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Research Article

A Note on The Generalized k-Fibonacci Sequence

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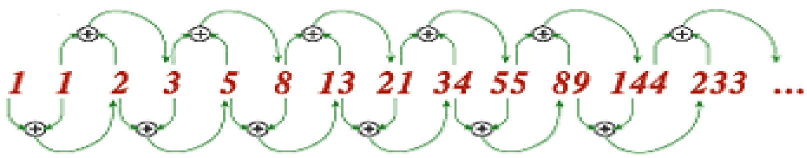
ABSTRACT: In this paper, we present a generalization of well-known k-Fibonacci sequence. Namely, we defined generalized k-Fibonacci sequence. This sequence generalizes others, k-Fibonacci sequence, classical Fibonacci sequence, Pell sequence and Jacobsthal sequence. We establish some of the interesting properties of generalized k-Fibonacci sequence. Also, we obtain a generating function for them.

Keywords: Generalized k-Fibonacci sequence, k-Fibonacci sequence, Fibonacci sequence, Binet's formula, Generating function.

1. INTRODUCTION

It is well-known that the Fibonacci sequence is most prominent examples of recursive sequence. The Fibonacci sequence is famous for possessing wonderful and amazing properties. The Fibonacci appear in numerous mathematical problems. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The book for which he is most famous in the "Liber abaci" published in 1202. In the third section of the book, he posed the equation of rabbit problem which is known as the first mathematical model for population growth. From the statement of rabbit problem, the famous Fibonacci numbers can be derived,



This sequence in which each number is the sum of the two preceding numbers has proved extremely fruitful and appears in different areas in Mathematics and Science.

The Fibonacci numbers F_n are terms of the sequence $\{0,1,1,2,3,5,...\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

The Fibonacci sequence [11], is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\ ,\ n \geq 2\ \ with\ F_0 = 0,\ F_1 = 1 \tag{1}$$

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The Lucas sequence [11],is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}\,,\ n \geq 2\ with\ L_0 = 2\,,\ L_1 = 1 \qquad (2)$$

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are the most prominent examples of recursive sequences. The second-order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [10], generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2}\,,\ n \geq 2 \quad with \quad F_0 = 0\,,\ F_1 = 1 \qquad (3)$$

Horadam [7], defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p\,,\ H_2 = p + q \qquad (4)$$

where p and q are arbitrary integers.

The k-Fibonacci numbers defined by Falcon and Plaza [2, 5], for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}\,,\ n \geq 2 \quad with \quad F_{k,0} = 0\,,\ F_{k,1} = 1 \qquad (5)$$

The k-Lucas numbers defined by Falcon [3],

$$L_{k,n} = k\,L_{k,n-1} + L_{k,n-2}\,,\ n \geq 2 \quad with \quad L_{k,0} = 2\,,\ L_{k,1} = k \qquad (6)$$

Most of the authors introduced Fibonacci pattern-based sequences in many ways which are known as Fibonacci-Like sequences and k-Fibonacci-like sequences [12, 13, 18, 20, 21].

Generalized Fibonacci sequence [6], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\,,\ k \geq 2 \quad with \quad F_0 = a\,,\ F_1 = b \qquad (7)$$

(p,q) - Fibonacci numbers [15], is defined as

$$F_{p,q,n} = pF_{p,q,n-1} + bF_{p,q,n-2}\,,\ n \geq 2 \quad with \quad F_{p,q,0} = 0\,,\ F_{p,q,n} = 1 \qquad (8)$$

(p,q) - Lucas numbers [16], is defined as

$$L_{p,q,n} = pL_{p,q,n-1} + bL_{p,q,n-2}\,,\ n \geq 2 \quad with \quad L_{p,q,0} = 2\,,\ L_{p,q,n} = p \qquad (9)$$

Generalized (p,q) -Fibonacci-Like sequence [17], is defined by recurrence relation

$$S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2}\,,\ n \geq 2 \quad with \quad S_{p,q,0} = 2k\,,\ S_{p,q,n} = 1 + kp \qquad (10)$$

Goksal Bilgici [1], defined new generalizations of Fibonacci and Lucas sequences

$$f_k = 2af_{k-1} + (b - a^2)f_{k-2}\ ,\ k \geq 2\ \ with\ \ f_0 = 0\,,\ f_1 = 1 \qquad (11)$$

$$l_k = 2al_{k-1} + (b - a^2)l_{k-2}\ ,\ k \geq 2\ \ with\ \ l_0 = 2\,,\ l_1 = 2a \qquad (12)$$

In this paper, we introduce a generalized $k-$Fibonacci sequence. The generalized $k-$Fibonacci numbers have lots of properties.

2. THE GENERALIZED k-FIBONACCI SEQUENCE

In this section, we define the generalized *k*-Fibonacci sequence and its particular cases.

Definition 1: Let k be any positive real number and p, q are positive integer. For $n \geq 2$, the generalized *k*-Fibonacci sequence $\{F_{k,n}\}$, is defined by

$$F_{k,n} = pkF_{k,n-1} + qF_{k,n-2} \tag{13}$$

with initial conditions $F_{k,0} = a$, $F_{k,1} = b$.

The first few generalized *k*-Fibonacci numbers are

$$F_{k,2} = pkb + aq$$

$$F_{k,3} = p^2k^2b + pkqa + qb$$

$$F_{k,4} = p^3k^3b + p^2k^2qa + 2pkqb + aq^2$$

$$F_{k,5} = p^4k^4b + p^3k^3qa + 3p^2k^2qb + 2pkaq^2 + bq^2$$

...

Particular cases of generalized *k*-Fibonacci sequence are

- If $a = 0$, $p = q = b = 1$, the *k*-Fibonacci sequence is obtained
  $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, for $n \geq 2$:
  $\{F_{k,n}\}_{n \in N} = \{0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, ...\}$
- If $a = 0$, $k = p = q = b = 1$, the classic Fibonacci sequence is obtained
  $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$:
  $\{F_n\}_{n \in N} = \{0, 1, 1, 2, 3, 5, 8, ...\}$
- If $a = 0$, $k = 2$, $p = q = b = 1$, the classic Pell sequence appears
  $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$:
  $\{P_n\}_{n \in N} = \{0, 1, 2, 5, 12, 29, 70, ...\}$
- If $a = 0$, $q = 2$, $p = k = b = 1$, the classic Jacobsthal sequence appears
  $J_0 = 0$, $J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$:
  $\{J_n\}_{n \in N} = \{0, 1, 1, 3, 5, 11, 21, ...\}$
- If $a = 0$, $k = 3$, $p = q = b = 1$, the following sequence appears
  $H_0 = 0$, $H_1 = 1$ and $H_n = 3H_{n-1} + H_{n-2}$, for $n \geq 2$:

  $\{H_n\}_{n \in N} = \{0, 1, 3, 10, 33, 109, ...\}$

3. PROPERTIES OF THE GENERALIZED *k*-FIBONACCI SEQUENCE

In this section, we introduce and prove some interesting properties of the generalized *k*-Fibonacci sequence.

3.1. First Explicit Formula for The Generalized *K*-Fibonacci Sequence

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet's formula allows us to express the generalized *k*-Fibonacci numbers in the function of the roots $\Re_1^n$ and $\Re_2^n$ of the following characteristic equation, associated with the recurrence relation (13):

$$x^2 = pkx + q \tag{14}$$

3.1.1. Binet's formula

Theorem 1: The nth generalized *k*-Fibonacci number is given by

$$F_{k,n} = A\Re_1^n + B\Re_2^n \tag{15}$$

where $\Re_1^n$ and $\Re_2^n$ are the roots of the characteristic equation (13) , $\Re_1 > \Re_2$ and

$$A = \frac{b - a\beta}{\sqrt{p^2k^2 + 4q}} \text{ and } B = \frac{a\alpha - b}{\sqrt{p^2k^2 + 4q}}.$$

Proof: The roots of the characteristic equation (13) are

$$\Re_1 = \frac{pk + \sqrt{p^2k^2 + 4q}}{2} \text{ and } \Re_2 = \frac{pk - \sqrt{p^2k^2 + 4q}}{2},$$

we use the Principle of Mathematical Induction (PMI) on n. The result is true for $n = 0$ *and* $n = 1$ by hypothesis. Assume that it is true for r such that $0 \le r \le s + 1$, then

$$F_{k,r} = A\Re_1^r + B\Re_2^r$$

It follows from the definition of generalized *k*-Fibonacci numbers and equation (15)

$$F_{k,s+2} = pkF_{k,s+1} + qF_{k,s} = A\Re_1^{s+2} + B\Re_2^{s+2}$$

Thus, the formula is true for any positive integer.

Particular cases are:

- If $a = 0$, $p = q = b = 1$, we obtained *k*-Fibonacci numbers and then $\sigma = \frac{k + \sqrt{k^2 + 4}}{2}$ is known as the *k*-metallic ratio.
- If $a = 0$, $k = p = q = b = 1$, we obtained classic Fibonacci numbers and then $\tau = \frac{1 + \sqrt{5}}{2}$ is well known as the golden ratio, τ is also denoted by α .
- If $a = 0$, $k = 2,$ $p = q = b = 1$, we obtained classic Pell numbers and then $\alpha = 1 + \sqrt{2}$ is well known as the silver ratio.

- If $a=0$, $k=3,\ p=q=b=1,$ for the sequence $\{H_n\}$, and then $\sigma=\dfrac{3+\sqrt{13}}{2}$ is known as the bronze ratio.

Proposition 2: For any integer $n \ge 1$,

$$\begin{aligned}\Re_1^{n+2} &= pk\Re_1^{n+1}+q\Re_1^{n}\\ \Re_2^{n+2} &= pk\Re_2^{n+1}+q\Re_2^{n}\end{aligned} \tag{16}$$

Proof: Since $\Re_1^n$ and $\Re_2^n$ are the roots of the characteristic equation (13), then

$\Re_1^2 = pk\Re_1+q$, $\Re_2^2 = pk\Re_2+q$

now, multiplying both sides of these equations by $\Re_1^n$ and $\Re_2^n$ respectively, we obtain the desired result.

Lemma 3: If r is a positive integer then $\dfrac{\Re_1^r-\Re_2^r}{\Re_1-\Re_2}=\dfrac{bF_{k,r}-aF_{k,r+1}}{b^2-a^2q-abpk}$ (17)

Proof: Using the Principle of Mathematical Induction (PMI) on n, the proof is clear.

Theorem 4: If $V=\left(\dfrac{\Re_1^n-\Re_2^n}{\Re_1-\Re_2}\right)^2-\left(\dfrac{\Re_1^{n-r}-\Re_2^{n-r}}{\Re_1-\Re_2}\right)\left(\dfrac{\Re_1^{n+r}-\Re_2^{n+r}}{\Re_1-\Re_2}\right)$, then

$$V=\left(-q\right)^{n-r}\left(\frac{bF_{k,r}-aF_{k,r+1}}{b^2-a^2q-abpk}\right)^2 \tag{18}$$

Proof: Using the roots of the characteristic equation (13), the proof is clear.

Proposition 5: $\left(A^2\Re_1^2+B^2\Re_2^2\right)\left(p^2k^2+4q\right)=\left(p^2k^2+2q\right)+2ab\left(aq+bpk\right)$ (19)

Proof: Since $\Re_1^n$ and $\Re_2^n$ are the roots of the characteristic equation (13) and

$A=\dfrac{b-a\beta}{\sqrt{p^2k^2+4q}}$ and $B=\dfrac{a\alpha-b}{\sqrt{p^2k^2+4q}}$, then

$$\left(A^2\Re_1^2+B^2\Re_2^2\right)=\frac{b^2\left(\Re_1^2+\Re_2^2\right)+2a^2\left(\Re_1\Re_2\right)^2+2ab\Re_1\Re_2\left(\Re_1+\Re_2\right)}{\left(p^2k^2+4q\right)}$$

$$\left(A^2\Re_1^2+B^2\Re_2^2\right)\left(p^2k^2+4q\right)=b^2\left\{pk\left(\Re_1+\Re_2\right)+2q\right\}+2a^2q+2abqpk$$

Finally, by simplifying the last expression, Eq. (19) is proven.

Theorem 6: If $T=\sum_{i=0}^{n}\dfrac{\Re_1^i-\Re_2^i}{\Re_1-\Re_2}$, then

$$T = \frac{1}{pk+q-1}\left(\frac{b\left(F_{k,n+1}+F_{k,n}\right)-a\left(F_{k,n+2}+F_{k,n}\right)}{b^2-a^2q-abpk}-1\right) \tag{20}$$

Proof: Since $\Re_1^n$ and $\Re_2^n$ are the roots of the characteristic equation (13), now by summing up the geometric partial sums $\sum_{i=0}^{n}\Re_j^i$ for $j=1,2$, we obtain

$$T = \frac{1}{\Re_1-\Re_2}\left(\frac{\Re_1^{n+1}-1}{\Re_1-1}-\frac{\Re_2^{n+1}-1}{\Re_2-1}\right)$$

$$= \frac{1}{pk+q-1}\left(\frac{\Re_1^{n+1}-\Re_2^{n+1}}{\Re_1-\Re_2}+\frac{\Re_1^{n}-\Re_2^{n}}{\Re_1-\Re_2}-1\right)$$

$$T = \frac{1}{pk+q-1}\left(\frac{b\left(F_{k,n+1}+F_{k,n}\right)-a\left(F_{k,n+2}+F_{k,n}\right)}{b^2-a^2q-abpk}-1\right)$$

This completes the proof.

3.1.2. Limit of the quotient of two consecutive terms

A useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation.

Proposition 7: $\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \Re_1$ (21)

Proof: Using Eq. (13), $\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \lim_{n\to\infty}\frac{A\Re_1^r+B\Re_2^r}{A\Re_1^r+B\Re_2^r} = \lim_{n\to\infty}\frac{1+\frac{B}{A}\left(\frac{\Re_2}{\Re_1}\right)^n}{\frac{1}{\Re_1}+\frac{B}{A}\left(\frac{\Re_2}{\Re_1}\right)^n\frac{1}{\Re_2}}$

and taking into account that $\lim_{n\to\infty}\left(\frac{\Re_2}{\Re_1}\right)^n = 0$, since $\left|\Re_2\right| < \Re_1$, Eq. (21) is obtained.

Particular cases are:

- If $a=0$, $k=p=q=b=1$, we obtained, $\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \tau$.
- If $a=0$, $k=2,\ p=q=b=1$, we obtained $\lim_{n\to\infty}\frac{P_n}{P_{n-1}} = \alpha$.
- If $a=0$, $k=3,\ p=q=b=1$, we obtained $\lim_{n\to\infty}\frac{H_n}{H_{n-1}} = \sigma$.

3.1.3. Catalan's identity

Catalan's identity for Fibonacci numbers was founded in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory.

Proposition 8: (Catalan's identity) $F_{k,n}^2 - F_{k,n+r}F_{k,n-r} = (-q)^{n-r}\dfrac{(bF_{k,n} - aF_{k,r+1})^2}{(b^2 - a^2q - abpk)}$ (22)

Proof: By using Eq. (13) in the left-hand side (LHS) of Eq. (22), and taking into account that $\Re_1\Re_2 = -q$ it is obtained

$$\begin{aligned}
(\text{LHS}) &= \left(A\Re_1^n + B\Re_2^n\right)^2 - \left(A\Re_1^{n+r} + B\Re_2^{n+r}\right)\left(A\Re_1^{n-r} + B\Re_2^{n-r}\right) \\
&= AB\left(\Re_1\Re_2\right)^n\left(2 - \Re_1^r\Re_2^{-r} - \Re_1^{-r}\Re_2^r\right) \\
&= AB(-q)^n\left\{2 - \left(\frac{\Re_1^r}{\Re_2^r}\right) - \left(\frac{\Re_2^r}{\Re_1^r}\right)\right\} \\
&= (-AB)(-q)^n\frac{\left(\Re_1^r - \Re_2^r\right)^2}{(-q)^r} \\
&= (b^2 - a^2q - abpk)(-q)^{n-r}\left(\frac{\Re_1^r - \Re_2^r}{\Re_1 - \Re_2}\right)^2
\end{aligned}$$

Finally, by using Eq. (17),

$$= (-q)^{n-r}\frac{(bF_{k,n} - aF_{k,r+1})^2}{(b^2 - a^2q - abpk)}$$

This completes the proof.

3.1.4. Cassini's identity

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer.

Proposition 9: (Cassini's identity or Simpson's identity)

$$F_{k,n}^2 - F_{k,n+1}F_{k,n-1} = (-q)^{n-1}(b^2 - a^2q - abpk) \tag{23}$$

Proof: Taking $r = 1$ in Catalan's identity the proof is completed.

3.1.5. d'ocagnes's identity

Proposition 10: (d'ocagnes's Identity)

$$F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-q)^{n-1}(bF_{k,m-n} - aF_{k,m-n+1}) \tag{24}$$

where $n \leq m$ integers.

Proof: Using the Binet's formula, the proof is clear.

- If $a=0$, $k=p=q=b=1$, we obtained d'ocagnes's Identity for classic Fibonacci numbers, $F_m F_{n+1} - F_{m+1} F_n = (-1)^{n-1} F_{m-n}$.

3.2. Generalized İdentity for Generalized *K*-Fibonacci Sequence

In this section, we present generalized identity for generalized *k*-Fibonacci sequence, from which we obtain Catlan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 11: If $Y = F_{k,m} F_{k,n} - F_{k,m-r} F_{k,m+r}$ and $F_{k,n}$ be the generalized k-Fibonacci numbers,

then $$Y = (-q)^{m-r} \frac{\left(bF_{k,r} - aF_{k,r+1}\right)\left(bF_{k,n+r-m} - aF_{k,n+r-m+1}\right)}{\left(b^2 - a^2 q - abpk\right)} \tag{25}$$

where n, m, r nonnegative integers.

Proof: Using the Binet's formula, the proof is clear.

Eq. (25), is generalized of Catalan's, Cassini's and d'ocagnes's identities.

- If $a=0,\ k=p=q=b=1,$ we have $F_m F_n - F_{m-r} F_{n+r} = (-1)^{m-r} F_r F_{n+r-m}$

which is given for Fibonacci numbers by Spivey in [14].

3.3. A Second Formula for The Generalized K-Fibonacci Sequence in Terms of Their Characteristic Roots

Theorem 12: $$\Re_1^n - \Re_2^n = \frac{1}{2^n - 1} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} (pk)^{n-1-2i} \left(p^2k^2 + 4q\right)^i \tag{26}$$

where $\lfloor a \rfloor$ is the floor function of a that is $\lfloor a \rfloor = \sup.\{n \in N \mid n \le a\}$ and says the integer part of a, for $a \ge 0$.

Proof: Since $\Re_1^n$ and $\Re_2^n$ are the roots of the characteristic equation $x^2 - pkx - q = 0$, using the value of $\Re_1^n$ and $\Re_2^n$, we get $\Re_1^n - \Re_2^n = \left[\left(\frac{pk + \sqrt{p^2k^2 + 4q}}{2}\right)^n - \left(\frac{pk - \sqrt{p^2k^2 + 4q}}{2}\right)^n\right]$

from where, by developing the nth powers, it follows:

$$\Re_1^n - \Re_2^n = \frac{1}{2^n - 1} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} (pk)^{n-1-2i} \left(p^2k^2 + 4q\right)^i$$

Particular cases are:

- If $p=q=1$, for the classical Fibonacci sequence, we have

  $$\Re_1^n - \Re_2^n = \frac{1}{2^n - 1} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} (5)^i$$

- If $k=2,\ p=q=1$, for the Pell sequence, we have

$$\Re_1^n-\Re_2^n=\frac{1}{2^n-1}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}(2)^{n-1-2i}(8)^i=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}(2)^i$$

- If $k=3,\ p=q=1$, for the sequence $\{H_n\}$, we have

$$\Re_1^n-\Re_2^n=\frac{1}{2^n-1}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}(3)^{n-1-2i}(13)^i=\left(\frac{3}{2}\right)^{n-1}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}\left(\frac{13}{9}\right)^i$$

4. SUM OF FIRST TERMS OF THE GENERALIZED *k*-FIBONACCI SEQUENCE

Theorem 13: Let $F_{k,n}$ be the nth generalized *k*-Fibonacci number then

$$\sum_{s=1}^{n}F_{k,s}=\frac{F_{k,n}+qF_{k,n-1}+apk-a-b}{pk+q-1} \tag{27}$$

Proof: Using the Binet's formula for the generalized *k*-Fibonacci numbers,

$$\sum_{s=1}^{n}F_{k,s}=\sum_{s=1}^{n}A\Re_1^s+B\Re_2^s$$

$$=A\sum_{s=1}^{n}\Re_1^s+B\sum_{s=1}^{n}\Re_2^s$$

$$=A\left(\frac{1-\Re_1^s}{1-\Re_1}\right)+B\left(\frac{1-\Re_2^s}{1-\Re_2}\right)$$

$$=\frac{(A+B)-(A\Re_1^s+B\Re_2^s)-(A\Re_2+B\Re_1)+\Re_1\Re_2(A\Re_1^{s-1}+B\Re_2^{s-1})}{(1-\Re_1)(1-\Re_2)}$$

$$=\frac{F_{k,n}+qF_{k,n-1}+apk-a-b}{pk+q-1}$$

Particular cases are:

- If $a=0$, $k=p=q=b=1$, for the classic Fibonacci sequence, we have:
  $\sum_{s=1}^{n}F_s=F_{n+2}-1$.
- If $a=0$, $k=2, p=q=b=1$, for the Pell sequence, we have:
  $\sum_{s=1}^{n}P_s=\frac{1}{2}(P_{n+1}+P_n-1)$.
- If $a=0$, $k=3, p=q=b=1$, the sum of the first elements of the sequence $\{H_n\}$ is:
  $\sum_{s=1}^{n}H_s=\frac{1}{3}(H_{n+1}+H_n-1)$

5. GENERATING FUNCTION FOR GENERALIZED *k*-FIBONACCI SEQUENCE

The function is $F(x)=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n+...$ called the generating function for the sequence $\{a_0,a_1,a_2,...\}$. Generating functions provide a powerful tool for solving linear recurrence relations with constant coefficients.

Let $$F_{k,n}(x)=\frac{a+x(b-apk)}{1-pkx-qx^2} \tag{28}$$

Particular cases are:

- If $a=0$, $p=q=b=1$, generating function of the *k*- Fibonacci sequence:
  $F_{k,n}(x)=\dfrac{x}{1-kx-x^2}$.
- If $a=0$, $k=p=q=b=1$, generating function of the classic Fibonacci sequence:
  $F_n(x)=\dfrac{x}{1-x-x^2}$.
- If $a=0$, $k=2, p=q=b=1$, generating function of the Pell sequence:
  $P_n(x)=\dfrac{x}{1-2x-x^2}$.
- If $a=0$, $k=3, p=q=b=1$, generating function of the sequence $\{H_n\}$:
  $H_n(x)=\dfrac{x}{1-3x-x^2}$.

6. CONCLUSIONS

In this study new generalized *k*-Fibonacci sequences have been introduced and studied. Many of the properties of these sequences are proved by simple algebra. Compactly and directly many formulas of such numbers have been deduced.

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