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Contact Hamiltonian Description of 1D Frictional Systems

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Abstract
In this paper, we consider contact Hamiltonian description of 1D frictional dynamics with no conserved force. Friction forces that are monomials of velocity, and the sum of two monomials are considered. For that purpose, quite general forms of contact Hamiltonians are taken into account. We conjecture that it is impossible to give a contact Hamiltonian description of dissipative systems where the friction force is not in the form considered in this paper.

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1. Introduction and Preliminaries

We already know that the fundamentals of many applications used in physics go through mathematical calculations. Since the theory of manifolds is used as configuration space in both mathematics and physics, the differential geometric methods used in the theory of manifolds are very important.

The geometry of the contact manifolds is done with the help of odd-dimensional manifolds. The fact that contact geometry can be applied in odd-dimensional manifolds has earned a prominent place in physics as well as differential geometry. Both contact and symplectic manifolds have found application in classical mechanics. In line with these studies, contact geometry was found to be under many physical phenomena and related to many other mathematical structures. Andrew Mclnerney's "First Steps in Differential Geometry" (Ref. [1]) is an important resource for the history of contact geometry and its significance in physics.

Lie [2] was the first to study contact structures systematically. Contact structures were considered in Gibbs' study of thermodynamics [3], Huygens' theory of light, geometric optics and Hamiltonian dynamics [4, 5].

Although the study of mathematical methods in classical mechanics dates back to old times [6], the issue of expressing Hamiltonian dynamics with contact equations is quite new. If we mention some works on Hamiltonian systems with contact equations, in 2016 contact Hamiltonian mechanics have been introduced by Bravetti and et al. [7]. In that paper, authors have focused on the major features of standard symplectic Hamiltonian dynamics and they have showed that all of them can be generalized to the contact case. Later, in Liu's work, the connections between the notions of Hamiltonian system, contact Hamiltonian system and nonholonomic system from the perspective of differential equations and dynamical systems have been described [8]. Also in [9], Dündar has provided a simple contact Hamiltonian description of a system with exponentially vanishing (or zero) potential under a friction term that is quadratic in velocity.

In the light of these previous studies with this present paper, we have provided for contact Hamiltonian description of 1D frictional dynamics with no conserved force. In this way, we have applied contact geometric methods in systems with frictional force (where friction force is not linearly dependent on velocity or where it is a polynomial of velocity). Friction forces that are monomials of velocity, and the sum of two monomials are considered. For that purpose, quite general forms of contact Hamiltonians are taken into account. Furthermore, we have given a conjecture that it is impossible to give a contact Hamiltonian description dissipative systems where the friction force is not in the form considered.

In this article we consider a 1D frictional system. The independent variables are q,p,S. The contact 1-form is given as follows: $\eta = dS + pdq$ [7]. It is easy to check that this expression satisfies the nondegeneracy condition $\eta \wedge d\eta = dS \wedge dp \wedge dq \neq 0$. Moreover, the readers may recognize the pdq term in the contact form as presymplectic potential.

In order to define contact system, we need a contact Hamiltonian. Contact Hamiltonian is a function of positions, momenta and an extra variable S as opposed to usual Hamiltonian function which is a function of positions and momenta. The extra variable, S, helps one describe dissipative systems. Now, we give a basic definition that we will use throughout this study. Let H be a contact Hamiltonian, depending on three variables: q,p,S. The equations of motion are then as follows [7]:

$$\dot{q} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S},$$
$$\dot{S} = p\frac{\partial H}{\partial p} - H.$$

In this paper, we will investigate various forms of contact Hamiltonians to account for friction terms with no potential function. The organization of the paper is as follows: In Section 2 we consider a friction term that is a monomial of $\dot{q}$, in Section 3 we handle the case where the friction term is a sum of two monomials of $\dot{q}$, in Section 4 we give an applciation of Section 3 to a friction term that has linear and quadratic dependence on $\dot{q}$, and finally in Section 5 we conclude the paper by also giving a conjecture.

2. Friction term that is a monomial of $\dot{q}$

The goal of this Section is to find a contact Hamiltonian that will yield a friction term which is a monomial of $\dot{q}$, that is an equation of motion as seen in Equation (2.4). We consider a contact Hamiltonian of the following form:

$$H = \frac{p^2}{2m} + \lambda p S^a.$$

The case $a = 1$ gives a quadratic dependence on $\dot{q}$ for the friction term, which is investigated in Ref. [9]. As an ansatz, we let $p = \alpha m\dot{q}$ and $S = S(\dot{q})$. The contact equations of motion are as follows:

$$\dot{q} = \frac{p}{m} + \lambda S^a, \tag{2.1}$$
$$\dot{p} = -a\lambda p^2 S^{a-1}, \tag{2.2}$$
$$\dot{S} = \frac{p^2}{2m}. \tag{2.3}$$

Using the ansatz for p in Equation (2.1) gives us $\lambda S^a = (1-\alpha)\dot{q}$. We want our contact Hamiltonian to produce the following equation of motion:

$$m\ddot{q} + \gamma\dot{q}^n = 0. \tag{2.4}$$

Let $S'(\dot{q}) = \partial_{\dot{q}} S(\dot{q})$. Using Equation (2.3) yields:

$$\ddot{q}S' = \frac{1}{2}m\alpha^2\dot{q}^2,$$

Use $\ddot{q} = -(\gamma/m)\dot{q}^n$

$$S' = -\frac{1}{2}\frac{m^2\alpha^2}{\gamma}\dot{q}^{2-n},$$
$$S = -\frac{1}{2}\frac{m^2\alpha^2}{\gamma}\frac{\dot{q}^{3-n}}{3-n}$$

We omit the integration constant. With an extra integration constant, the form we found here would not match $\lambda S^a = (1-\alpha)\dot{q}$. Let us use the result we found so far in Equation (2.2) and obtain:

$$\begin{aligned}\alpha m\ddot{q} &= -a\lambda p^2 S^{a-1},\\ &= -a(\alpha m\dot{q})^2\frac{\lambda S^a}{S},\\ &= 2a\gamma(1-\alpha)(3-n)\dot{q}^n,\end{aligned}$$

Use $\ddot{q} = -(\gamma/m)\dot{q}^n$

$$2a = \frac{\alpha}{1-\alpha}\frac{1}{n-3}$$

Choose $\alpha = 2$

$$a = \frac{1}{3-n}$$

We finally obtain λ in terms of m, n, γ using the expression for S and $\lambda S^a = (1-\alpha)\dot{q}$:

$$\lambda = -\left(\frac{2m^2}{\gamma}\frac{1}{n-3}\right)^{\frac{1}{n-3}}.$$

The contact Hamiltonian is as follows:

$$H = \frac{p^2}{2m} - \left(\frac{2m^2}{\gamma}\frac{1}{n-3}\right)^{\frac{1}{n-3}} pS^{1/(3-n)}.$$

This contact Hamiltonian gives us the following equation of motion:

$$m\ddot{q} + \gamma\dot{q}^n = 0,$$

for $n = 2$ and $n > 3$.

3. Friction term that is the sum of two monomials of $\dot{q}$

Our aim in this Section is to find a contact Hamiltonian that will yield a friction term that is a sum of two monomials of $\dot{q}$, that is, an equation of motion of the form $m\ddot{q} + \gamma_A\dot{q}^{n_A} + \gamma_B\dot{q}^{n_B} = 0$. However we will soon see that the only allowed combination is Equation (3.1). Let us consider the following contact Hamiltonian:

$$H = \frac{p^2}{2m} + \sum_k \lambda_k p^{b_k} S^{a_k},$$

where k runs over natural numbers (or any other countable set). The extra terms includes all analytic functions of p, S as well as other type functions with singularities. One can also absorb the first term into the sum, so this form is very general. This type of contact Hamiltonian, though seems quite general, can only model the following type of a differential equation:

$$m\ddot{q} + \gamma_A\dot{q}^{n_A} + \gamma_B\dot{q}^{n_A+1} = 0. \tag{3.1}$$

Let us write down the equations of motion for q, p, S:

$$\dot{q} = \frac{p}{m} + \sum_k b_k \lambda_k p^{b_k - 1} S^{a_k}, \tag{3.2}$$

$$\dot{p} = -\sum_k a_k \lambda_k p^{b_k + 1} S^{a_k - 1}, \tag{3.3}$$

$$\dot{S} = \frac{p^2}{2m} + \sum_k (b_k - 1) \lambda_k p^{b_k} S^{a_k}. \tag{3.4}$$

As an ansatz, let us write $p = \alpha m \dot{q}$ for some constant α. Then the first equation becomes:

$$\dot{q} = \alpha \dot{q} + \sum_k b_k \lambda_k (\alpha m \dot{q})^{b_k - 1} S^{a_k}.$$

Then let $S = \beta \dot{q}^c$ for some constants β, c. We obtain:

$$(1 - \alpha)\dot{q} = \sum_{b_k \neq 0} b_k \lambda_k (\alpha m \dot{q})^{b_k - 1} (\beta \dot{q}^c)^{a_k}.$$

So we obtain the following condition by equating the powers of $\dot{q}$:

$$c a_k + b_k = 2, \quad \text{if } b_k \neq 0, \tag{3.5}$$

and the remaining equation is the following:

$$1 - \alpha = \sum_{b_k \neq 0} b_k \lambda_k (\alpha m)^{b_k - 1} \beta^{a_k}. \tag{3.6}$$

The Equation (3.3) becomes:

$$\begin{aligned} \alpha m \ddot{q} &= -\sum_k a_k \lambda_k (\alpha m \dot{q})^{b_k + 1} (\beta \dot{q}^c)^{a_k - 1}, \\ &= -\sum_{b_k = 0} a_k \lambda_k \alpha m \beta^{a_k - 1} \dot{q}^{1 + c(a_k - 1)} \\ &\quad - \sum_{b_k \neq 0} a_k \lambda_k (\alpha m)^{b_k + 1} \beta^{a_k - 1} \dot{q}^{3 - c}. \end{aligned}$$

Hence we obtain:

$$\begin{aligned} m \ddot{q} = &-\sum_{b_k = 0} a_k \lambda_k m \beta^{a_k - 1} \dot{q}^{1 + c(a_k - 1)} \\ &- \sum_{b_k \neq 0} a_k \lambda_k \frac{(\alpha m)^{b_k + 1}}{\alpha} \beta^{a_k - 1} \dot{q}^{3 - c}. \end{aligned} \tag{3.7}$$

3.1 When $c \neq 0$

In this Subsection we suppose $c \neq 0$.[1] Then Equation (3.4) yields:

$$\beta c \dot{q}^{c-1} \ddot{q} = \frac{1}{2} m \alpha^2 \dot{q}^2 + \sum_k (b_k - 1) \lambda_k (\alpha m \dot{q})^{b_k} (\beta \dot{q}^c)^{a_k}.$$

[1] The case where $c = 0$ is investigated in Subsection 3.2 and causes a vanishing Hamiltonian, but included in this article for completeness.

From this, we obtain:

$$m\ddot{q} = \frac{1}{2}\frac{\alpha^2 m^2}{\beta c}\dot{q}^{3-c} + \sum_k (b_k - 1)\lambda_k (\alpha m)^{b_k}\frac{m}{c}\beta^{a_k-1}\dot{q}^{ca_k+b_k+1-c}.$$

When collected as a sum over $b_k = 0$ and $b_k \neq 0$ we obtain:

$$m\ddot{q} = \frac{1}{2}\frac{\alpha^2 m^2}{\beta c}\dot{q}^{3-c} - \sum_{b_k=0}\lambda_k\frac{m}{c}\beta^{a_k-1}\dot{q}^{1+c_k(a_k-1)} + \sum_{b_k\neq 0}(b_k-1)\lambda_k\frac{(\alpha m)^{b_k}m}{c}\beta^{a_k-1}\dot{q}^{3-c}. \tag{3.8}$$

We now have two equations of motion $m\ddot{q}$. They need to be consistent with each other. So we equate (and suppose $1+c(a_k-1)\neq 3-c$ or $ca_k \neq 2$.) Equation (3.7) and Equation (3.8):

$$\sum_{b_k=0}\lambda_k m\left(a_k - \frac{1}{c}\right)\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} = 0, \tag{3.9}$$

from which we obtain:

$$a_k = 1/c, \quad \text{if } b_k = 0, \tag{3.10}$$

and

$$\frac{1}{2}\frac{\alpha^2 m^2}{\beta c} + \sum_{b_k\neq 0}(b_k-1)\lambda_k\frac{(\alpha m)^{b_k}m}{c}\beta^{a_k-1} = -\sum_{b_k\neq 0}a_k\lambda_k\frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1},$$

which yields

$$-\frac{1}{2} = \sum_{b_k\neq 0}\lambda_k(\alpha m)^{b_k-2}m\beta^{a_k}.$$

Since c is a constant, we see that we can only obtain two powers of $\dot{q}$ in the equation of motion. The first is a power of $3-c$ (when $b_k \neq 0$) and the second is a power of $1+c(a_k-1) = 2-c$ (when $b_k = 0$). As a result, it is sufficient to consider two types of variables $(b_k, a_k) \in \{(0, a_A), (b_B, a_B)\}$. In this case the contact Hamiltonian is as follows:

$$H = \frac{p^2}{2m} + \lambda_A S^{a_A} + \lambda_B p^{b_B} S^{a_B}.$$

We equate Equation (3.7) to $-\gamma_A\dot{q}^{n_A} - \gamma_B\dot{q}^{n_B}$ and obtain:

$$-\sum_{b_k=0}a_k\lambda_k m\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} - \sum_{b_k\neq 0}a_k\lambda_k\frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1}\dot{q}^{3-c} = -\gamma_A\dot{q}^{n_A} - \gamma_B\dot{q}^{n_B}$$

Putting the relations between the constants we found and their values we get:

$$n_A = 2-c, \tag{3.11}$$
$$n_B = 3-c, \tag{3.12}$$
$$\lambda_A = \frac{c\gamma_A}{m}\beta^{1-1/c}, \tag{3.13}$$
$$\lambda_B = \frac{\gamma_B}{a_B m}\frac{\beta^{1-a_B}}{(\alpha m)^{b_B}}. \tag{3.14}$$

Moreoever for λ_B we have one more equation, namely Equation (3.6), which gives us the following constraint:

$$\lambda_B = \frac{1-\alpha}{b_B}\frac{(\alpha m)^{1-b_B}}{\beta^{a_B}} \tag{3.15}$$

By equating Equation (3.14) and Equation (3.15) we obtain:

$$\beta = \frac{m^2\alpha(1-\alpha)}{\gamma_B}\frac{a_B}{b_B} = \frac{m^2\alpha(1-\alpha)}{\gamma_B}\frac{a_B}{2-ca_B}.$$

All in all we have the following relations:

$$a_A = \frac{1}{2-n_A} = \frac{1}{3-n_B},$$
$$n_B = n_A + 1,$$
$$n_A = 2 - \frac{1}{a_A},$$
$$\lambda_A = \frac{\gamma_A}{ma_A}\beta^{1-a_A},$$
$$\lambda_B = \frac{\gamma_B}{ma_B}\frac{\beta^{1-a_B}}{(\alpha m)^{b_B}}.$$

Only λ_A can vanish. By letting $\lambda_A = 0$ one can model a system where frictional force is proportional to a monomial of $\dot{q}$.

3.2 When $c = 0$

In this Subsection we consider the case $c = 0$. Under this situation we have $S = \beta$. Hence $\dot{S} = 0$. Equation (3.5) yields the following:

$$\text{if } b_k \neq 0, b_k = 2.$$

When used in Equation (3.4) we obtain the following two equations:

$$\sum_{b_k=0}\lambda_k\beta^{a_k} = 0, \tag{3.16}$$

$$\sum_{b_k=2}\lambda_k\beta^{a_k} = -\frac{1}{2m} \tag{3.17}$$

Since there are two options, we can restrict the set of (b_k, a_k) to two values: $(b_k, a_k) \in \{(0, a_D), (2, a_E)\}$. Equation (3.16) gives us $\lambda_D = 0$. So the contact Hamiltonian is of the following form:

$$H = \frac{p^2}{2m} + \lambda_E p^2 S^{a_E},$$

with $\lambda_E = -\beta^{-a_E}/(2m)$ obtained from Equation (3.17). Let us write the equation of motion derived by $\dot{p}$ (Equation (3.7)):

$$m\ddot{q} = -\sum_{b_k=0} a_k\lambda_k m\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} - \sum_{b_k\neq 0} a_k\lambda_k\frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1}\dot{q}^{3-c},$$
$$= -a_D\lambda_D m\beta^{a_D-1}\dot{q} - a_E\lambda_E\frac{(\alpha m)^3}{\alpha}\beta^{a_E-1}\dot{q}^3$$

First term vanishes since $\lambda_D = 0$. The second term is proportional to the derivative of $\lambda_E\beta^{a_E} = 1/(2m)$ (see Equation (3.17)) with respect to β and is thus zero.

$$= 0.$$

Finally we obtain the dynamics of a free particle in 1D. As a result, this case is not interesting and does not cause frictional dynamics to appear.

4. Friction term that has linear and quadratic dependence on $\dot{q}$

In this Section, we give an application of Section 3 to the case where there are linear and quadratic dependencies of the friction force on speed ($\dot{q}$). The equation of motion is the following:

$$m\ddot{q}+\gamma_A\dot{q}+\gamma_B\dot{q}^2=0.$$

The contact Hamiltonian is the following:

$$H=\frac{p^2}{2m}+\lambda_A S^{a_A}+\lambda_B p^{b_B}S^{a_B}.$$

Using the results of Section 3 we obtain:

$$\begin{aligned}n_A&=1,\\ n_B&=2,\\ a_A&=1,\end{aligned}$$

Let us choose $a_B=b_B=1$ and $\alpha=2$

$$\begin{aligned}a_B&=1,\\ b_B&=1,\\ \alpha&=2,\end{aligned}$$

Then we obtain:

$$\begin{aligned}\lambda_A&=\frac{\gamma_A}{m},\\ \lambda_B&=\frac{\gamma_B}{2m^2}.\end{aligned}$$

So the contact Hamiltonian is the following:

$$H=\frac{p^2}{2m}+\frac{\gamma_A}{m}S+\frac{\gamma_B}{2m^2}pS,$$

and it yields the following equation of motion:

$$m\ddot{q}+\gamma_A\dot{q}+\gamma_B\dot{q}^2=0.$$

This is an important step, because in classical mechanics when the speed is low the friction is linear in velocity and when the speed is high the frictional force is quadratic in velocity due to effect of turbulence.

5. Conclusion

In this paper, we mainly focused on contact Hamiltonian description 1D frictional systems. The contact Hamiltonians of the form $H=p^2/2m+\lambda pS^a$ can describe a situation where friction force is a monomial of $\dot{q}$:

$$m\ddot{q}+\gamma\dot{q}^n=0, \tag{5.1}$$

for $n=2$ and $n>3$. The case for $n=1$ is given in Ref. [7] and the contact Hamiltonian for that case is $H=p^2/2m+V(q)+\gamma S$ and it is the only contact Hamiltonian found so far to include an arbitrary potential. An exponentially decreasing potential in the case of quadratic dependence on velocity of the friction term is found in Ref. [9].

On the other hand, what we found is that a quite general contact Hamiltonian of the form $H = p^2/2m + \sum_k \lambda_k p^{b_k} S^{a_k}$ (which includes all analytic functions of p, S) can at most describe a dissipative system in the following form:

$$m\ddot{q} + \gamma_A \dot{q}^{n_A} + \gamma_B \dot{q}^{n_A+1} = 0. \tag{5.2}$$

In order to solve the contact equations of motion, we considered p, S to be functions of $\dot{q}$ and used two different ansatzes for this purpose. We conjecture that it is impossible to model a dissipative system with no potential that is not of the form appearing in Equation (5.1) or Equation (5.2).

We also have given the contact Hamiltonian description of the following equation of motion:

$$m\ddot{q} + \gamma_A \dot{q} + \gamma_B \dot{q}^2 = 0.$$

This form of frictional force is the most prevalent in nature. When the speed is low the linear term is dominant, and when the speed is high the quadratic term becomes dominant due to turbulence.

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Author's contributions

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