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Fixed Point Theorems for Multi-valued α-F-contractions in Partial metric spaces with Some Application

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Abstract

This paper aims to prove a fixed point theorem for multi-valued mapping using α-F-contraction in partial metric spaces. Furthermore, we prove a fixed point theorem for F-Hardy-Roger's multi-valued mappings in ordered partial metric spaces. Specifically, this paper intends to generalize the theorems by Ali and Kamran, Sgroi and Vetro and Kumar. We also provided illustrative examples and some applications to integral equations.

Keywords: multi-valued mapping, α-F-contraction, Hardy-Rogers contraction, partial metric spaces
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1. Introduction

In 1969, Nadler [27] introduced multi-valued contraction mappings using the Hausdorff metric and extended Banach's contraction principle [8] from single valued to multi-valued mappings. Since then, several researchers were influenced by his work and generalized results for multi-valued mappings in various spaces. The theory of multi-valued mappings has many applications in diverse areas such as in control theory, approximation theory, differential equations and economics.

In 1973, Hardy-Rogers [17] gave a generalization of the Reich fixed point theorem [37]. Since then, several authors have been using different Hardy-Rogers contractive type conditions in order to obtain fixed point results. Some of them are [10, 11, 28, 35, 42].

In 1994, Matthews [24] came up with a generalization of the metric space called the partial metric space by relaxing the zero self distance axiom for the metric space. He extended the Banach contraction principle

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to partial metric space and found applications in computer networking, data structure, and computer programming languages. In recent years a number of researchers have extended fixed point theorems in metric spaces to partial metric spaces [see [6, 31, 32, 41]].

In 2004, Ran and Reurings [36] followed by Nieto and Rodriguez-Lopez [29] in 2006 introduced the study of fixed point theorems for partially ordered sets along with relevant. Recently Abbas *et al.* [1] introduced the analogue of F-contraction to establish ordered-theoretical results. On the other hand, Durmaz *et al.* [13] introduced the concept of ordered metric space by using the results of Ran and Reurings [36] (also one can refer to [22, 29] the reference therein).

In 2012, Wardowski [44] introduced a generalization of Banach contraction principle in metric spaces. After massive influence, research was carried out on F-contraction for single and multivalued mappings in various spaces. For literature, one can see [2, 5, 15, 23, 25, 33, 34, 39, 45] and the references therein. Karapinar and Samet [21] generalised Banach contraction principle by proving the results using α-ψ-contraction. For more detail we refer the reader to [16, 18, 19]. In 2016, Ali and Kamran [3] proved a fixed point in metric spaces by combining the concepts of α-admissible mappings and F-contractions to get a generalized contraction named α-F-contraction. For more details one can refer to [1, 7, 12, 14, 26, 43].

2. Preliminaries

We now introduce preliminaries that will be of use in this paper.

First, we describe the partial metric space and some of its properties.

Definition 2.1. [24] *A partial metric on a non-empty set X is a mapping $p : X \times X \to \mathbb{R}_+$, such that for all $x, y, z \in X$*

(P1) $0 \le p(x,x) \le p(x,y)$,

(P2) $x = y$ *if and only if* $p(x,x) = p(x,y) = p(y,y)$,

(P3) $p(x,y) = p(y,x)$ *and*

(P4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is said to be a partial metric space.

As an example, let $X = \mathbb{R}^+$ and let $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Lemma 2.2. [24] *If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

for all $x, y \in X$, defines a metric on X.

Definition 2.3. [24]

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a p-Cauchy sequence if only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).

(iii) A partial metric space (X,p) is said to be complete if every p-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

We take note of the following lemma.

Lemma 2.4. [24]. *Let (X,p) be a partial metric space.*

(i) A sequence $\{x_n\}$ is p-Cauchy in a partial metric space (X,p) if and only if it is a Cauchy in the metric space (X,p^s).

(ii) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Moreover

$$\lim_{n\to\infty} p^s(x,x_n) \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

We obtain the description and properties of the partial Hausdorff metric from Aydi *et al.* [6].

Let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of a partial metric space (X,p), induced by the partial metric p. Furthermore, the set A is said to be a bounded subset in (X,p) if there exists $x_0 \in X$ and $N \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,N)$

$$p(x_0,a) \leq p(a,a) + N.$$

For all $A,B \in CB^p(X)$ and $x \in X$, we define:

$$\begin{aligned} p(x,A) &= \inf\{p(x,a) : a \in A\}; \\ \delta_p(A,B) &= \sup\{p(a,B) : a \in A\}; \\ \delta_p(B,A) &= \sup\{p(b,A) : b \in B\}. \end{aligned}$$

Note that

$$p(x,A) = 0 \implies p^s(x,A) = 0, \tag{1}$$

where

$$p^s(x,A) = \inf\{p^s(x,A), x \in A\}.$$

We define the partial Hausdorff metric $H_p : CB^p \times CB^p \to \mathbb{R}^+$ as

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

We state some properties of the partial Hausdorff metric H_p.

Lemma 2.5. [6] *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b(a) \in B$ such that*

$$p(a,b) \leq hH_p(A,B).$$

Proposition 2.6. [6] *Let (X,p) be a partial metric space, then for any $A,B,C \in CB^p(X)$, we have*

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;

(ii) $\delta_p(A,A) \leq \delta_p(A,B)$;

(iii) $\delta_p(A,B) = 0 \to A \subseteq B$;

(ii) $\delta_p(A,B) = \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 2.7. [6] *Let (X,p) be a partial metric space. For all $A,B,C \in CB^p(X)$, we have*

$(H1)$ $H_p(A,A) \leq H_p(A,B)$;
$(H2)$ $H_p(A,B) = H_p(B,A)$;
$(H3)$ $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\to\ C} p(c,c)$.

It is easy to see that $H_p(A,B) = 0 \to A = B$.

Remark 2.8. [4] *Let (X, p) be partial metric space and A be a nonempty subset of X. Then $a \in \bar{A}$ if and only if*

$$p(a, A) = p(a, a),$$

where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $\bar{A} = A$.

The following explanations for developing the definition of the F-contraction are obtained from Wardowski and Dung [45].

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

(F_1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

(F_2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) There exists $k \in (0, 1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the family of all functions F satisfying conditions $F_1 - F_3$ by $\mathfrak{F}$. Some examples of functions $F \in \mathfrak{F}$ are:

(1) $F(a) = \ln a$;
(2) $F(a) = a + \ln a$.

Definition 2.9. [38] *Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.*

Definition 2.10. [9] *Let A and B be two non-empty subsets of $(X, \preceq)$, the relation between A and B are denoted and defined as follows:*

(1) *$A \prec_1 B$: if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$,*
(2) *$A \prec_2 B$: if for every $b \in B$ there exists $a \in A$ we have $a \preceq b$,*
(3) *$A \prec_3 B$: if $A \prec_1 B$ and $A \prec_2 B$.*

Theorem 2.11. [40] *Let $(X, d, \preceq)$ be an ordered complete metric space and Let $T : X \to CB(X)$. Assume that there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that*

$$\begin{aligned} 2\tau + F\big(H(Tx, Ty)) \quad &\leq \quad F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \\ &\qquad \delta d(x, Ty) + L d(y, Tx), \end{aligned}$$

for all comparable $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. If the following condition are satisfied:

(*i*) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$;*

(*ii*) *for $x, y \in X$, $x \preceq y$ implies $Tx \prec_2 Ty$;*

(*iii*) *X is regular;*

then T has a fixed point.

Kumar [22] extended the results due to Durmaz *et al.* [13] where he introduced the following definition and theorem on ordered partial metric spaces using two compatible mappings:

Definition 2.12. [22] *Let$(X, \preceq, p)$ be an ordered partial metric space and $T : X \to X$ be a mapping. Also let $Y = \{(x, y) \in X \times X : x \preceq y, p(Tx, Ty) > 0\}$. We say that T is an ordered F-contraction if $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that for all $(x, y) \in Y$, we have*

$$\tau + F(p(Tx, Ty)) \leq F(p(x, y)). \tag{2}$$

Theorem 2.13. [22] *Let $(X, \preceq)$ be partial ordered set and suppose that there exists a partial metric space on X such that (X, p) is a complete partial metric space. Suppose T and g are continuous self F-contraction mappings on X, $T(X) \subseteq g(X)$, T is monotone g-non decreasing mapping and*

$$\tau + F(p(Tx, Ty)) \leq F(\mathbb{M}(x, y)),$$

where

$$\mathbb{M}(x, y) = \max\left\{p(gx, gy), p(gx, Tx), p(gy, Ty), \frac{1}{2}[p(gx, Ty) + p(gy, Tx)]\right\},$$

for all $x, y \in X$ for which gx and gy are comparable and $\tau > 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$ and T and g are compatible, then T and g have a coincident point.

In this paper, we develop a fixed point theorem for multi-valued α-F contraction mappings in partial metric spaces. We also construct a fixed theorem for multi-valued Hardy-Rogers type F-contraction in ordered partial metric spaces. Besides, we provided examples of the use of theorems and an application to integral equations.

3. Main Results

3.1. Fixed point theorem for multi-valued α-F-contraction mappings in partial metric spaces

We start our first results by slightly modifying the Definition 2.9 given in [38].

Definition 3.1. *Let $\alpha : X \times X \to [0, \infty)$ be a function in a partial metric space (X, p). A mapping $T : X \to CB^p(X)$ is said to be strictly α-admissible if for each $x \in X$ and $y \in Tx$ such that $\alpha(x, y) > 1$ we have $\alpha(y, z) > 1$ for each $z \in Ty$.*

From Ali and Kamran [3], we get the following definition of a α-F- contraction mapping:

Definition 3.2. [3] Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be function. A mapping $T : X \to CB(X)$ is α-F-contraction if there exists a continuous function F in $\mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(\mathbb{M}(x, y)),$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), \mathbb{M}(x, y)\} > 0$, where

$$\mathbb{M}(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

and $L \geq 0$.

We consider the following theorem by Ali and Kamran [3]

Theorem 3.3. [3] *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an α-F-contraction satisfying the following conditions:*

- *(i) T is strictly α-admissible mapping;*
- *(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;*
- *(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

In order to develop our main result, we modify Definition 3.2 as follows:

Definition 3.4. Let (X,p) be a partial metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to CB^p(X)$ is an α-F-contraction if there exists a continuous function $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha(x,y)H_p(Tx,Ty)) \leq F(\mathbb{M}(x,y)), \tag{3}$$

for each $x, y \in X$, whenever $\min\{\alpha(x,y)H_p(Tx,Ty), \mathbb{M}(x,y)\} > 0$ and $q, r \geq 2$, where

$$\mathbb{M}(x,y) = \max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{q}, \frac{p(x,Ty)+p(y,Tx)}{r}\right\}.$$

By extending Theorem 3.3, we prove following results:

Theorem 3.5. *Let (X,p) be a complete partial metric space, and $T : X \to CB^p(X)$ be an α-F-contraction satisfying the following conditions:*

(i) T is strictly α-admissible mapping;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) > 1$;
(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n,x) > 1$ for each $n \in \mathbb{N}$.

Then there exists $x^* \in X$ such that $Tx^* = x^*$ and $p(x^*,x^*) = 0$. x^* is a fixed point of T.

Proof. Let $x_0 \in X$ be an arbitrary point and choose $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed.

If however $x_1 \notin Tx_1$, then apply (3) with $x = x_0$ and $y = x_1$ as follows:

$$\tau + F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big) \leq F[\max\{\mathbb{M}(x_0,x_1)\}], \tag{4}$$

where

$$\begin{aligned}
&\mathbb{M}(x_0,x_1)\\
&\quad= \max\left\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\right\}\\
&\quad\leq \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,Tx_1)}{q}, \frac{p(x_0,Tx_1)+p(x_1,x_1)}{r}\right\}
\end{aligned}$$

because $x_1 \in Tx_0, x_2 \in Tx_1$, we have

$$\leq \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{q}, \frac{p(x_0,x_2)+p(x_1,x_1)}{r}\right\},$$

by $P4$ of Definition 2.1, we have

$$\begin{aligned}
\leq\ & \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,Tx_1)}{q},\right.\\
&\left.\frac{p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1)+p(x_1,x_1)}{r}\right\},
\end{aligned}$$

using $P1$ and (1) in above inequality, we get

$$\begin{aligned}
\leq\ & \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{q},\right.\\
&\left.\frac{p(x_0,x_1)+p(x_1,x_2)}{r}\right\},\\
\Rightarrow \mathbb{M}(x_0,x_1)\ \leq\ & p(x_0,x_1).
\end{aligned} \tag{5}$$

We substitute (5) into (4) and get

$$\tau + F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big) \quad \leq \quad F\big(p(x_0,x_1)\big). \tag{6}$$

As $\alpha(x_0,x_1) > 1$, by Lemma (2.5) there exists $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \leq \alpha(x_0,x_1)H_p(Tx_0,Tx_1). \tag{7}$$

As F is an increasing function we have

$$F\big(p(x_1,x_2)\big) \leq F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big). \tag{8}$$

Inserting (7) in (6) we get

$$\tau + F\big(p(x_1,x_2)\big) \leq F\big(p(x_0,x_1)\big). \tag{9}$$

Since T is strictly α- admissible, according to Definition (3.1), we have $\alpha(x_0,x_1) > 1 \Rightarrow \alpha(x_1,x_2) > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point and the proof is completed. Suppose $x_2 \notin Tx_2$. We apply Equation (3) with $x = x_1, y = x_2$ and get

$$\tau + F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big) \leq F[\max\{\mathbb{M}(x_1,x_2)\}], \tag{10}$$

where

$$\begin{aligned} &\mathbb{M}(x_1,x_2) \\ &\quad = \max\left\{p(x_1,x_2), \frac{p(x_1,Tx_1)+p(x_2,Tx_2)}{q}, \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{r}\right\} \\ &\Rightarrow \mathbb{M}(x_1,x_2) \leq p(x_1,x_2). \end{aligned} \tag{11}$$

On applying (11) to (10) and get

$$\tau + F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big) \leq F\big(p(x_1,x_2)\big). \tag{12}$$

As $\alpha(x_1,x_2) > 1$, by Lemma (2.5) there exists $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \leq \alpha(x_1,x_2)H_p(Tx_1,Tx_2). \tag{13}$$

F is an increasing function, therefore

$$F\big(p(x_2,x_3)\big) \leq F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big). \tag{14}$$

On applying (14) to (12), we get

$$\tau + F(p(x_2,x_3)) \leq F\big(p(x_1,x_2)\big). \tag{15}$$

Therefore (15) becomes

$$\begin{aligned} &\tau + F(p(x_2,x_3)) \leq F(p(x_1,x_2)) \\ &\Rightarrow F(p(x_2,x_3)) \leq F(p(x_1,x_2)) - \tau \\ &\Rightarrow F(p(x_2,x_3)) \leq F(p(x_0,x_1)) - 2\tau, \text{ by (9)}. \end{aligned} \tag{16}$$

Continuing in the same manner, we form a sequence $\{x_n\}$ which reaches one the following scenarios. Either $x_n \in Tx_n$ for some $n \in \mathbb{N}$. In this case, x_n is the fixed point and the proof is completed.

Otherwise, we have for all $n \in \mathbb{N}$, $x_n \notin Tx_n, x_n \in Tx_{n-1}$,
$\alpha(x_{n-1},x_n) > 1$ and

$$F\big(p(x_n,x_{n+1})\big) \leq F\big(p(x_0,x_1)\big) - n\tau. \tag{17}$$

We determine the limit $n \to \infty$ of (17) and get

$$\lim_{n\to\infty} F\big(p(x_n, x_{n+1})\big) = -\infty.$$

By condition (F_2), this implies

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{18}$$

Let $\alpha_n = p(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. By condition $(F3)$, there exist $k \in (0,1)$ and such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n).$$

From (17) we have

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -n\alpha_n^k \tau < 0 \text{ for each } n \in \mathbb{N}. \tag{19}$$

Letting $n \to \infty$ in (19) we get

$$\lim_{n\to\infty} n\alpha_n^k = 0. \tag{20}$$

This implies there exists $n_1 \in \mathbb{N}$ such that $n\alpha_n^k < 1$ for all $n > n_1$. Therefore we have

$$\alpha_n < \frac{1}{n^{1/k}}. \tag{21}$$

We now show that $\{x_n\}$ is a p-Cauchy sequence. Consider $m, n \in \mathbb{N}, m < n < n_1$. By $P3$ of Definition 2.1, we have

$$\begin{aligned} p(x_m, x_n) &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) - \sum_{i=m+1}^{n-1} p(x_i, x_i) \\ &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{\infty} p(x_i, x_{i+1}) \\ &= \sum_{i=m}^{\infty} \alpha_i \\ &\leq \sum_{i=m}^{\infty} \frac{1}{i^{1/k}}, \text{ from (21).} \end{aligned}$$

The series $\sum_{i=m}^{\infty} \dfrac{1}{i^{1/k}}$ converges as it is a p-series with an exponent greater than one. This implies $\lim_{m,n\to\infty} p(x_m, x_n) =$ 0. This makes $\{x_n\}$ a Cauchy sequence by (ii) of Definition (2.3).

As (X, p) is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$. By (18), this means $p(x^\star, x^\star) = 0$. Also by condition (iii) of Theorem (3.5), we have $\alpha(x_n, x^\star) > 1$ for all $n \in \mathbb{N}$.

We claim that $x^\star$ is a fixed point of T, that is $p(x^\star, Tx^\star) = p(x^\star, x^\star) = 0$. Suppose $p(x^\star, Tx^\star) > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $p(x_n, Tx^\star) > 0$ for all $n > n_0$. By (3.4), for all $n > n_0$ and $q, r \geq 0$, we have

$$\begin{aligned} &\tau + F\big(p(x_{n+1}, Tx^\star)\big) \\ &\quad \leq \tau + \alpha(x_n, x^\star) F\left(H_p(Tx_n, Tx^\star)\right) \\ &\quad \leq F\left(\max\left\{p(x_n, x^\star), \frac{p(x_n, Tx_n) + p(x^\star, Tx^\star)}{q}, \frac{p(x_n, Tx^\star) + p(x^\star, Tx_n)}{r}\right\}\right) \end{aligned} \tag{22}$$

We let $n \to \infty$ in (22) and get

$$\tau + F\big(p(x^\star, Tx^\star)\big) \leq F\big(p(x^\star, x^\star)\big). \tag{23}$$

since $\tau > 0$, the above inequality yield a contradiction. Hence $p(x^\star, Tx^\star) = 0$. $\square$

Example 3.6. *Consider the partial metric space* (X,p) *where* $X=\{0,1,2,\dots\}$ *and* $p(x,y)=|x-y|+\max\{x,y\}$ *for all* $x,y\in X$*. Define the multivalued function* $T:X\to CB^p(X)$ *as*

$$Tx=\begin{cases}\{0,1\} & \textit{for } 0\le x\le 1;\\ \\ \{x-1,x\} & \textit{for } x>1.\end{cases}$$

Let $\alpha : X\times X\to[0,\infty)$ *be defined as*

$$\alpha(x,y)=\begin{cases}2 & \textit{if } x,y\in\{0,1\};\\ \frac{1}{2} & \textit{if } x,y>1;\\ 0 & \textit{otherwise.}\end{cases}$$

Now, we show that T *is strictly* α*-admissible with the following cases:*

Case 1 *Assume that* $x=x_0$ *and* $y=x_1$*. Let* $x_0=0$ *and* $x_1=1$*, then* $x_1\in Tx_0=\{0,1\}$*, such that* $\alpha(x_0,x_1)>1$*. Also we choose* x_2 *such that* $x_2\in Tx_1$*,* $x_2=0\in Tx_1=\{0,1\}$*, thus* $\alpha(x_1,x_2)>1$*.*

Case 2 *We define* $F(x)=x+\ln(x), x\in(0,\infty)$*. Under this* F*, the Equation (4) simplifies to*

$$\frac{\alpha(x,y)H_p(Tx,Ty)}{\mathbb{M}(x,y)}e^{\alpha(x,y)H_p(Tx,Ty)-\mathbb{M}(x,y))}\le e^{-\tau}. \tag{24}$$

We now calculate $H_p(Tx,Ty)$ *for* $x>y>1$ *and* $q,r\ge 2$*.*

$$\begin{aligned}Tx=\{x-1,x\},&\quad Ty=\{y-1,y\};\\ p(x-1,y-1)=2x-y-1,&\quad p(x-1,y)=2x-y-2\\ p(x,y-1)=2x-y+1,&\quad p(x,y)=2x-y.\end{aligned}$$

$$\begin{aligned}p(x-1,Ty)&=\min\{p(x-1,a),a\in Ty\}\\ &=\min\{p(x-1,y-1),p(x-1,y)\}\\ &=\min\{2x-y-1,2x-y-2\}\\ &=2x-y-2.\end{aligned}$$

In the same manner we get

$$p(x,Ty)=2x-y,\ \ p(y-1,Tx)=2x-y-1,\ \ p(y,Tx)=2x-y-2.$$

$$\begin{aligned}\delta_p(Tx,Ty)&=\max\{p(a,Ty),a\in Tx\}\\ &=\max\{p(x-1,Ty),p(x,Ty)\}\\ &=\max\{2x-y-2,2x-y\}\\ &=2x-y.\end{aligned}$$

Similarly

$$\delta_p(Ty,Tx)=2x-y-1.$$

Hence

$$\begin{aligned}H_p(Tx,Ty)&=\max\{\delta_p(Tx,Ty),\delta_p(Ty,Tx)\}\\ &=\max\{2x-y,2x-y-1\}\\ &=2x-y.\end{aligned}$$

We note that for $x > y > 1$, $\min\{\alpha(x,y)H(Tx,Ty), \mathbb{M}(x,y)\} > 0$ and $\mathbb{M}(x,y) \geq p(x,y) = 2x - y$. Hence (4) becomes

$$\begin{aligned}\frac{1}{2}\cdot\frac{2x-y}{\mathbb{M}(x,y)}e^{\frac{1}{2}(2x-y)-\mathbb{M}(x,y)} &\leq \frac{1}{2}\cdot\frac{2x-y}{2x-y}e^{\frac{1}{2}(2x-y)-(2x-y)}\\ &\leq \frac{1}{2}e^{-3/2},\ \textit{because } x>y>1,\\ &\leq e^{-\tau}\ \textit{for } \tau\geq\frac{3}{2}.\end{aligned}$$

for $\tau \geq \frac{3}{2}$.

This shows that T is a multivalued α-F-contraction with contractive factor $\tau = \frac{3}{2}$ and $F(a) = \ln a + a$. For $x_0 = 0$ and $x_1 \in Tx_0 = \{0, 1\}$, we obtain $\alpha(0, 1) > 1$. Furthermore, we see that T is strictly α- admissible map and for any sequences $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 3.5, T has a fixed point in X.

3.2. Fixed Point Theorem for Multi-valued F-contraction mappings in Ordered Partial metric space

In order to prove our second main result, we first define an ordered relation as follows.

Definition 3.7. [9] *Let A and B be two non-empty subsets of $(X, \preceq)$, the relation between A and B are denoted and defined as follows:*

1. *$A \prec_1 B$: if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$,*
2. *$A \prec_2 B$: if for every $b \in B$ there exists $a \in A$ we have $a \preceq b$,*
3. *$A \prec_3 B$: if $A \prec_1 B$ and $A \prec_2 B$.*

Theorem 3.8. [40] *Let $(X, d, \preceq)$ be an ordered complete metric space and Let $T : X \to CB(X)$. Assume that there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that*

$$\begin{aligned}2\tau + F\big(H(Tx,Ty)) \ &\leq\ F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) +\\ &\qquad \delta d(x,Ty) + L d(y,Tx),\end{aligned}$$

for all comparable $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. If the following condition are satisfied:

(*i*) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$;*

(*ii*) *for $x, y \in X$, $x \preceq y$ implies $Tx \prec_2 Ty$;*

(*iii*) *X is regular;*

then T has a fixed point.

Kumar [22] extended the results due to Durmaz *et al.* [13] where he introduced the following definition and theorem on ordered partial metric spaces using two compatible mappings:

Definition 3.9. [22] *Let$(X, \preceq, p)$ be an ordered partial metric space and $T : X \to X$ be a mapping. Also let $Y = \{(x, y) \in X \times X : x \preceq y, p(Tx, Ty) > 0\}$. We say that T is an ordered F-contraction if $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that for all $(x, y) \in Y$, we have*

$$\tau + F(p(Tx, Ty)) \leq F(p(x, y)). \tag{25}$$

Theorem 3.10. [22] *Let $(X, \preceq)$ be partial ordered set and suppose that there exists a partial metric space on X such that (X, p) is a complete partial metric space. Suppose T and g are continuous self F-contraction mappings on X, $T(X) \subseteq g(X)$, T is monotone g-non decreasing mapping and*

$$\tau + F(p(Tx, Ty)) \leq F(\mathbb{M}(x, y)),$$

where

$$\mathbb{M}(x, y) = \max\left\{p(gx, gy), p(gx, Tx), p(gy, Ty), \frac{1}{2}[p(gx, Ty) + p(gy, Tx)]\right\},$$

for all $x, y \in X$ for which gx and gy are comparable and $\tau > 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$ and T and g are compatible, then T and g have a coincident point.

We give the extendend version of Definition 3.9 to an ordered multi-valued Hardy-Rogers F-contraction in partial metric space as follows:

Definition 3.11. *Let $(X, \preceq, p)$ be an ordered partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. We say that T is an ordered multi-valued Hardy-Rogers F-contraction if $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that for all $x, y \in X$, we have*

$$2\tau + F\big(H_p(Tx, Ty)\big) \leq F\big(\mathbb{M}(x, y)\big), \tag{26}$$

where

$$\mathbb{M}(x, y) = \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty) + \delta p(x, Ty) + Lp(y, Tx)$$

for $x \preceq y \Leftrightarrow Tx \preceq Ty$, $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + \delta = 1$ and $\gamma \neq 1$.

By extending Theorem 3.8, we prove following theorem:

Theorem 3.12. *Let $(X, \preceq)$ be a partial ordered set and suppose that there exists a partial metric p such that (X, p) is a complete partial metric space. Let $T : X \to CB^p(X)$ be a multi-valued map. Assume that there exists $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}_+$ such that T is a multi-valued Hardy-Rogers-type F-contraction which satisfy the following conditions:*

(*i*) *there exists $x_0 \in X$ such that $x_0 \prec_1 Tx_0$;*
(*ii*) *for $x, y \in X$, $x \preceq y \implies Tx \prec_2 Ty$;*
(*iii*) *if $x_n \to x$ is a non decreasing sequence in X, for all n and*

$$2\tau + F(H_p(Tx, Ty)) \leq F(\mathbb{M}(x, y)), \tag{27}$$

where

$$\mathbb{M}(x, y) = \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty) + \delta p(x, Ty) + Lp(y, Tx)$$

for $x, y \in X$, $\tau > 0$, $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + \delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

Proof. From assumption (i), there exists $x_0 \in X$ such that $x_0 \prec_1 Tx_0$. Choosing $x_1 \in Tx_0$, by(ii) we have $x_0 \preceq x_1 \Rightarrow Tx_0 \prec_2 Tx_1$. If $x_1 \in Tx_1$ then x_1 is a fixed point of T and we have completed our proof.

Suppose $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exist a real number $h > 1$ such that

$$F(hH_p(Tx_0, Tx_1)) < F(H_p(Tx_0, Tx_1)) + \tau.$$

Now, from $F(p(x_1, Tx_1)) < F(H_p(Tx_0, Tx_1))$ and $Tx_0 \prec_2 Tx_1$, by this case we choose $x_2 \in Tx_1$ such that $F(p(x_1, x_2)) \leq F(H_p(Tx_0, Tx_1))$ and by use of Lemma 2.5 as a results, we get

$$p(x_1, x_2) \leq hH_p(Tx_0, Tx_1) < H_p(Tx_0, Tx_1) + \tau,$$

$$F(p(x_1, x_2)) \leq F(hH_p(Tx_0, Tx_1)) < F(H_p(Tx_0, Tx_1)) + \tau,$$

we apply (27) with $x = x_0, y = x_1$ to get

$$\begin{aligned} 2\tau + F(p(x_1,x_2)) &\leq 2\tau + F(H_p(Tx_0,Tx_1)) + \tau, \\ &\leq F\big(\mathbb{M}(x_0,x_1)\big) + \tau, \\ &= F\big(\alpha p(x_0,x_1) + \beta p(x_0,Tx_0) + \gamma p(x_1,Tx_1), \\ &\quad +\delta p(x_0,Tx_1) + Lp(x_1,Tx_0)\big) + \tau, \end{aligned}$$

from $x_1 \in Tx_0, x_2 \in Tx_1$, we have

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1), +\beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_2) + Lp(x_0,x_1)\big) + \tau, \end{aligned}$$

by $P4$ of Definition 2.1, we get

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1) + \beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_1) + \delta p(x_1,x_2) - \delta p(x_1,x_1) + Lp(x_1,x_1)\big) + \tau \\ &= F\big((\alpha+\beta+\delta+L)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big) \\ &\quad +\tau. \end{aligned}$$

using $P1$ and (1), we get

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1) + \beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_1) + \delta p(x_1,x_2)\big) + \tau \\ &= F\big((\alpha+\beta+\delta)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big) \\ &\quad +\tau. \end{aligned}$$

$$\tau + F(p(x_1,x_2)) \leq F\big((\alpha+\beta+\delta)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big). \tag{28}$$

As F is an increasing function, by (F_1) (28) implies

$$\begin{aligned} \Rightarrow F(p(x_1,x_2)) &< F\big((\alpha+\beta+\delta)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big) \\ \Rightarrow p(x_1,x_2) &< (\alpha+\beta+\delta)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2) \\ \Rightarrow (1-\gamma-\delta)p(x_1,x_2) &< (\alpha+\beta+\delta)p(x_0,x_1). \end{aligned} \tag{29}$$

From the assumption we have

$$\alpha+\beta+\gamma+\delta = 1, L = 0 \text{ implying } 1-\gamma-\delta = \alpha+\beta+\delta.$$

Hence, (29) implies

$$p(x_1,x_2) < p(x_0,x_1). \tag{30}$$

Using (30) in (28) we get

$$F\big(p(x_1,x_2)\big) \leq F\big(p(x_0,x_1)\big) - \tau. \tag{31}$$

If $x_2 \in Tx_2$ then x_2 is a fixed point of T and the proof is completed. However, suppose $x_2 \notin Tx_2$. As $Tx_0 \prec_2 Tx_1$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, we have $x_1 \preceq x_2 \Rightarrow Tx_1 \prec_2 Tx_2$. Let us choose $x_3 \in Tx_2$. Therefore, by Lemma 2.5, we get

$$p(x_2,x_3) \leq hH_p(Tx_1,Tx_2) < H_p(Tx_1,Tx_2) + \tau.$$

$$F(p(x_2,x_3)) \leq F(hH_p(Tx_1,Tx_2)) < F(H_p(Tx_1,Tx_2)) + \tau.$$

We apply (27) with $x = x_1, y = x_2$, we get

$$\begin{aligned} 2\tau + F(p(x_2, x_3)) &\leq 2\tau + F(H_p(Tx_1, Tx_2)) + \tau, \\ &\leq F\big(\mathbb{M}(x_1, x_2)\big) + \tau, \\ &= F\big(\alpha p(x_1, x_2), +\beta p(x_1, Tx_1) + \gamma p(x_2, Tx_2), \\ &\quad +\delta p(x_1, Tx_2) + Lp(x_2, Tx_1)\big) + \tau, \end{aligned}$$

Similar, one obtains,

$$\tau + F\big(p(x_2, x_3)\big) \leq \big((\alpha + \beta + \delta)p(x_1, x_2) + (\gamma + \delta)p(x_2, x_3)\big)$$

As F is an increasing function, by (F_1), we get

$$p(x_2, x_3) < p(x_1, x_2). \tag{32}$$

Using (32) in (28) and (31) we get

$$F\big(p(x_2, x_3)\big) \leq F\big(p(x_1, x_2)\big) - \tau \leq F\big(p(x_0, x_1)\big) - 2\tau. \tag{33}$$

Continuing in the same manner we get the sequence $\{x_n\}$ with $x_1 \prec x_2 \prec x_3 \dots$. If $x_n \in Tx_n$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the proof is completed. Suppose $x_n \notin Tx_n$ for all $x \in \mathbb{N}$. In this case we have

$$F\big(p(x_n, x_{n+1})\big) \leq F\big(p(x_0, x_1)\big) - n\tau. \tag{34}$$

We notice that (34) is identical to (17). Next, proceeding as in the proof of Theorem 3.5, we obtained that $\{x_n\}$ is Cauchy sequence. Also, since (X, p) is a complete partial metric space, there is $x^\star \in X$ such that $x_n \to x^\star$, and $p(x^\star, x^\star) = 0$.

We claim that $x^\star$ is a fixed point of T. We do this by showing that $p(x^\star, Tx^\star) = p(x^\star, x^\star) = 0$. Suppose $p(x^\star, Tx^\star) > 0$. Then there exists some $n_0 \in \mathbb{N}$ such that $p(x_n, Tx^\star) > 0$ for all $n > n_0$. By (25) we have

$$\begin{aligned} 2\tau + F\big(p(x_{n+1}, Tx^\star)\big) &\leq 2\tau + F\big(H_p(Tx_n, Tx^\star)\big) + \tau \\ &\leq F\big(\mathbb{M}(x_n, x^\star)\big) \\ &= F\big(\alpha p(x_n, x^\star) + \beta p(x_n, Tx_n) \\ &\quad + \gamma p(x^\star, Tx^\star) + \delta\big(p(x_n, Tx^\star) + p(x^\star, Tx_n)\big) + Lp(x^*, Tx_n) + \tau. \end{aligned} \tag{35}$$

Taking $n \to \infty$ in (35) and applying the fact that F is an increasing function, we get

$$\begin{aligned} 2\tau + F\big(p(x^\star, Tx^\star)\big) & \\ &\leq F\big(\alpha p(x^\star, x^\star) + \beta p(x^\star, Tx^\star) + \gamma p(x^\star, Tx^\star) + 2\delta p(x^\star, Tx^\star) \\ &\quad + Lp(x^*, Tx^*)\big) + \tau, \\ &\leq F\big((\alpha + \beta + \gamma + \delta + L)p(x^\star, Tx^\star)\big) + \tau, \\ 2\tau + F\big(p(x^\star, Tx^\star) &\leq F\big(p(x^\star, Tx^\star)\big) + \tau, \\ 2\tau + F\big(p(x^\star, Tx^\star) &\leq F\big(p(x^\star, Tx^\star)\big) + \tau, \\ F\big(p(x^\star, Tx^\star) &\leq F\big(p(x^\star, Tx^\star)\big) - \tau. \end{aligned}$$

Since $\tau > 0$, the above inequality yield a contradiction. Hence $p(x^\star, Tx^\star) = 0$ making $x^\star$ a fixed point of T. The proof is completed. □

Now, we give an example to illustrate the use of Theorem 3.12.

Example 3.13. Consider partial metric spaces (X, p), where set $X = \{0, 1, 2, ...\}$ and

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}.$$

for all $x, y \in X$. Let $(X, \preceq)$ be partial ordered set where

$$y \preceq x \implies x \geq y.$$

Define the multivalued function $T : X \to CB^p(X)$ as

$$Tx = \begin{cases} \{x-2, x-1\}, & \text{for } x \ \geq 2, \\ \\ \{0, x+1\}, & \text{for } x \in \{0,1\}. \end{cases}$$

We note that $x \geq 2$, $x \prec_1 Tx$, $x \preceq y \implies Tx \prec_2 Ty$ and $x \notin Tx$. We define $F \in \mathfrak{F}$ as $F(a) = \ln a + a$. The condition (27) becomes

$$\frac{H_p(Tx,Ty)}{\mathbb{M}(x,y)} e^{H_p(Tx,Ty) - \mathbb{M}(x,y)} \leq e^{-2\tau}. \tag{36}$$

We now calculate $H_p(Tx, Ty)$ for $x > y \geq 2$.

$$Tx = \{x-2, x-1, \}, Ty = \{y-2, y-1, \}.$$

$$p(x-1, y-2) = \frac{3x-y}{4} - \frac{1}{4},\ p(x-1, y-1) = \frac{3x-y}{4} - \frac{1}{2}.$$

$$p(x-2, y-2) = \frac{3x-y}{4} - 1,\ p(x-2, y-1) = \frac{3x-y}{4} - \frac{5}{4}.$$

$$\begin{aligned} p(x-2, Ty) &= \min\{p(x-2, a); a \in Ty\} \\ &= \min\{p(x-2, y-1), p(x-2, y-2)\}, \\ &= \min\{\frac{3x-y}{4} - 1, \frac{3x-y}{4} - \frac{5}{4}\}, \\ &= \frac{3x-y}{4} - \frac{5}{4}. \end{aligned}$$

In a similar manner we calculate

$$\begin{aligned} p(x-1, Ty) &= \frac{3x-y}{4} - \frac{1}{2}. \\ p(x-2, Tx) &= \frac{3x-y}{4} - 1. \\ p(y-1, Tx) &= \frac{3x-y}{4} - \frac{5}{2}. \end{aligned}$$

$$\begin{aligned} \delta_p(Tx, Ty) &= \max\{p(a, Ty); a \in Tx\} \\ &= \max\{p(x-2, Ty), p(x-1, Ty)\}, \\ &= \max\{\frac{3x-y}{4} - \frac{5}{4}, \frac{3x-y}{4} - \frac{1}{2}\}, \\ &= \frac{3x-y}{4} - \frac{1}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \delta_p(Ty, Tx) &= \max\{p(a, Tx); a \in Ty\} \\ &= \max\{p(y-2, Tx), p(y-1, Tx)\}, \\ &= \max\{\frac{3x-y}{4} - 1, \frac{3x-y}{4} - \frac{5}{4}\}, \\ &= \frac{3x-y}{4} - 1. \end{aligned}$$

$$\begin{aligned} H_p(Tx,Ty) &= \max\{\delta_p(Tx,Ty),\delta_p(Ty,Tx)\}, \\ &= \max\left\{\frac{3x-y}{4}-\frac{1}{2},\frac{3x-y}{4}-1\right\}, \\ &= \frac{3x-y}{4}-\frac{1}{2}. \end{aligned}$$

We note that

$$\mathbb{M}(x,y)=\alpha p(x,y)=p(x,y)=\frac{3x-y}{4}.$$

Applying to (36) we get

$$\frac{3x-y-2}{3x-y}e^{-\frac{1}{2}}\leq e^{-2\tau}.$$

which is true for $\tau=\frac{1}{4}$. The mapping has a fixed point at $x=0$. This shows that T is a multivalued Hardy-Rogers-type F-contraction with contractive factor τ. Hence, satisfy Theorem 3.12.

4. Some Applications

In this section, we will provide an application of our theorem for Hardy Rogers type contraction in ordered partial metric spaces. We will use Volterra type integral equation to illustrate the results. Consider the Volterra type integral equation :

$$x(t)=\ g(t)+\int_0^t f(t,s,x(s))ds,\ t\in[0,K], \tag{37}$$

where $K>0$. Let $X=C([0,K],\mathbb{R})$ be the space of all continuous functions defined on $[0,K]$. Notice that $(C([0,K])$ endowed with partial metric.

$$p(x,y)=\|x-y\|_\infty=\max_{t\in[0,K]}|x(t)-\ y(t)|, \tag{38}$$

is a complete partial metric space and X can be equipped with the partial order $\preccurlyeq$ given by $x,y\in X$, $(x\preceq y)\implies(x(t)\preccurlyeq y(t)$ and $\|x\|_\infty,\|\ y\|_\infty\leq 1)$, or $(x(t)=y(t))$ for all $t\in[0,K]$. It was shown by Nieto and Rodrigurz-Lopez [29] that $(X,\preccurlyeq)$ is regular. From Equation 37, x is the solution of $x^{'}(t)=f(t,y(s))$ satisfying initial condition $x(t_0)=x_0$ if and only if

$$x(t)=\ g(t)+\int_0^t f(t,s,x(s))ds,\ t\in[0,K]. \tag{39}$$

We consider Volterra integral equation as

$$\begin{cases} x^{'}(t)=\ f(t,x(s)), \\ x(t_0)=x_0. \end{cases}$$

Equation (39) may be formulated as a fixed point equation

$$x=Tx.$$

Let << be a partial order relation on $\mathbb{R}$. Define a mapping $T:X\to X$ by

$$Tx(t)=\ g(t)+\int_0^t f(t,s,x(s))ds,\ t\in[0,K]. \tag{40}$$

Theorem 4.1. *Let* $X=C([0,K]\times\mathbb{R},\mathbb{R})$ *for all value* $x,y\in X$ *;*

(i) *$f(t,s,x(s)) : \mathbb{R} \to \mathbb{R}$ is increasing for all $t \in [0,K]$ and for $x, y \in X$, $x \prec y \Leftrightarrow Tx \prec_1 Ty$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \prec_1 Tx_0$;*

$$x_0(t) \prec_1 \ g(t) + \int_0^t f(t,s,x(s))ds, \ t \in [0,K] :$$

(iii) *there exist $\tau \in [1,\infty]$ such that*

$$|f(t,s,x(s)) - f(t,s,y(s))| \le L(t,s)|x(s) - y(s)|,$$

where $L(t,s) = \alpha\tau e^{-2\tau}$, for all $t \in [0,K]$ and $x, y \in \mathbb{R}$ with $x \prec y$.

(*iv*) *if $x_n \to x$ is a non decreasing sequence in X, for all n and*

$$2\tau + F(Hp(Tx,Ty)) \le F(\mathbb{M}(x,y)), \tag{41}$$

where

$$\mathbb{M}(x,y) = \alpha \ p(x,y) + \beta p(x,Tx) + \gamma p(y,Ty) + \delta p(x,Ty) + \ Lp(y,Tx)$$

for $x, y \in X$, $\tau > 0$, $\alpha, \beta, \delta \le$, $L \ge 0, \alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \ne 1$. Then T has a fixed point. Therefore Equation (37) has at least one fixed point $x \in X$.

Proof: Using (i), let K be a kernel function on $G = [0,K] \times [0,K]$ and is increasing on G. Then is bounded function on G. For $t, s \in [0,K]$, where $K : [0,K] \times [0,K] \times \mathbb{R} \to \mathbb{R}$ and $f(t), x(s), y(s) : [0,K] \to \mathbb{R}$ are continuous functions. Hence $x \prec y \Leftrightarrow Tx \prec_2 Ty$. From ($ii$) take $x_0 \in X$ as an initial point on $[0,K]$ we note that there is point x^* which is the limit of iterative sequence $(x_0, x_1, x_2, x_3, ...x_{n+1})$ where x_0 is any continuous function on X and for $(n = 0, 1, 2, ...)$, we have

$$x_{n+1}(t) = \ g(t) + \int_0^t f(t,s,x(s))ds, \ t \in [0,K].$$

Suppose we start with $x_0 = 1 = g(t)$ we get the following iteration of a sequence

$$\begin{aligned}
x_1(t) &= 1 + \int_{t_0}^t 1.ds = 1 + t, \\
x_2(t) &= 1 + \int_0^t x_1(s)ds = 1 + t + \frac{t^2}{2}, \\
x_3(t) &= 1 + \int_0^t x_2(s)ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}, \\
... &= ... \\
x_n(t) &= 1 + \int_0^t x_{n-1}(s)ds = \sum_{n=0}^n \frac{t^n}{n!}.
\end{aligned}$$

The limit of this sequence

$$\lim_{n\to\infty} x_n(t) = e^t, \ \forall \ t \in [0,K].$$

For arbitrary $x \in X$, define $|x|_\tau = \max_{t\in[0,K]} \{|x|e^{-\tau t}\}$, where $\tau \ge 1$ is taken randomly. Since $\|.\|_\tau$ is a Banach space norm equivalent to maximum norm and $(X, \|.\|_\tau)$ endowed with a metric d_τ given as below by O'Regan and Petrusel [30]. Also one can see [37, 40]

$$d_\tau(x,y) \ = \ \max_{t\in[0,K]} \{|x(t) - \ y(t)|\}e^{-\tau t}, \tag{42}$$

for all $x, y \in X$. Next, assume that X endowed with partial metric defined by Paesano and Vetro [33] as follows:

$$p_\tau(x,y) = \begin{cases} d_\tau(x,y) & if \ \ \|x\|_\infty, \|y\|_\infty \le 1, \\ d_\tau(x,y) + \tau & otherwise. \end{cases}$$

Therefore (X,p) is 0 - complete partial metric. Also

$$p_\tau(x,y)=\begin{cases} d_\tau(x,y) & if\ \|x\|_\infty,\|y\|_\infty\le 1\ or\ (\|y\|_\infty>1),\\ d_\tau(x,y)+\tau & otherwise,\end{cases}$$

and consequently (X,p^s_τ) is 0 - complete. Consider partial order defined on X by $x,y\in C([0,K]\times\mathbb{R}^n,\mathbb{R})$, $x\preceq y$ if and only if $x(t)\preceq\ y(t)$, for $t\in[0,K]$. Then $(X,\|.\|_\tau),\preceq)$ is complete partial ordered metric space and for any increasing sequence $\{x_n\}$ in X, it has the limit $x^*\in X$, we have $x_n\preceq x^*$ for any $t\in[0,K]$.

Assume that the initial condition of Equation (27) are $x_0(t)=x_0$ for $t\in[0,K]$ has a unique solution. The solution of Volterra equation is the fixed point of T. Thus, (i) and (ii) satisfied. From condition (iv) the operator T is surely increasing. Now we have to justify condition of Equation (40) by comparing $Tx\prec_2 Ty$ and $x,y\in X$ such that $x\preceq y$. On using condition (i) and (iii), we reach on the following results

$$\begin{aligned}
|Tx(t)-Ty(t)| &= |\int_0^t f(t,s,x(s)ds)-\int_0^t f(t,s,y(s)ds|\\
&\le \int_0^t |f(t,s,x(s))-f(t,s,y(s))|ds\\
&\le \alpha\tau e^{-2\tau}\int_0^t |x(s)-y(s)|ds\\
&\le \alpha\tau e^{-2\tau}\int_0^t |x(s)-y(s)|e^{-\tau s}e^{\tau s}ds\\
&\le \alpha\tau e^{-2\tau}\int_0^t e^{\tau s}|x(s)-y(s)|e^{-\tau s}ds\\
&\le \alpha\tau e^{-2\tau}\left(\int_0^t e^{\tau s}ds\right)|x(s)-y(s)|e^{-\tau s}\\
&\le \alpha\tau e^{-2\tau}\left(\int_0^t e^{\tau s}ds\right)\|x(s)-y(s)\|_\tau\\
&\le \alpha\tau e^{-2\tau}\frac{e^{\tau t}}{\tau}\ \|x(s)-y(s)\|_\tau,\\
&\le \alpha e^{-2\tau}\ \|x(s)-y(s)\|_\tau.
\end{aligned}$$

After all, since $x,y\in X$ such that $x\preccurlyeq y$, from $\|x\|_\tau,\|y\|_\tau\le 1$, we have

$$|Tx(t)-Ty(t)|e^{-\tau t}\ \le\ \alpha e^{-2\tau}\ \|x-y\|_\tau,$$

or equivalently,

$$p_\tau(Tx,Ty)\ \le\ \alpha e^{-2\tau}\ p_\tau(x,y).$$

Taking naturai logarithm to both sides, we obtain

$$\ln(p_\tau(Tx,Ty))\ \le\ \ln(\alpha e^{-2\tau}\ p_\tau(x,y)),$$

which is equivalently,

$$2\tau+F(p_\tau(Tx,Ty))\ \le\ F(\alpha p_\tau(x,y)).$$

for $\alpha=1$, we have

$$2\tau+F(p_\tau(Tx,Ty))\ \le\ F(p_\tau(x,y)).$$

Through observation for a function $F:\mathbb{R}^+\to\mathbb{R}$ defined by $F(a)=\ln a$, for all $x\in X$, belong to $\mathfrak{F}$ and so we deduced that operator T satisfies condition of Equation (39) with $\alpha=1$ and $\beta=\gamma=\delta=0, L=0$. Hence by Theorem 4.1, we obtained that operator T has a fixed point $x^*\in X$, which is the solution of Volterra integral Equation (37).

5. Conclusion

The main contribution of this study to fixed point theory is the fixed point for multivalued result given in Theorem 3.5, Theorem 3.12 and Theorem Theorem 4.1. These theorems provides the fixed point conditions for a substantial class of Hady-Rogers contraction mappings on various abstract spaces.

We prove a fixed point theorem for multi-valued mapping using α-F-contraction in partial metric spaces. Furthermore, a fixed point theorem is proved for F-Hardy-Rogers multi-valued mappings in ordered partial metric spaces. Specifically, this paper motivated by the works by Ali and Kamran [3], Sgroi and Vetro [40] and Kumar [22]. We also provided illustrative examples and an application to integral equations. which generalizes some well-known results in the literature. These results have some applications in many areas of applied mathematics, especially in the Volterra type integral equation.

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