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New Kind Frenet Curves in Minkowski Space

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Abstract: In this study, we define new vector fields along a Frenet curve with nonvanishing curvatures in 4-dimensional Minkowski space R_1^4. By using these vector fields we obtain some new planes and curves. We show that these planes play the role of the Darboux vector. We characterized that, osculating curves of the first kind and rectifying curves in Minkowski space R_1^4 can be given as space curves whose position vectors always lie in a two-dimensional subspace.

Keywords: Rectifying curve, Frenet curve, Darboux vector.

1. Introduction

The classical differential geometry of curves has been studied by several authors. İlarslan and Boyacıoğlu studied position vectors of a timelike and a null helice in R_1^3 [5]. İlarslan and Nesovic gave the necessary and sufficient conditions for null curves in E_1^4 to be osculating curves in terms of their curvature functions [6].

İlarslan, Nesovic and Petrovic-Torgasev characterized rectifying curves in R_1^3 [7]. Ali and Önder characterized rectifying spacelike curves with curvature functions in Minkowski spacetime [1]. Keleş, Perktaş and Kılıç studied Biharmonic Curves in LP-Sasakian manifolds [8].

Vector fields have always been used for characterizing differential geometry of curves and surfaces in 3-dimensional and higher dimensional spaces. Natural vector fields in space, Frenet vector fields along curves and the Darboux vector field of a curve in 3-dimensional and 4-dimensional spaces are well known. These vector fields determine most geometric properties of curves and spaces. Frenet vector fields along a curve constitute an orthonormal frame. This frame is called the Frenet frame and it includes all the information about the curve. The rate of change of the Frenet frame is given by Frenet formulas. These formulas can be rewritten as vector products by means of the Darboux vector field which determines the instantaneous axis of rotation

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of Frenet frame. Therefore, the Darboux vector field plays an important role for space curves in 3-dimensional and 4-dimensional spaces.

In the literature, we can find a generalized Darboux vector in E^n and as a special case in E^4 [2]. Izumiya and Takeuchi [4] defined new special curves in Euclidean 3-space which they called slant helices and conical geodesic curves. Yaylı, Gök and Hacısalihoğlu [12] gave some relations between non-helical extended rectifying curves and their Darboux vector fields using any orthonormal frame along the curves.

Along a space curve with nonvanishing curvatures in E^4 Düldül introduced four special vector fields. Later, by using the introduced vector fields, he defined some new planes and curves. Düldül showed that the determined new planes play the role of the Darboux vector [3].

This paper is organized as follows: In the second section we give basic notions for curves and vector fields along Frenet curves. We give main results, theorems and corollaries for new kind Frenet curves in the third section.

2. Preliminaries

The Minkowski space R_1^4 is the standart vector space equipped with an indefinite flat metric g given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of R_1^4. A vector v in R_1^4 is called a spacelike, timelike or null (lightlike) if respectively hold $g(v,v) > 0$, $g(v,v) < 0$ or $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$ [7].

An arbitrary curve $\alpha : I \to R_1^4$ can locally be spacelike, timelike or null if respectively all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null.

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame along the curve $\alpha(s)$ in R_1^4. Then the vector fields T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively.

Let α be a spacelike curve in R_1^4, parametrized by arc length function of s. The following cases occur for the spacelike curve α [10].

Case I: Let the vector N be spacelike and B_1 be timelike. In this case there exists only one Frenet frame $\{T, N, B_1, B_2\}$ for which $\alpha(s)$ is a spacelike curve with Frenet equations

$$\begin{aligned} \nabla_T T &= k_1 N \\ \nabla_T N &= -k_1 T + k_2 B_1 \\ \nabla_T B_1 &= k_2 N + k_3 B_2 \\ \nabla_T B_2 &= k_3 B_1 \end{aligned} \tag{2}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying the equations

$$g(N,N) = g(T,T) = g(B_2,B_2) = 1, \quad g(B_1,B_1) = -1 \tag{3}$$

Case II: Let the vector N be timelike. In this case there exists only one Frenet frame $\{T, N, B_1, B_2\}$ for which $\alpha(s)$ is a spacelike curve with Frenet equations

$$\begin{aligned} \nabla_T T &= k_1 N \\ \nabla_T N &= k_1 T + k_2 B_1 \\ \nabla_T B_1 &= k_2 N + k_3 B_2 \\ \nabla_T B_2 &= -k_3 B_1 \end{aligned} \tag{4}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1, \quad g(N,N) = -1 \tag{5}$$

Recall that the functions $k_1 = k_1(s)$, $k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called the first, the second and the third curvature of the spacelike curve $\alpha(s)$, respectively and we will assume throughout this work that all the three curvatures satisfy $k_i(s) \neq 0$, $1 \leq i \leq 3$.

Definition 2.1 *[7] Let γ be a Frenet curve in R_1^4. γ is called as a rectifying curve if its position vector lies always in the orthogonal complement of its principal normal vector field.*

Definition 2.2 *[9] Let γ be a Frenet curve in R_1^4. γ is called as an oscullating curve of first kind if its position vector lies always in the orthogonal complement of its first binormal vector field.*

Definition 2.3 *[11] Let (e_1, e_2, e_3, e_4) be the standart basis of R_1^4. The vector*

$$X \otimes Y \otimes Z = \begin{vmatrix} e_1 & e_2 & e_3 & -e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{6}$$

is called the ternary product of the vectors $X = \sum_{i=1}^4 x_i e_i$, $Y = \sum_{i=1}^4 y_i e_i$ and $Z = \sum_{i=1}^4 z_i e_i$.

3. Main Results

In this section, we define some new special vector fields along a regular curve in R_1^4. By using these vector fields we obtain some characterizations for new Frenet curves. Moreover, we use these characterizations on these curves to be rectifying curves and osculating curve of the first kind.

Let γ be a unit speed curve as given in the Case I and $\{T, N, B_1, B_2\}$ be the Frenet frame along the curve γ where the curvatures k_1, k_2, k_3 are non-zero everywhere. Now we can define following vector fields along γ:

$$\begin{aligned} D_1 &= B_2 \\ D_2 &= k_2 T + k_1 B_1 \\ D_3 &= k_3 N + k_2 B_2 \\ D_4 &= -T \end{aligned} \tag{7}$$

where $\{D_1, D_2, D_3, D_4\}$ is linearly independent along γ. Furthermore, we see that the spaces $\{D_1, D_2\}$, $\{D_2, D_3\}$ and $\{D_3, D_4\}$ are ortogonal spaces. We call that $Sp\{D_1, D_2\}$, $Sp\{D_2, D_3\}$ and $Sp\{D_3, D_4\}$ as D_1D_2-plane, D_2D_3-plane and D_3D_4-plane, respectively. So, we obtain the new Frenet equations as

$$\begin{aligned} \nabla_T T &= D_1 \otimes D_2 \otimes T \\ \nabla_T N &= D_1 \otimes D_2 \otimes N \\ \nabla_T B_1 &= D_3 \otimes D_4 \otimes B_1 \\ \nabla_T B_2 &= D_3 \otimes D_4 \otimes B_2. \end{aligned} \tag{8}$$

We see that the vector fields T and N rotate around the D_1D_2-plane, and the vector fields B_1 and B_2 rotate around the D_3D_4-plane. These two planes play the role that the Darboux vector d plays in 3-dimensional space. If the position vector of a space curve always lie in its D_1D_2-plane (D_2D_3-plane, D_3D_4-plane), then we call such a curve as D_1D_2-curve (D_2D_3-curve, D_3D_4-curve).

From the above literature we can give the following theorems for such curves.

Theorem 3.1 *Let γ be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4. If γ is a D_1D_2-curve in R_1^4, then it satisfies one of the following forms*

$$\gamma(s) = \frac{1}{k_2(s)}(s+c)D_2 \tag{9}$$

where c is a constant and $k_1(s) = 0$ or

$$\gamma(s) = c_1 D_1 + \frac{1}{k_2(s)}(s+c_2)D_2 \tag{10}$$

where c_1, c_2 are constants and $k_1(s) = k_3(s) = 0$.

Proof Let γ be a D_1D_2-curve with nonvanishing curvatures in R_1^4. From the definition of D_1D_2-curve, we have

$$\gamma(s) = v(s)D_1(s) + w(s)D_2(s) \tag{11}$$

for some differentiable functions $v(s)$ and $w(s)$. If we differentiate (11) according to s and use the Frenet equations of the curve γ, then we obtain

$$\begin{cases} (w(s)k_2(s))' = 1 \\ 2w(s)k_1(s)k_2(s) = 0 \\ (w(s)k_1(s))' + v(s)k_3(s) = 0 \\ v'(s) + w(s)k_1(s)k_3(s) = 0. \end{cases} \tag{12}$$

From the first equation of (12), we find $w(s) = \frac{s+c}{k_2(s)}$ and if we use the second equation, then we have $k_1(s) = 0$. Then from the third equation, we have $v(s) = 0$ or $k_3(s) = 0$. If $v(s) = 0$, then the position vector of the curve $\gamma(s)$ can be defined as

$$\gamma(s) = \frac{1}{k_2(s)}(s+c)D_2 \tag{13}$$

where c is a constant. If $k_3(s) = 0$, then the position vector of the curve $\gamma(s)$ can be defined as

$$\gamma(s) = c_1D_1 + \frac{1}{k_2(s)}(s+c_2)D_2 \tag{14}$$

where c_1, c_2 are constants. □

Corollary 3.2 *Let γ be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4 with non-zero curvature functions. Then $\gamma(s)$ is a D_1D_2-curve if and only if $\gamma(s)$ is a rectifying curve in R_1^4.*

Corollary 3.3 *If we consider Definition 2.1, the position vector of a rectifying curve in R_1^4 always lies in the subspace $Sp\{T, B_1, B_2\}$. However from Corollary 3.2 a rectifying curve in R_1^4 can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called D_1D_2-plane.*

Theorem 3.4 *Let γ be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4. γ is a D_3D_4-curve in R_1^4 if and only if the non-zero curvature functions k_1, k_2, k_3 satisfy the equation*

$$c\left[\frac{1}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)'\right]' + c\frac{k_1(s)k_3(s)}{k_2(s)} - 1 = 0 \tag{15}$$

where c is a constant.

Proof Let γ be a D_3D_4-curve with nonvanishing curvatures in R_1^4. From the definition of D_3D_4-curve, we have

$$\gamma(s) = \lambda(s)D_3(s) + \mu(s)D_4(s) \tag{16}$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$. If we differentiate (16) according to s and use the Frenet equations of the curve γ, then we obtain

$$\begin{cases} \lambda(s)k_1(s)k_3(s) - \mu'(s) = 1 \\ (\lambda(s)k_2(s))' = 0 \\ (\lambda(s)k_3(s))' + \mu(s)k_1(s) = 0. \end{cases} \tag{17}$$

From the second equation of (17), we find $\lambda(s) = \frac{c}{k_2(s)}$ and if we use the third equation, then we obtain $\mu(s) = -\frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)'$. Then the position vector of the curve $\gamma(s)$ can be defined as

$$\gamma(s) = \frac{c}{k_2(s)}D_3(s) - \frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)' D_4(s). \tag{18}$$

If we use $\lambda(s)$ and $\mu(s)$ in the first equation of (17), then we get (15).

Conversely, we assume that (15) holds. Let us consider the vector given

$$Z(s) = \gamma(s) - \frac{c}{k_2(s)}D_3(s) + \frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)' D_4(s). \tag{19}$$

Differentiating vector Z and considering the equation of (15), we obtain

$$\frac{dZ}{ds} = 0. \tag{20}$$

Thus Z is a constant vector and so, the curve $\gamma(s)$ is congruent to a D_3D_4-curve. □

Corollary 3.5 *Let γ be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4 with non-zero curvature functions. Then $\gamma(s)$ is a D_3D_4-curve if and only if $\gamma(s)$ is an osculating curve of the first kind in R_1^4.*

Corollary 3.6 *If we consider Definition 2.2, the position vector of an oscullating curve of the first kind in R_1^4 always lies in the subspace $Sp\{T, N, B_2\}$. However from Corollary 3.5 an oscullating curve of the first kind in R_1^4 can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called D_3D_4-plane.*

Corollary 3.7 *If we consider the equation (15) and substitute*

$$\nu = \frac{k_3(s)}{k_2(s)} \tag{21}$$

by using exchange variable $t = \int k_1(s)ds = h(s)$ *, then we find*

$$\frac{d^2\nu}{dt^2} + \nu = f(t) \tag{22}$$

where $f(t) = \frac{1}{ck_1(h^{-1}(t))}$ *. The general solution of the differential equation is*

$$\nu(s) = \left(c_1 + \int f(t)\sin t dt\right)\cos t + \left(c_2 - \int f(t)\cos t dt\right)\sin t \tag{23}$$

where c_1, c_2 *are constants. Thus the solution of the equation (22) is obtained as*

$$\frac{k_3(s)}{k_2(s)} = \left(c_1 - \frac{1}{c}\int \sin(\int k_1(s)ds)ds\right)\cos(\int k_1(s)ds) \tag{24}$$

$$+\left(c_2 + \frac{1}{c}\int \cos(\int k_1(s)ds)ds\right)\sin(\int k_1(s)ds).$$

Theorem 3.8 *Let* γ *be a spacelike Frenet curve parametrized by the arc length parameter* s *in* R_1^4*.* γ *is a* D_2D_3*-curve in* R_1^4 *if and only if the non-zero curvature functions* k_1, k_2, k_3 *satisfy the equation*

$$c\left(\frac{k_2(s)}{k_1(s)}\right)' + \left[\frac{ck_3^2(s) + 2k_2^2(s)}{k_2(s)k_3'(s) - k_2'(s)k_3(s)}\right]k_1(s)k_3(s) - 1 = 0 \tag{25}$$

where c *is a constant.*

Proof Let γ be a D_2D_3-curve with nonvanishing curvatures in R_1^4. From the definition of D_2D_3-curve, we have

$$\gamma(s) = \pi(s)D_2(s) + \rho(s)D_3(s) \tag{26}$$

for some differentiable functions $\pi(s)$ and $\rho(s)$. If we differentiate (26) according to s and use the Frenet equations of the curve γ, then we obtain

$$\begin{cases} \rho(s)k_1(s)k_3(s) + (\pi(s)k_2(s))' = 1 \\ (\pi(s)k_1(s))' = 0 \\ 2\pi(s)k_1(s)k_2(s) - (\rho(s)k_3(s))' = 0 \\ \pi(s)k_1(s)k_3(s) + (\rho(s)k_2(s))' = 0. \end{cases} \tag{27}$$

From the second equation of (27), we find $\pi(s) = \frac{c}{k_1(s)}$ and if we use the third equation, then we obtain

$$\rho(s) = \frac{ck_3^2(s) + 2k_2^2(s)}{k_2(s)k_3'(s) - k_2'(s)k_3(s)}. \tag{28}$$

Then the position vector of the curve $\gamma(s)$ can be defined as

$$\gamma(s) = \frac{c}{k_1(s)}D_2(s) + \left(\frac{ck_3^2(s) + 2k_2^2(s)}{k_2(s)k_3'(s) - k_2'(s)k_3(s)}\right)D_3(s). \tag{29}$$

If we use $\pi(s)$ and $\rho(s)$ in the first equation of (27), then we get (25).

Conversely, we assume that (25) holds. Let us consider the vector given

$$Y(s) = \gamma(s) - \frac{c}{k_1(s)} D_2(s) - \left(\frac{c k_3^2(s) + 2k_2^2(s)}{k_2(s)k_3'(s) - k_2'(s)k_3(s)} \right) D_3(s). \tag{30}$$

Differentiating vector Y and considering the equation of (25), we obtain

$$\frac{dY}{ds} = 0. \tag{31}$$

Thus Y is a constant vector and so, the curve $\gamma(s)$ is congruent to a D_2D_3-curve. □

Now let β be a unit speed curve as given in the Case II and $\{T, N, B_1, B_2\}$ be the Frenet frame along the curve β where the curvatures k_1, k_2, k_3 are non-zero everywhere. Now we can define following vector fields along β:

$$\begin{aligned} D_1 &= B_2 \\ D_2 &= k_2 T - k_1 B_1 \\ D_3 &= k_3 N - k_2 B_2 \\ D_4 &= T \end{aligned} \tag{32}$$

where $\{D_1, D_2, D_3, D_4\}$ is linearly independent along β. If we use the initial literature of this section and new Frenet equations in (8), then we obtain the following theorems:

Theorem 3.9 *Let β be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4. β is a D_1D_2-curve in R_1^4 if and only if the non-zero curvature functions k_1, k_2, k_3 satisfy the equation*

$$\left[-\frac{1}{k_3(s)} \left((s+c) \frac{k_1(s)}{k_2(s)} \right)' \right]' - \frac{k_1(s)k_3(s)}{k_2(s)} (s+c) = 0 \tag{33}$$

where c is a constant.

Proof Let β be a D_1D_2-curve with nonvanishing curvatures in R_1^4. From the definition of D_1D_2-curve, then we have

$$\beta(s) = v(s)D_1(s) + w(s)D_2(s) \tag{34}$$

for some differentiable functions $v(s)$ and $w(s)$. If we differentiate (34) according to s and use the Frenet equations of the curve β, then we obtain

$$\begin{cases} (w(s)k_2(s))' = 1 \\ (w(s)k_1(s))' + v(s)k_3(s) = 0 \\ v'(s) - w(s)k_1(s)k_3(s) = 0. \end{cases} \tag{35}$$

From the first equation of (35), we find $w(s) = \frac{s+c}{k_2(s)}$ and if we use the second equation, then we obtain $v(s) = -\frac{1}{k_3(s)}\left((s+c)\frac{k_1(s)}{k_2(s)}\right)'$. Then the position vector of the curve $\beta(s)$ can be defined as

$$\beta(s) = \left[-\frac{1}{k_3(s)}\left((s+c)\frac{k_1(s)}{k_2(s)}\right)'\right]D_1(s) + \left[\frac{s+c}{k_2(s)}\right]D_2(s). \tag{36}$$

If we use $v(s)$ and $w(s)$ in the third equation of (35), then we get (33).

Conversely, we assume that (33) holds. Let us consider the vector given

$$U(s) = \beta(s) + \left[\frac{1}{k_3(s)}\left((s+c)\frac{k_1(s)}{k_2(s)}\right)'\right]D_1(s) - \left[\frac{s+c}{k_2(s)}\right]D_2(s). \tag{37}$$

Differentiating vector U and considering the equation of (33), we obtain

$$\frac{dU}{ds} = 0. \tag{38}$$

Thus U is a constant vector and so, the curve $\beta(s)$ is congruent to a D_1D_2-curve. □

Corollary 3.10 *Let β be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4 with non-zero curvature functions. Then $\beta(s)$ is a D_1D_2-curve if and only if $\beta(s)$ is a rectifying curve in R_1^4.*

Corollary 3.11 *If we consider Definition 2.1, the position vector of a rectifying curve in R_1^4 always lies in the subspace $Sp\{T, B_1, B_2\}$. However from Corollary 3.10 a rectifying curve in R_1^4 can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called D_1D_2-plane.*

Corollary 3.12 *If we consider the equation (33) and substitute*

$$\nu = \frac{k_1(s)}{k_2(s)}(s+c) \tag{39}$$

by using exchange variable $t = \int k_3(s)ds$, then we find

$$\frac{d^2\nu}{dt^2} + \nu = 0 \tag{40}$$

which has the general solution $\nu = c_1 \cos t + c_2 \sin t$ where c_1, c_2 are constants. Thus the solution of the equation (40) is obtained as

$$\frac{k_1(s)}{k_2(s)}(s+c) = c_1 \cos(\int k_3(s)ds) + c_2 \sin(\int k_3(s)ds). \tag{41}$$

Theorem 3.13 *Let β be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4. β is a D_3D_4-curve in R_1^4 if and only if the non-zero curvature functions k_1, k_2, k_3 satisfy the equation*

$$\left[\frac{-c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)'\right]' + c\frac{k_1(s)k_3(s)}{k_2(s)} - 1 = 0 \tag{42}$$

where c is a constant.

Proof Let β be a D_3D_4-curve with nonvanishing curvatures in R_1^4. From the definition of D_3D_4-curve, then we have

$$\beta(s) = \lambda(s)D_3(s) + \mu(s)D_4(s) \tag{43}$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$. If we differentiate (43) according to s and use the Frenet equations of the curve β, then we obtain

$$\begin{cases} \lambda(s)k_1(s)k_3(s) + \mu'(s) = 1 \\ (\lambda(s)k_2(s))' = 0 \\ (\lambda(s)k_3(s))' + \mu(s)k_1(s) = 0. \end{cases} \tag{44}$$

From the second equation of (44), we find $\lambda(s) = \frac{c}{k_2(s)}$ and if we use the third equation, then we obtain $\mu(s) = -\frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)'$. Then the position vector of the curve $\beta(s)$ can be defined as

$$\beta(s) = \frac{c}{k_2(s)}D_3(s) - \frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)' D_4(s). \tag{45}$$

If we use $\lambda(s)$ and $\mu(s)$ in the third equation of (44), then we get (42).

Conversely, we assume that (42) holds. Let us consider the vector given

$$Z(s) = \beta(s) - \frac{c}{k_2(s)}D_3(s) + \frac{c}{k_1(s)}\left(\frac{k_3(s)}{k_2(s)}\right)' D_4(s). \tag{46}$$

Differentiating vector Z and considering the equation of (42), we obtain

$$\frac{dZ}{ds} = 0. \tag{47}$$

Thus Z is a constant vector and so, the curve $\beta(s)$ is congruent to a D_3D_4-curve. □

Corollary 3.14 *Let β be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4 with non-zero curvature functions. Then $\beta(s)$ is a D_3D_4-curve if and only if $\beta(s)$ is an osculating curve of the first kind in R_1^4.*

Corollary 3.15 *If we consider Definition 2.2, the position vector of an oscullating curve of the first kind in R_1^4 always lies in the subspace $Sp\{T, N, B_2\}$. However from Corollary 3.14 an oscullating curve of the first kind in R_1^4 can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called D_3D_4-plane.*

Corollary 3.16 *If we consider the equation (42) and substitute*

$$\nu = \frac{k_3(s)}{k_2(s)} \tag{48}$$

by using exchange variable $t = \int k_1(s)ds = h(s)$, then we find

$$\frac{d^2\nu}{dt^2} - \nu = f(t) \tag{49}$$

where $f(t) = \frac{-1}{ck_1(h^{-1}(t))}$. The general solution of the differential equation is

$$\nu(s) = c_1e^t + c_2e^{-t} + ck_1(h^{-1}(t)) \tag{50}$$

where c_1, c_2 are constants. Thus the solution of the equation (49) is obtained as

$$\frac{k_3(s)}{k_2(s)} = c_1e^{\int k_1(s)ds} + c_2e^{-\int k_1(s)ds} + ck_1(s). \tag{51}$$

Theorem 3.17 *Let β be a spacelike Frenet curve parametrized by the arc length parameter s in R_1^4. β is a D_2D_3-curve in R_1^4 if and only if the non-zero curvature functions k_1, k_2, k_3 satisfy the equation*

$$\left[-c\frac{k_2(s)}{k_1(s)k_3(s)}\left(\frac{k_2(s)}{k_3(s)}\right)'\right]' + ck_1(s) - 1 = 0 \tag{52}$$

where c is a constant.

Proof Let β be a D_2D_3-curve with nonvanishing curvatures in R_1^4. From the definition of D_2D_3-curve, we have

$$\beta(s) = \pi(s)D_2(s) + \rho(s)D_3(s) \tag{53}$$

for some differentiable functions $\pi(s)$ and $\rho(s)$. If we differentiate (53) according to s and use the Frenet equations of the curve β, then we obtain

$$\begin{cases} \rho(s)k_1(s)k_3(s) + (\pi(s)k_2(s))' = 1 \\ (\rho(s)k_3)' = 0 \\ -(\pi(s)k_1(s))' + 2\rho(s)k_2(s)k_3(s) = 0 \\ (\rho(s)k_2(s))' + \pi(s)k_1(s)k_3(s) = 0. \end{cases} \tag{54}$$

From the second equation of (54), we find $\rho(s) = \frac{c}{k_3(s)}$ and if we use the third equation, then we obtain

$$\pi(s) = \frac{-c}{k_1(s)k_3(s)}\left(\frac{k_2(s)}{k_3(s)}\right)'. \tag{55}$$

Then the position vector of the curve $\beta(s)$ can be defined as

$$\beta(s) = \frac{-c}{k_1(s)k_3(s)}\left(\frac{k_2(s)}{k_3(s)}\right)' D_2(s) - \frac{c}{k_3(s)}D_3(s). \tag{56}$$

If we use $\pi(s)$ and $\rho(s)$ in the first equation of (54), then we get (52).

Conversely, we assume that (52) holds. Let us consider the vector given

$$Y(s) = \beta(s) + \frac{c}{k_1(s)k_3(s)}\left(\frac{k_2(s)}{k_3(s)}\right)' D_2(s) - \frac{c}{k_3(s)}D_3(s). \tag{57}$$

Differentiating vector Y and considering the equation of (52), we obtain

$$\frac{dY}{ds} = 0. \tag{58}$$

Thus Y is a constant vector and so, the curve $\beta(s)$ is congruent to a D_2D_3-curve. □

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