731)
Bayesian HPD CI	0.671665, 0.827000 (0.155335)	0.511690, 0.711954 (0.200264)

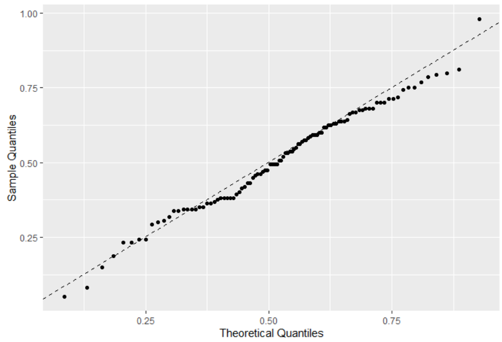

Figure 3. Q-Q plot of Data Set II.

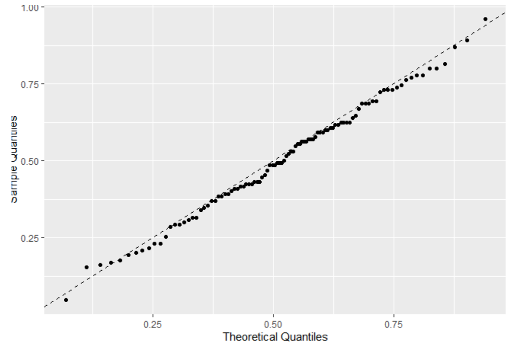

Figure 4. Q-Q plot of Data Set I.

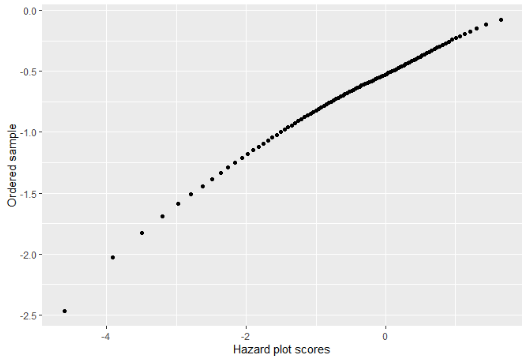

Figure 5. Hazard plot of Data Set II.

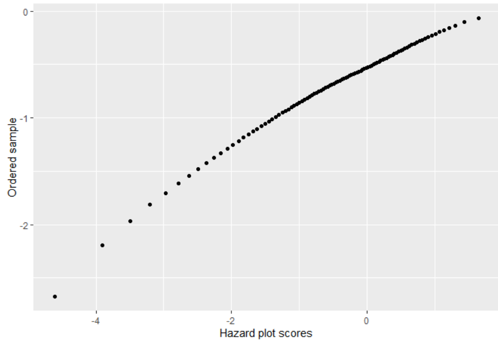

Figure 6. Hazard plot of Data Set I

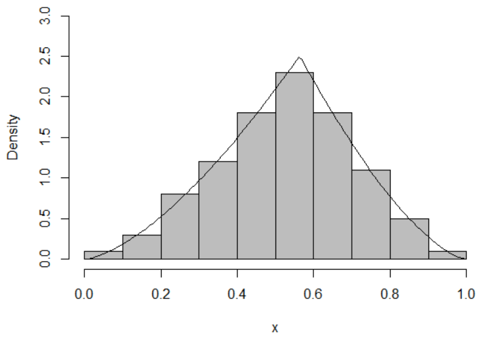

Figure 7. Data Set II and superimposed fit.

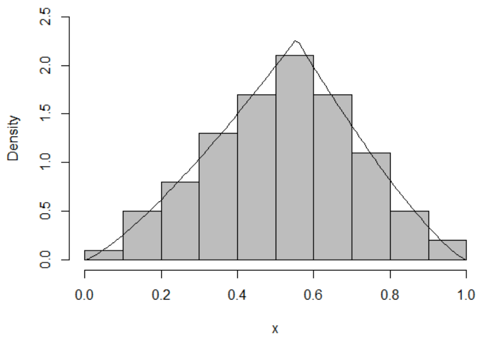

Figure 8. Data Set I and superimposed fit

5. Conclusions

In this paper, we have studied the multicomponent stress-strength reliability estimation based on the STSP distribution when both stress and strength random variables follow the same distribution. We considered the s-out-of-k:G system and we have compared the reliability estimations of such a system based on the MLE and Bayesian estimation procedures with respect to their mean squared errors and approximate confidence intervals. We compared the confidence intervals using boot-p confidence intervals of the MLEs and the highest posterior density credible intervals for the Bayes estimates. The simulation results show that the HPD credible intervals are better in large samples than bootstrap confidence intervals. In the case of small and moderate sample sizes, nonparametric boot-p gives the shortest confidence intervals. The MLE method overestimate and Bayes method underestimate the actual values. According to MSE values, for the large values of $R_{s,k}$, MLEs give smaller MSE for large samples and Bayes estimates for the small and moderate sample sizes. On the other hand, for the smaller values of $R_{s,k}$, Bayes estimates give the smallest MSE values in all sample sizes.

Table 4. Avarage estimates and corresponding mean squared errors (within paranthesis) of simulated MLEs and Bayes estimates of $R_{s,k}$, the average 95% bootstrap intervals and HPD CI with their lengths for different choises of (s,k) and (n,m) when $\alpha_1 = 3$, $\beta_1 = 0.7$, $\alpha_2 = 2$, $\beta_2 = 0.4$ where actual $R_{1,3} = 0.865535$ and $R_{2,4} = 0.793819$.

				Boot-p		Bayesian
(s,k)	(n,m)	$\hat{R}_{s,k}$	$\hat{R}^B_{s,k}$	Parametric	Nonparametric	HPD CI
(1,3)	5,10	0.882756	0.849905	(0.737311,0.982155)	(0.759215,0.982049)	(0.648389,0.912267)
		(0.005965)	(0.004352)	0.224843	0.222833	0.263879
	10,20	0.871194	0.854087	(0.761858,0.963208)	(0.777465,0.955518)	(0.596232,0.858642)
		(0.003645)	(0.002645)	0.201350	0.178052	0.262410
	20,30	0.866173	0.855601	(0.782365,0.950258)	(0.796253,0.945985)	(0.765190,0.941653)
		(0.002427)	(0.001798)	0.167893	0.149731	0.176463
	50,50	0.865686	0.860023	(0.795030,0.927910)	(0.803085,0.926303)	(0.778430,0.936704)
		(0.001402)	(0.001225)	0.132881	0.123218	0.158274
	100,100	0.865137	0.862425	(0.812179,0.908650)	(0.814684,0.909217)	(0.836752,0.913350)
		(0.000605)	(0.000625)	0.096471	0.094533	0.076799
(2,4)	5,10	0.831069	0.786600	(0.659510,0.973676)	(0.696075,0.968529)	(0.625819,0.920711)
		(0.011082)	(0.005886)	0.314166	0.272454	0.294893
	10,20	0.807798	0.783674	(0.683217,0.936728)	(0.698204,0.927716)	(0.557156,0.800152)
		(0.006590)	(0.004310)	0.253511	0.229513	0.242996
	20,30	0.799981	0.783549	(0.685837,0.910082)	(0.705141,0.901160)	(0.631234,0.901894)
		(0.004215)	(0.003272)	0.224246	0.196019	0.270660
	50,50	0.793580	0.788311	(0.707233,0.877018)	(0.713171,0.879028)	(0.715031,0.908621)
		(0.002304)	(0.002075)	0.169785	0.165856	0.193590
	100,100	0.793177	0.789792	(0.728364,0.855552)	(0.726781,0.850727)	(0.743813,0.859315)
		(0.001141)	(0.001071)	0.127187	0.123946	0.115502

Table 5. Avarage estimates and corresponding mean squared errors (within paranthesis) of simulated MLEs and Bayes estimates of $R_{s,k}$, the average 95% bootstrap intervals and HPD CI with their lengths for different choises of (s,k) and (n,m) when $\alpha_1 = 2$, $\beta_1 = 0.85$, $\alpha_2 = 2$, $\beta_2 = 0.15$ where actual $R_{1,3} = 0.933949$ and $R_{2,4} = 0.876481$.

				Boot-p		Bayesian
(s,k)	(n,m)	$\hat{R}_{s,k}$	$\hat{R}^B_{s,k}$	Parametric	Nonparametric	HPD CI
(1,3)	5,10	0.929858	0.891050	(0.766380,0.996325)	(0.783010,0.993631)	(0.752027,0.990607)
		(0.003976)	(0.005471)	0.229944	0.210622	0.238581
	10,20	0.930954	0.900902	(0.820047,0.988782)	(0.829812,0.986639)	(0.786813,0.973039)
		(0.002324)	(0.003267)	0.168734	0.156827	0.186225
	20,30	0.935575	0.913176	(0.851792,0.980834)	(0.856567,0.980087)	(0.854426,0.977148)
		(0.001170)	(0.001782)	0.129042	0.123520	0.122722
	50,50	0.934321	0.924147	(0.878626,0.972390)	(0.882668,0.970609)	(0.907999,0.979926)
		(0.000588)	(0.000806)	0.093765	0.087940	0.071926
	100,100	0.933493	0.930294	(0.898757,0.963150)	(0.897588,0.962862)	(0.917098,0.973513)
		(0.000302)	(0.000343)	0.064392	0.065273	0.056415
(2,4)	5,10	0.886827	0.827813	(0.681309,0.992165)	(0.693905,0.990573)	(0.612559,0.945347)
		(0.009017)	(0.009403)	0.310857	0.296669	0.332788
	10,20	0.880919	0.836645	(0.735747,0.978122)	(0.735593,0.975255)	(0.736839,0.967209)
		(0.005162)	(0.006299)	0.242375	0.239661	0.230371
	20,30	0.879047	0.847155	(0.764357,0.958965)	(0.767369,0.960321)	(0.683952,0.916676)
		(0.003081)	(0.004115)	0.194608	0.192953	0.232724
	50,50	0.877842	0.864185	(0.798070,0.942142)	(0.802498,0.944067)	(0.785530,0.936170)
		(0.001502)	(0.001773)	0.144072	0.141569	0.150640
	100,100	0.877498	0.871300	(0.825120,0.924853)	(0.825204,0.924873)	(0.801546,0.897265)
		(0.000677)	(0.000804)	0.099734	0.099669	0.095719

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Appendix

The following R [25] codes can be used for calculations.

```
library(zipfR) # Required R Package

a1<3;b1<0.7;a2<2;b2<0.4 # Initial parameter values
n<20;m<30;s<1;k<3 # Sample Sizes and System Type
h<250;B<200;M<3500;thinning<seq(503,M,3) # Iteration Numbers

rsk<function(a1,a2,b1,b2,s,k){ # Function of R_{s,k}
q1<a2/a1;q2<b2/b1;t1<0
for(i in s:k){
t2<0
for(j in 0:i){
t2[j+1]<choose(i,j)*(( 1)^j)*(b1^((1 a1)*(j+k i)))*
(pbeta(b1,a1*(k+j i)+1,a2)*beta(a1*(k+j i)+1,a2) pbeta(b2,a1*(k+j i)+1,a2)*
beta(a1*(k+j i)+1,a2))
}
t1[i]<choose(k,i)*q1*((((b1^(q1*(a1 1)))/(b2^(a2 1)))*
pbeta(b1*(q2^a1),k i+q1,i+1)*beta(k i+q1,i+1))+((1 b2)^(1 a2))*
(a1*sum(t2)+((1 b1)^((a1 1)*q1))*pbeta(1 b1,i+q1,k i+1)*beta(i+q1,k i+1)))
}
sum(t1[s:k])
}

r0<rsk(a1,a2,b1,b2,s,k)

fnk<function(xb){ # MLE's
zz<sort(xb);s<length(zz);t<0;k<matrix(0,1,s)
for(r in 1:s){
t1<zz/zz[r];t2< (1 zz)/(1 zz[r]);t1[r:s]<t2[r:s];t<t1;k[r]<prod(t)
}
r2<which.max(k);M_r<k[r2];alfa<( s)/log(M_r);beta<zz[r2]
c(alfa,beta)
}

rnd<function(as,b,ss){ # Function for generating random data
d<0;u<runif(ss)
for (i in 1:ss) {
```

```
if (u[i]<b) {
d[i] < ((u[i])^(1/as))*(b^(1 1/as))
}    else   {
d[i] < 1 ((1 u[i])^(1/as))*((1 b)^(1 1/as))
}
}
d
}


xm< matrix(0,h,n);ym< matrix(0,h,m)

for(j in 1:h){
xm[j,] < rnd(a1,b1,n)
ym[j,] < rnd(a2,b2,m)
}

par1 < matrix(0,h,2); par2 < matrix(0,h,2)

for(j in 1:h){
par1[j,] < fnk(xm[j,])
par2[j,] < fnk(ym[j,])
}

rlb < matrix(0,h,1)

for(j in 1:h){ # MLE of R
if(par2[j,2]<par1[j,2]){
rlb[j] < rsk(par1[j,1], par2[j,1], par1[j,2], par2[j,2], s, k)
}else{
rlb[j] < 2
}
}

#########PARAMETRIC BOOTSTRAP ##########################
prm1 < matrix(0, ncol=2,nrow=h); prm2 < matrix(0, ncol=2,nrow=h)
relboot1 < 0; npar < 0

for(i in 1:h){
prm1[i,] < fnk(xm[i,]); prm2[i,] < fnk(ym[i,])
j < 1
while(j<B+1){
xx < rnd(prm1[i,][1], prm1[i,][2], n)
yy < rnd(prm2[i,][1], prm2[i,][2],m)
xin < fnk(xx); yin < fnk(yy)
if(yin[2] < xin[2]){
relboot1[j] < rsk(xin[1], yin[1], xin[2], yin[2], s, k)
j=j+1
```

```
}else{
j=j
}
}
npar[i]<mean(relboot1);print(i)
}

CI_par<as.numeric(quantile(npar,c(0.025,0.975)))
Length_par<diff(CI_par)

#########NON PARAMETRIC BOOTSTRAP####################
relboot2<0;npar2<0

for( i in 1:h){
j<1
while(j<B+1){
xx<sample(xm[i,],n,replace=T)
yy<sample(ym[i,],m,replace=T)
xin<fnk(xx); yin<fnk(yy)
if(yin[2]<xin[2]){
relboot2[j]<rsk(xin[1],yin[1],xin[2],yin[2],s,k)
j=j+1
}else{
j=j
}
}
npar2[i]<mean(relboot2)
print(i)
}

CI_npar<as.numeric(quantile(npar2,c(0.025,0.975)));Length_npar<diff(CI_npar)

#########BAYESIAN ESTIMATIONS####################
relbay< 0;rlb_b<matrix(0,h,1000)

for(jj in 1:h){
t<2
alfa1<c(a1,rep(0,M 1));beta1<c(b1,rep(0,M 1))
alfa2<c(a2,rep(0,M 1));beta2<c(b2,rep(0,M 1))
x<sort(xm[jj,]);y<sort(ym[jj,])

while(t<M+1){
v1<0
for(j in 1:n){
xs<x[j]
v1[j]<runif(1,max(1 (xs/beta1[t 1])^(alfa1[t 1] 1),
1 ((1 xs)/(1 beta1[t 1]))^(alfa1[t 1] 1)),1)
}
```

```
v2 < 0
for (u in 1:m){
ys < y[u]
v2[u] < runif(1,max(1 (ys/beta2[t 1])^(alfa2[t 1] 1),
1 ((1 ys)/(1 beta2[t 1]))^(alfa2[t 1] 1)),1)
}

repeat {
k0 < runif(1,max(1 (1 x)/((1 v1)^(1/(alfa1[t 1] 1))),0),
min(x/((1 v1)^(1/(alfa1[t 1] 1))),1))
if((k0>min(x))&(k0<max(x))){
break
}
}
beta1[t] < k0
repeat {
l0 < runif(1,max(1 (1 y)/((1 v2)^(1/(alfa2[t 1] 1))),0),
min(y/((1 v2)^(1/(alfa2[t 1] 1))),1))
if((l0>min(y))&(l0<max(y))){
break
}
}
beta2[t] < l0

c < 100
qsa < max(which(x<beta1[t]))
be1 < min(1+log(1 v1[1:qsa])/log(x[1:qsa]/beta1[t]),
1+log(1 v1[(qsa+1):n])/log((1 x[(qsa+1):n])/(1 beta1[t])),c)
alfa1[t] < (runif(1)*((be1^(n+1)) 1)+1)^(1/(n+1))
qsb < max(which(y<beta2[t]))
be2 < min(1+log(1 v2[1:qsb])/log(y[1:qsb]/beta2[t]),
1+log(1 v2[(qsb+1):m])/log((1 y[(qsb+1):m])/(1 beta2[t])),c)
alfa2[t] < (runif(1)*((be2^(m+1)) 1)+1)^(1/(m+1))

if(beta2[t]<beta1[t]){
relbay[t] < rsk(alfa1[t],alfa2[t],beta1[t],beta2[t],s,k)
t=t+1
}else{
t=t
}
}
rlb_b[jj,] < relbay[thinning]
print(jj)
}

Rbys < 0;for(j in 1:h){Rbys[j] < mean(rlb_b[j,])}
HPD< matrix(0,h,2);for(i in 1:h){HPD[i,] < quantile(rlb_b[i,],c(0.025,0.975))}
```

```
CI_hpd < c(mean(HPD[ ,1]) ,mean(HPD[ ,2])); Length_hpd < diff(CI_hpd)

######PRINT RESULTS#########################
mean(rlb);mean(Rbys)
mean((rlb  r0)^2);mean((Rbys  r0)^2)
CI_par;CI_npar;CI_hpd
Length_par;Length_npar;Length_hpd
```