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Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions

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Abstract

The purpose of this paper, is studying the existence and nonexistence of positive solutions to a class of a following tripled system of fractional differential equations.

$$\begin{cases} D^{\alpha}u(\zeta)+a(\zeta)f(\zeta,v(\zeta),\omega(\zeta))=0, & u(0)=0, \quad u(1)=\int_0^1\phi(\zeta)u(\zeta)d\zeta,\\ D^{\beta}v(\zeta)+b(\zeta)g(\zeta,u(\zeta),\omega(\zeta))=0, & v(0)=0, \quad v(1)=\int_0^1\psi(\zeta)v(\zeta)d\zeta,\\ D^{\gamma}\omega(\zeta)+c(\zeta)h(\zeta,u(\zeta),v(\zeta))=0, & \omega(0)=0, \quad \omega(1)=\int_0^1\eta(\zeta)\omega(\zeta)d\zeta, \end{cases}$$

where $0\le\zeta\le 1$, $1<\alpha,\beta,\gamma\le 2$, $a,b,c\in C((0,1),[0,\infty))$, $\phi,\psi,\eta\in L^1[0,1]$ are nonnegative and $f,g,h\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$ and D is the standard Riemann-Liouville fractional derivative.
Also, we provide some examples to demonstrate the validity of our results.

Keywords: Tripled system, fractional differential equation, integral boundary conditions, existence and nonexistence of positive solutions.
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1. Introduction

E. Karapınar and coauthors obtained some fixed point results and applied them to proving the existence and uniqueness of positive solutions for functional boundary value problem (see [1]-[17], [15], [26]). In recent years, some systems of nonlinear fractional differential equations were examined by many authors, [18]-[25] and other references. In [27], Su investigated some conditions for the existence of solutions for a coupled system of two-point fractional boundary value problem.
In [31] the authors studied the existence and nonexistence of positive solutions to boundary values problem for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{cases} D^{\alpha}u(\zeta)+a(\zeta)f(\zeta,v(\zeta))=0, \qquad u(0)=0, \quad u(1)=\int_0^1\phi(\zeta)u(\zeta)d\zeta, \\ D^{\beta}v(\zeta)+b(\zeta)g(\zeta,u(\zeta))=0, \qquad v(0)=0, \quad v(1)=\int_0^1\psi(\zeta)v(\zeta)d\zeta, \end{cases} \tag{1}$$

where $0\le\zeta\le 1$, $1<\alpha,\beta\le 2$, $a,b\in C((0,1),[0,\infty))$, $\phi,\psi\in L^1[0,1]$ are nonnegative and $f,g\in C([0,1]\times[0,\infty),[0,\infty))$ and D is the standard Riemann-Liouville fractional derivative.

In this paper we study the equations

$$\begin{cases} D^{\alpha}u(\zeta)+a(\zeta)f(\zeta,v(\zeta),\omega(\zeta))=0, \qquad u(0)=0, \quad u(1)=\int_0^1\phi(\zeta)u(\zeta)d\zeta, \\ D^{\beta}v(\zeta)+b(\zeta)g(\zeta,u(\zeta),\omega(\zeta))=0, \qquad v(0)=0, \quad v(1)=\int_0^1\psi(\zeta)v(\zeta)d\zeta, \\ D^{\gamma}\omega(\zeta)+c(\zeta)h(\zeta,u(\zeta),v(\zeta))=0, \qquad \omega(0)=0, \quad \omega(1)=\int_0^1\eta(\zeta)\omega(\zeta)d\zeta, \end{cases} \tag{2}$$

where $0\le\zeta\le 1$, $1<\alpha,\beta,\gamma\le 2$, $a,b,c\in C((0,1),[0,\infty))$, $\phi,\psi,\eta\in L^1[0,1]$ are nonnegative and $f,g,h\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$ and D is the standard Riemann-Liouville fractional derivative.

Definition 1.1. *[28, 29] The Riemann-Liouville fractional derivative for a continuous function f is defined by*

$$D^{\nu}f(\tau)=\frac{1}{\Gamma(n-\nu)}(\frac{d}{d\tau})^n\int_0^{\tau}\frac{f(\zeta)}{(\tau-\zeta)^{\nu-n+1}}d\zeta, \qquad (n=[\nu]+1)$$

where the right-hand side is point-wise defined on $(0,\infty)$.

Definition 1.2. *[28, 29] Let $[a,b]$ be an interval in $\mathbb{R}$ and $\nu>0$. The Riemann-Liouville fractional order integral of a function $f\in L^1([a,b],\mathbb{R})$ is defined by*

$$I_a^{\nu}f(\tau)=\frac{1}{\Gamma(\nu)}\int_a^{\tau}\frac{f(\zeta)}{(\tau-\zeta)^{1-\nu}}d\zeta,$$

whenever the integral exists.

Lemma 1.3. *(Nonlinear Differentiation of Leray-Schauder Type, [32]). Let E be a Banach space with $C\subset E$ closed and convex. Let U be a relatively open subset of C with $0\in U$ and let $T:U\to C$ be a continuous and compact mapping. Then either*
(a) the mapping T has a fixed point in U,
or (b) there exist $u\in\partial U$ and $\lambda\in(0,1)$ with $u=\lambda Tu$.

Lemma 1.4. *(Fixed-Point Theorem of Cone Expansion and Compression of Norm Type, See [33]). Let P be a cone of real Banach space E, and let Ω_1 and Ω_2 be two bounded open sets in E such that $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$. Let operator $A:P\cap(\overline{\Omega_2}-\Omega_1)\to P$ be completely continuous operator. Suppose that one of the two conditions holds:*

(i_1) $\|Au\| \leq \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \geq \|u\|$, *for all* $u \in P \cap \partial\Omega_2$;
(i_2) $\|Au\| \geq \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \leq \|u\|$, *for all* $u \in P \cap \partial\Omega_2$.

Then A *has at least one fixed point in* $P \cap (\overline{\Omega_2} - \Omega_1)$.

Lemma 1.5. *Assume that* $\int_0^1 \zeta^{\nu-1}\phi(\zeta)d\zeta \neq 1$. *Then for any* $\sigma \in C[0,1]$, *the unique solution of boundary value problem*

$$\begin{cases} D^\nu u(\zeta) + \sigma(\zeta) = 0, & 0 < \tau < 1, \\ u(0) = 0, & u(1) = \int_0^1 \phi(\zeta)u(\zeta)d\zeta, \end{cases}$$

is given by

$$u(\zeta) = \int_0^1 G_{1\nu}(\zeta,\tau)\sigma(\tau)d\tau$$

where

$$G_{1\nu}(\zeta,\tau) = G_{2\nu}(\zeta,\tau) + G_{3\nu}(\zeta,\tau), \quad (\zeta,\tau) \in [0,1]\times[0,1], \tag{3}$$

with

$$G_{2\nu}(\zeta,\tau) = \frac{1}{\Gamma(\nu)} \begin{cases} \zeta^{\nu-1}(1-\tau)^{\nu-1} - (\zeta-\tau)^{\nu-1}, 0 \leq \tau \leq \zeta \leq 1, \\ \zeta^{\nu-1}(1-\tau)^{\nu-1}, \qquad 0 \leq \zeta \leq \tau \leq 1 \end{cases}$$

and

$$G_{3\nu}(\zeta,\tau) = \frac{\zeta^{\nu-1}}{1-\int_0^1 \phi(\zeta)\zeta^{\nu-1}d\zeta} \int_0^1 G_{2\nu}(\zeta,\tau)\phi(\zeta)d\zeta.$$

We call $G = (G_{1\nu}, G_{1\nu'}, G_{1\nu''})$ the Green's functions of the boundary value problem (2).

Lemma 1.6. *If* $\int_1^0 \varphi(\tau)\tau^{\nu-1}d\tau \in [0,1)$, *the function* $G_{1\nu}(\tau,\zeta)$ *defined by* (3) *satisfies*

(i_1) $G_{1\nu}(\tau,\zeta) \geq 0$ *is continuous for all* $\tau,\zeta \in [0,1]$, $G_{1\nu}(\tau,\zeta) > 0$ *for all* $\tau,\zeta \in (0,1)$;

(i_2) $G_{1\nu}(\tau,\zeta) \leq G_{1\nu}(\zeta)$ *for each* $\tau,\zeta \in (0,1)$, *and*
$\min_{\tau\in[\theta,1-\theta]} G_{1\nu}(\tau,\zeta) \geq G_{1\nu}(\zeta)$, *where* $\theta \in (0,\frac{1}{2})$ *and*

$$G_{1\nu}(\zeta) = G_{2\nu}(\zeta,\zeta) + G_{3\nu}(1,\zeta), \qquad \Upsilon_\nu = \theta^{\nu-1}.$$

We will discuss the existence of positive solutions for boundary value problem (2). First of all, we define the Banach space

$$X = \{u(\zeta) | u(\zeta) \in C[0,1]\} \quad \textit{endowed with the norm} \|u\|_X = \max_{\zeta\in[0,1]} |u|,$$

$$Y = \{v(\zeta) | v(\zeta) \in C[0,1]\} \quad \textit{endowed with the norm} \|v\|_Y = \max_{\zeta\in[0,1]} |v|,$$

$$Z = \{\omega(\zeta) | \omega(\zeta) \in C[0,1]\} \quad \textit{endowed with the norm} \|\omega\|_Z = \max_{\zeta\in[0,1]} |\omega|.$$

For $(u,v,\omega) \in X\times Y\times Z$, let $\|(u,v,\omega)\|_{X\times Y\times Z} = \max\{\|u\|_X, \|v\|_Y, \|\omega\|_Z\}$. Clearly, $(X\times Y\times Z, \|(u,v,\omega)\|_{X\times Y\times Z})$ is a Banach space. Define, $P = \{(u,v,\omega) \in X\times Y\times Z | u(\zeta) \geq 0, v(\zeta) \geq 0, \omega(\zeta) \geq 0\}$, then the cone $P \subset X\times Y\times Z$. Let $J_\theta = [\theta, 1-\theta]$ for $\theta \in (0,\frac{1}{2})$ and

$$K = \left\{ \begin{array}{l} (u,v,\omega) \in P, \min_{\tau\in J_\theta} u(\tau) \geq \Upsilon_\alpha \|u\|, \\ \min_{\tau\in J_\theta} v(\tau) \geq \Upsilon_\beta \|v\|, \min_{\tau\in J_\theta} \omega(\tau) \geq \Upsilon_\gamma \|\omega\| \end{array} \right\},$$

$$K_r = \left\{ (u,v,\omega) \in K, \|(u,v,\omega)\| \leq r \right\},$$

$$\partial K_r = \left\{ (u,v,\omega) \in K, \|(u,v,\omega)\| = r \right\}.$$

From Lemma 1.5, we can obtain the following lemma.

Lemma 1.7. *Suppose that $f(t,v,\omega)$, $g(t,u,\omega)$ and $h(t,u,v)$ are continuous, then $(u,v,\omega)\in X\times Y\times Z$ is a solution of B.V.P (2) if and only if*
$(u,v,\omega)\in X\times Y\times Z$ is a solution of the integral equations

$$\begin{cases} u(\zeta)=\int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),\omega(\tau))d\tau, \\ v(\zeta)=\int_0^1 G_{1\beta}(\zeta,\tau)b(\tau)g(\tau,u(\tau),\omega(\tau))d\tau, \\ \omega(\zeta)=\int_0^1 G_{1\gamma}(\zeta,\tau)c(\tau)h(\tau,u(\tau),v(\tau))d\tau. \end{cases}$$

Let $T:X\times Y\times Z\to X\times Y\times Z$ be the operator defined as

$$T(u,v,\omega)=\Big(\int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),\omega(\tau))d\tau, \tag{4}$$

$$\int_0^1 G_{1\beta}(\zeta,\tau)b(\tau)g(\tau,u(\tau),\omega(\tau))d\tau, \int_0^1 G_{1\gamma}(\zeta,\tau)c(\tau)h(\tau,u(\tau),v(\tau))d\tau\Big)$$
$$=:(T_1u(\zeta),T_2v(\zeta),T_3\omega(\zeta)), \tag{5}$$

then by Lemma (1.7), the fixed point of operator T coincides with the solution of system (2).

Lemma 1.8. *Let $f(\tau,v,v)$, $g(\tau,u,u)$ and $h(\tau,\omega,\omega)$ be continuous on*
$[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$, then $T:P\to P$, $T:K\to K$ defined by (4) are completely continuous.

Proof. Since Lemma (1.8) is similar to Lemma (1.8) in [2] and [30] we omit the proof Lemma (1.8). □

Theorem 1.9. *Assume that $a(\tau)$, $b(\tau)$ and $c(\tau)$, are continuous on*
$(0,1)\to[0,+\infty)$ and $f(\tau,v(\tau),\omega(\tau))$, $g(\tau,u(\tau),\omega(\tau))$ and $h(\tau,u(\tau),v(\tau))$ are continuous on $[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$, and there exist three positive functions $m(\tau)$, $n(\tau)$ and $k(\tau)$ that satisfy

(L_1) $f(\tau,v_2,\omega_2)-f(\tau,v_1,\omega_1)\le m(\tau)\max\{|v_2-v_1|,|\omega_2-\omega_1|\}$,
(L_2) $g(\tau,u_2,\omega_2)-g(\tau,u_1,\omega_1)\le k(\tau)\max\{|u_2-u_1|,|\omega_2-\omega_1|\}$,
(L_3) $h(\tau,u_2,v_2)-h(\tau,u_1,v_1)\le n(\tau)\max\{|v_2-v_1|,|u_2-u_1|\}$,
for $\tau\in(0,1)$, $v_1,v_2,\omega_1,\omega_2,u_1,u_2\in(0,+\infty)$.

Then system (2) has a unique positive solution if

$$\rho=\int_0^1 G_{1\alpha}(\tau)a(\tau)m(\tau)d\tau<1,$$
$$\theta=\int_0^1 G_{1\beta}(\tau)b(\tau)k(\tau)d\tau<1,$$
$$\kappa=\int_0^1 G_{1\gamma}(\tau)c(\tau)n(\tau)d\tau<1. \tag{6}$$

Proof. For all $(u,v,\omega)\in P$ by the nonnegativeness of $G(\zeta,\tau)$ and $a(\tau)$, $b(\tau)$, $c(\tau)$, $f(\tau,v(\tau),\omega(\tau))$, $g(\tau,u(\tau),\omega(\tau))$, $h(\tau,u(\tau),v(\tau))$, we have
$T(u,v,\omega)\ge 0$. Hence, $T(P)\subset P$. From Lemma 1.6, we obtain

$$\|T_1v_2-T_1v_1\|=\max_{\zeta\in[0,1]}|T_1v_2-T_1v_1|$$
$$=\max_{\zeta\in[0,1]}|\Big(\int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)[f(\tau,v_2(\tau),\omega(\tau))-f(\tau,v_1(\tau),\omega(\tau))]\Big)d\tau|$$
$$\le\Big(\int_0^1 G_{1\alpha}(\tau)a(\tau)m(\tau)\Big)d\tau|\|v_2-v_1\|=\rho\|v_2-v_1\| \tag{7}$$

Similarly,

$$\|T_2u_2 - T_2u_1\| \leq \theta\|u_2 - u_1\| \tag{8}$$

and

$$\|T_3\omega_2 - T_3\omega_1\| \leq \kappa\|\omega_2 - \omega_1\| \tag{9}$$

From (7), (8) to (9), we get

$$\|T(u_2, v_2, \omega_2) - T(u_1, v_1, \omega_1)\| \leq max(\rho, \theta, \kappa)\|(u_2, v_2, \omega_2) - (u_1, v_1, \omega_1)\|$$

From Lemma (1.8), T is completely continuous, by Banach fixed point theorem, the operator T has a unique fixed point in P, which is the unique positive solution of system (2). This completes the proof. □

Theorem 1.10. *Assume that* $a(\tau)$, $b(\tau)$ *and* $c(\tau)$, *are continuous on* $(0,1) \to [0,+\infty)$ *and* $f(\tau, v(\tau), \omega(\tau))$, $g(\tau, u(\tau), \omega(\tau))$ *and* $h(\tau, u(\tau), v(\tau))$ *are continuous on* $[0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$, *and satisfy*

(L_4) $|f(\tau, v(\tau), \omega(\tau))| \leq a_1(\tau) + a_2(\tau)\max\{|v(\tau)|, |\omega(\tau)|\}$,
(L_5) $|g(\tau, u(\tau), \omega(\tau))| \leq b_1(\tau) + b_2(\tau)\max\{|u(\tau)|, |\omega(\tau)|\}$,
(L_6) $|h(\tau, u(\tau), v(\tau))| \leq c_1(\tau) + c_2(\tau)\max\{|v(\tau)|, |u(\tau)|\}$,
(L_7) $A_1 = \int_0^1 G_{1\alpha}(\tau)a(\tau)a_2(\tau)d\tau < 1, B_1 = \int_0^1 G_{1\alpha}(\tau)a(\tau)a_1(\tau)d\tau < \infty$,
(L_8) $A_2 = \int_0^1 G_{1\beta}(\tau)b(\tau)b_2(\tau)d\tau < 1, B_2 = \int_0^1 G_{1\beta}(\tau)b(\tau)b_1(\tau)d\tau < \infty$,
(L_9) $A_3 = \int_0^1 G_{1\gamma}(\tau)c(\tau)c_2(\tau)d\tau < 1, B_3 = \int_0^1 G_{1\gamma}(\tau)c(\tau)c_1(\tau)d\tau < \infty$.

Then the system (2) *has at least one positive solution* (u, v, ω) *in*

$$Q = \left\{ (u, v, \omega) \in P : \|(u, v, \omega)\| < \min(\tfrac{A_1}{1-B_1}, \tfrac{A_2}{1-B_2}, \tfrac{A_3}{1-B_3}) \right\}.$$

Proof. Let $Q = \left\{ (u, v, \omega) \in P : \|(u, v, \omega)\| < r \right\}$ with

$$r = \min(\frac{A_1}{1-B_1}, \frac{A_2}{1-B_2}, \frac{A_3}{1-B_3}).$$

Define the operator $T : Q \to P$ as (4). Let $(u, v, \omega) \in Q$, that is, $\|(u, v, \omega)\| < r$. Then

$$\begin{aligned}\|T_1u\| &= \max_{\zeta\in[0,1]} |\int_0^1 G_{1\alpha}(\zeta, \tau)a(\tau)f(\tau, v(\tau), \omega(\tau))d\tau| \\ &\leq \int_0^1 G_{1\alpha}(\tau)a(\tau)(a_1(\tau) + a_2(\tau)|v(\tau)|)d\tau \\ &\leq \int_0^1 G_{1\alpha}(\tau)a(\tau)a_1(\tau)d\tau + \int_0^1 G_{1\alpha}(\tau)a(\tau)a_2(\tau)d\tau\|v(\tau)\| \\ &= B_1 + A_1\|v(\tau)\| \leq r.\end{aligned}$$

Similarly, $\|T_2v\| \leq r$, $\|T_3\omega\| \leq r$. So, $T(u, v, \omega) \leq (r, r, r)$ and hence $T(u, v, \omega) \in \overline{Q}$. From Lemma (1.8), we have $T : Q \to \overline{Q}$ is completely continuous. Consider the eigenvalue problem

$$(u, v, \omega) = \lambda T(u, v, \omega), \qquad \lambda \in (0, 1). \tag{10}$$

Under the assumption that (u, v, ω) is a solution of (10) for $\lambda \in (0,1)$, we have

$$\begin{aligned}\|u\| = \|\lambda T_1 u\| &= \max_{\zeta\in[0,1]} \Big| \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau, v(\tau), \omega(\tau))d\tau \Big| \\ &\le \int_0^1 G_{1\alpha}(\tau)a(\tau)(a_1(\tau) + a_2(\tau)|v(\tau)|)d\tau \\ &\le \int_0^1 G_{1\alpha}(\tau)a(\tau)a_1(\tau)d\tau + \int_0^1 G_{1\alpha}(\tau)a(\tau)a_2(\tau)d\tau\|v(\tau)\| \\ &= B_1 + A_1\|v(\tau)\| \le r.\end{aligned}$$

Similarly, $\|v\| = \|T_2\lambda v\| \le r$, $\|\omega\| = \|T_3\lambda\omega\| \le r$, so, $\|(u, v, \omega)\| \le r$, which shows that $(u, v, \omega) \in \partial Q$. By Lemma 1.3, T has a fixed point in Q. We complete the proof of theorem 1.10. □

Remark 1.11. *In the following we need the following assumptions and some notations:*

(B_1) $a, b, c \in C((0,1), [0,\infty))$, $a(\tau) \neq 0$, $b(\tau) \neq 0$, $c(\tau) \neq 0$ *on any subinterval of* $(0,1)$ *and*

$$0 < \int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau < \infty,$$

$0 < \int_0^1 G_{1\beta}(\tau)b(\tau)d\tau < \infty$ *and* $0 < \int_0^1 G_{1\gamma}(\tau)c(\tau)d\tau < \infty$ *where* $G_{1\alpha}$, $G_{1\beta}$ *and* $G_{1\gamma}$ *are defined in Lemma* 1.6*;*

(B_2) $f, g, h \in C([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$ *and* $f(\zeta, 0, 0) = 0$, $g(\zeta, 0, 0) = 0$ *and* $h(\zeta, 0, 0) = 0$ *uniformly with respect to* ζ *on* $[0,1]$*;*

(B_3) $\lambda, \mu, \nu \in [0,1)$ *where* λ, μ, ν *is defined as follows:*

$$\lambda = \int_0^1 \phi(\zeta)\zeta^{\alpha-1}d\zeta, \quad \mu = \int_0^1 \psi(\zeta)\zeta^{\beta-1}d\zeta \quad \text{and} \quad \nu = \int_0^1 \varphi(\zeta)\zeta^{\gamma-1}d\zeta.$$

let

$$f^\delta = \limsup_{u\to\delta} \max_{\zeta\in[0,1]} \frac{f(\zeta,u,u)}{u}, \qquad f_\delta = \liminf_{u\to\delta} \min_{\zeta\in[0,1]} \frac{f(\zeta,u,u)}{u},$$

where δ *denotes* 0 *or* ∞, *and*

$$\sigma_1 = \int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau, \quad \sigma_2 = \int_0^1 G_{1\beta}(\tau)b(\tau)d\tau \ \text{ and } \sigma_3 = \int_0^1 G_{1\gamma}(\tau)c(\tau)d\tau.$$

Theorem 1.12. *Assume that* $(B_1)-(B_3)$ *hold. And supposes that one of the following conditions is satisfied:*

(H_1) $f_0 > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$ *and* $f^\infty < \frac{1}{\sigma_1}$ *(particularly,* $f^0 = \infty$ *and* $f^\infty = 0$*);*
$g_0 > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$ *and* $g^\infty < \frac{1}{\sigma_2}$ *(particularly,* $g^0 = \infty$ *and* $g^\infty = 0$*);*
$h_0 > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$ *and* $h^\infty < \frac{1}{\sigma_3}$ *(particularly,* $h^0 = \infty$ *and* $h^\infty = 0$*).*

(H_2) *There exist two constants* r_2, R_2 *with* $0 < r_2 \le R_2$ *such that* $f(\zeta, ., .)$, $g(\zeta, ., .)$ *and* $h(\zeta, ., .)$ *are nondecreasing on* $[0, R_2]$ *for all* $\zeta \in [0,1]$,

$$f(\zeta, \Upsilon_\alpha r_2, \Upsilon_\alpha r_2) \ge \frac{r_2}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau},$$

$$g(\zeta, \Upsilon_\beta r_2, \Upsilon_\beta r_2) \ge \frac{r_2}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau},$$

$$h(\zeta, \Upsilon_\gamma r_2, \Upsilon_\gamma r_2) \ge \frac{r_2}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$$

and $f(\zeta, R_2, R_2) \le \frac{R_2}{\sigma_1}$, $g(\zeta, R_2, R_2) \le \frac{R_2}{\sigma_2}$, $h(\zeta, R_2, R_2) \le \frac{R_2}{\sigma_3}$ *for all* $\zeta \in [0,1]$. *Then boundary value problem* (2) *has at least one positive solution.*

Proof. Let T be cone preserving completely continuous that is defined by (4).

$Case 1$. The condition (H_1) holds. Considering $f_0 > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$, there exists $r_1 > 0$ such that $f(t,v,v) = (f_0 - \varepsilon_1)v$, for all $t \in [0,1]$, $v \in [0, r_1]$, where $\varepsilon_1 > 0$, satisfies

$$(f_0 - \varepsilon_1)\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau \geq 1.$$

Then, for $t \in [0,1], (u,v,\omega) \in \partial K_{r_1}$, we get

$$\begin{aligned} T_1 v(t) &= \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\ &\geq \Upsilon_\alpha \int_0^1 G_{1\alpha}(\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\ &\geq \Upsilon_\alpha \int_0^1 G_{1\alpha}(\tau)a(\tau)(f_0-\varepsilon_1)v(\tau)d\tau \\ &\geq (f_0-\varepsilon_1)\Upsilon_\alpha^2 \int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau \|v\| \\ &\geq \|v\|. \end{aligned}$$

Similarly, we have $T_2\omega(t) \geq \|\omega\|$, $T_3 u(t) \geq \|u\|$ that is
$(u,v,\omega) \in \partial K_{r_1}$ implies that

$$\|T(u,v,\omega)\| \geq \|(u,v,\omega)\|. \tag{11}$$

On the other hand, for $f^\infty < 1/\sigma_1$, there exists $\overline{R}_1 > 0$ such that $f(t,v,v) = (f_\infty + \varepsilon_2)v$, for $t \in [0,1]$, $v \in (R_1, +\infty)$, where $\varepsilon_2 > 0$ satisfies $\sigma_1(f^\infty + \varepsilon_2) = 1$. Set $M = max_{t\in[0,1], v\in[0,R_1]} f(t,v,v)$, then $f(t,v,v) = M + (f^\infty + \varepsilon_2)v$. Choose
$R_1 > max\{r_1, \overline{R}_1, M\sigma_1(1-\sigma_1(f^\infty+\varepsilon_2))^{-1}\}$. Then, for $t \in [0,1]$, $(u,v,\omega) \in \partial K_{R_1}$, we get

$$\begin{aligned} T_1 v(t) &= \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\ &\leq \int_0^1 G_{1\alpha}(\tau)a(\tau)(M+(f^\infty+\varepsilon_2))v(\tau)d\tau \\ &\leq M\int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau + \int_0^1 G_{1\alpha}(\tau)a(\tau)(f^\infty+\varepsilon_2)d\tau\|v\| \\ &\leq R_1 - \sigma_1(f^\infty+\varepsilon_2)R_1 + (f^\infty+\varepsilon_2)\sigma_1\|v\| \\ &\leq R_1. \end{aligned}$$

Similarly, we have $T_3 u(t) \leq \|u\|$, $T_2\omega(t) \leq \|\omega\|$ that is
$(u,v,\omega) \in \partial K_{R_1}$ implies that

$$\|T(u,v,\omega)\| \leq \|(u,v,\omega)\|. \tag{12}$$

$Case 2$. The condition (H_2) holds. For $(u,v,\omega) \in K$, from the definition of K, we obtain that

$$min_{t\in J_\theta} u(t) \geq \Upsilon_\alpha\|u\|, min_{t\in J_\theta} v(t) \geq \Upsilon_\beta\|v\|, min_{t\in J_\theta}\omega(t) \geq \Upsilon_\gamma\|\omega\|.$$

Therefore, for $(u,v,\omega) \in \partial K_{r_2}$, we have $\|(u,v,\omega)\| = r_2$ for $t \in J_\theta$. From (H_2), we have

$$\begin{aligned}
T_1 v(t) &= \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\
&\geq \Upsilon_\alpha \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\
&\geq \Upsilon_\alpha \frac{r_2}{\Upsilon_\alpha \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau} \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau \\
&= r_2.
\end{aligned}$$

Similarly, we have $T_3u(t) \geq r_2$, $T_2\omega(t) \geq r_2$ that is $(u,v,\omega) \in \partial K_{r_2}$ implies that

$$\|T(u,v,\omega)\| \geq \|(u,v,\omega)\| \tag{13}$$

On the other hand, for $(u,v,\omega) \in \partial K_{R_2}$, we have that $(u,v,\omega) = R_2$ for $t \in [0,1]$, from (H_2), we obtain

$$\begin{aligned}
T_1 v(t) &= \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\
&\leq \Upsilon_\alpha \int_0^1 G_{1\alpha}(\tau)a(\tau)f(\tau,v(\tau),v(\tau))d\tau \\
&\leq \frac{R_2}{\sigma_1}\Upsilon_\alpha \int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau \\
&= R_2.
\end{aligned}$$

Similarly, we have $T_3u(t) \leq R_2$, $T_2\omega(t) \leq R_2$ that is $(u,v,\omega) \in \partial K_{R_2}$ implies that

$$\|T(u,v,\omega)\| \leq \|(u,v,\omega)\|. \tag{14}$$

Applying Lemma 1.4 to (11) and (12), or (13) and (14), yields that T has a fixed point $(\overline{u},\overline{v},\overline{\omega}) \in \overline{K}_{r,R}$ or $(\overline{u},\overline{v},\overline{\omega}) \in \overline{K}_{r_i,R_i}(i=1,2)$ with $\overline{u}(t) = \Upsilon_\alpha\|u\| > 0$, $\overline{v}(t) = \Upsilon_\beta\|\overline{v}\| > 0$ and $\overline{\omega}(t) = \Upsilon_\gamma\|\overline{\omega}\| > 0$. Thus it follows that boundary value problems (1.1) has a positive solution $(\overline{u},\overline{v},\overline{\omega})$. We complete the proof of Theorem 1.12.

□

Similarly, we have the following result.

Theorem 1.13. *Assume that $(B_1)-(B_3)$ hold. And supposes that the following three conditions are satisfied:*

(H_3) $f^0 < \frac{1}{\sigma_1}$ *and* $f_\infty > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$ *(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*);*
$g^0 < \frac{1}{\sigma_2}$ *and* $g_\infty > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$ *(particularly,* $g^0 = 0$ *and* $g_\infty = \infty$*);*
$h^0 < \frac{1}{\sigma_3}$ *and* $h_\infty > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$ *(particularly,* $h^0 = 0$ *and* $h_\infty = \infty$*).*

Then boundary value problem (2) *has at least one positive solution.*

Theorem 1.14. *Assume that $(B_1)-(B_3)$ hold. And supposes that the following two conditions are satisfied:*

(H_4) $f_0 > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$ *and* $f_\infty > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$
(particularly, $f^0 = f_\infty = \infty$*);*
$g_0 > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$ *and* $g_\infty > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$
(particularly, $g^0 = g_\infty = \infty$*);*
$h_0 > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$ *and* $h_\infty > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$
(particularly, $h^0 = h_\infty = \infty$*).*

(H_5) *there exists* $b > 0$ *such that*

$$\max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_b} f(\zeta,u,u) < b/\sigma_1, \max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_b} g(\zeta,v,v) < b/\sigma_2$$

and

$$\max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_b} h(\zeta,\omega,\omega) < b/\sigma_3.$$

Then boundary value problem (2) *has at least two positive solutions* (u_1, v_1, ω_1), (u_2, v_2, ω_2), *which satisfy*

$$0 < \|(u_1, v_1, \omega_1)\| < b < \|(u_2, v_2, \omega_2)\|. \tag{15}$$

Proof. We consider condition (H_4). Choose r, R with $0 < r < b < R$.
If $f_0 > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$, $g_0 > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$ and $h_0 > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$, then similar to the proof of (11), we have

$$\|T(u,v,\omega)\| \geq \|(u,v,\omega)\|,$$

for

$$(u,v,\omega) \in \partial K_r. \tag{16}$$

If $f_\infty > \frac{1}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau}$, $g_\infty > \frac{1}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau}$ and $h_\infty > \frac{1}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau}$, then similar to the proof of (3.6), we have

$$\|T(u,v,\omega)\| \geq \|(u,v,\omega)\|, \quad for \quad (u,v,\omega) \in \partial K_R. \tag{17}$$

On the other hand, together with (H_5), $(u,v,\omega) \in \partial K_b$, we have

$$\begin{aligned} T_1 v(\zeta) &= \int_0^1 G_{1\alpha}(\zeta,\tau)a(\tau)f(\tau,v(\tau),\omega(\tau))d\tau \\ &\leq \int_0^1 G_{1\alpha}(\tau)a(\tau)f(\tau,v(\tau),\omega(\tau))d\tau \\ &< \frac{b}{\sigma_1}\int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau \\ &= b. \end{aligned}$$

Similarly, we have $T_3 u(\zeta) < b$, $T_2 \omega(\zeta) < b$, that is $(u,v,\omega) \in \partial K_b$ implies that

$$\|T(u,v,\omega)\| < \|(u,v,\omega)\|. \tag{18}$$

Applying Lemma 1.4 to $(16)-(18)$ yields that T has a fixed point $(u_1, v_1, \omega_1) \in \overline{\partial K_{r,b}}$, and a fixed point $(u_2, v_2, \omega_2) \in \overline{\partial K_{b,R}}$. Thus it follows that boundary value problem (2) has at least two positive solutions (u_1, v_1, ω_1) and (u_2, v_2, ω_2). Noticing (18), we have $(u_1, v_1, \omega_1) \neq b$ and $(u_2, v_2, \omega_2) \neq b$. Therefore (15) holds, and the proof is complete. □

Similarly, we have the following results.

Theorem 1.15. *Assume that $(B_1)-(B_3)$ hold. And supposes that the following conditions is satisfied:*

(H_6) $f^0 < 1/\sigma_1$ *and* $f^\infty < 1/\sigma_1$*;* $g^0 < 1/\sigma_2$ *and* $g^\infty < 1/\sigma_2$*;* $h^0 < 1/\sigma_3$ *and* $h^\infty < 1/\sigma_3$*.*

(H_7) *there exists $B > 0$ such that*

$$\max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_B} f(\zeta, u, u) > \frac{B}{\Upsilon_\alpha \int_\theta^{1-\theta} G_{1\alpha}(\tau)a(\tau)d\tau},$$

$$\max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_B} g(\zeta, v, v) > \frac{B}{\Upsilon_\beta \int_\theta^{1-\theta} G_{1\beta}(\tau)b(\tau)d\tau},$$

$$\max_{\zeta\in[0,1],(u,v,\omega)\in\partial K_B} h(\zeta, \omega, \omega) > \frac{B}{\Upsilon_\gamma \int_\theta^{1-\theta} G_{1\gamma}(\tau)c(\tau)d\tau},$$

Then boundary value problem (2) has at least two positive solutions (u_1, v_1, ω_1), (u_2, v_2, ω_2), which satisfy

$$0 < \|(u_1, v_1, \omega_1)\| < B < \|(u_2, v_2, \omega_2)\|.$$

Theorem 1.16. *Assume that $(B_1)-(B_3)$ hold. If there exist $3l$ positive numbers d_k, D_k, $k = 1, 2, \cdots, l$ with*

$$d_1 < \Upsilon_\alpha D_1 < D_1 < d_2 < \Upsilon_\alpha D_2 < D_2 < \cdots < d_l < \Upsilon_\alpha D_l < D_l,$$

$$d_1 < \Upsilon_\beta D_1 < D_1 < d_2 < \Upsilon_\beta D_2 < D_2 < \cdots < d_l < \Upsilon_\beta D_l < D_l$$

and

$$d_1 < \Upsilon_\gamma D_1 < D_1 < d_2 < \Upsilon_\gamma D_2 < D_2 < \cdots < d_l < \Upsilon_\gamma D_l < D_l,$$

such that

(H_8)

$$f(\zeta, u, u) > \frac{d_k}{\Upsilon_\alpha \int_0^1 G_{1\alpha}(\tau)a(\tau)d\tau},$$

for

$$(\zeta, u, u) \in [0, 1] \times [\Upsilon_\alpha d_k, d_k] \times [\Upsilon_\alpha d_k, d_k]$$

and

$$f(\zeta, u, u) = \sigma_1^{-1} D_k$$

for

$$(\zeta, u, u) \in [0, 1] \times [\Upsilon_\alpha D_k, D_k] \times [\Upsilon_\alpha D_k, D_k], k = 1, 2, \cdots, l.$$

Also

$$g(\zeta, v, v) > \frac{d_k}{\Upsilon_\beta \int_0^1 G_{1\beta}(\tau)b(\tau)d\tau},$$

for

$$(\zeta, v, v) \in [0,1] \times [\Upsilon_\beta d_k, d_k] \times [\Upsilon_\beta d_k, d_k]$$

and

$$g(\zeta, v, v) = \sigma_1^{-1} D_k$$

for

$$(\zeta, v, v) \in [0,1] \times [\Upsilon_\beta D_k, D_k] \times [\Upsilon_\beta D_k, D_k], k = 1, 2, \cdots, l.$$

And also

$$h(\zeta, \omega, \omega) > \frac{d_k}{\Upsilon_\gamma \int_0^1 G_{1\gamma}(\tau) c(\tau) d\tau};$$

for

$$(\zeta, \omega, \omega) \in [0,1] \times [\Upsilon_\gamma d_k, d_k] \times [\Upsilon_\gamma d_k, d_k]$$

and

$$h(\zeta, \omega, \omega) = \sigma_1^{-1} D_k$$

for

$$(\zeta, \omega, \omega) \in [0,1] \times [\Upsilon_\gamma D_k, D_k] \times [\Upsilon_\gamma D_k, D_k], k = 1, 2, \cdots, l.$$

Then boundary value problem (2) has at least l positive solutions (u_k, v_k, ω_k) which satisfy

$$d_k < \|(u_k, v_k, \omega_k)\| < D_k, \quad k = 1, 2, \cdots, l.$$

Theorem 1.17. *Assume that $(B_1) - (B_3)$ hold. If there exist $3l$ positive numbers d_k, D_k, $k = 1, 2, \cdots, l$ with $d_1 < D_1 < d_2 < D_2 < \cdots < d_l < D_l$ such that*
(H_9) $f(\zeta, ., .)$, $g(\zeta, ., .)$ and $h(\zeta, ., .)$ are nondecreasing on $[0, D_l]$ for all $t \in [0,1]$.
(H_{10})

$$f(\zeta, \Upsilon_\alpha d_k, \Upsilon_\alpha d_k) \geq \frac{d_k}{\Upsilon_\alpha \int_\theta^{1-\theta} G_{1\alpha}(\tau) a(\tau) d\tau},$$

and

$$f(\zeta, D_k, D_k) \leq \sigma_1^{-1} D_k, k = 1, 2, \cdots, l.$$

Also

$$g(\zeta, \Upsilon_\beta d_k, \Upsilon_\beta d_k) \geq \frac{d_k}{\Upsilon_\beta \int_\theta^{1-\theta} G_{1\beta}(\tau) b(\tau) d\tau},$$

and

$$g(\zeta, D_k, D_k) \leq \sigma_1^{-1} D_k, k = 1, 2, \cdots, l.$$

And also

$$h(\zeta, \Upsilon_\gamma d_k, \Upsilon_\gamma d_k) \geq \frac{d_k}{\Upsilon_\gamma \int_\theta^{1-\theta} G_{1\gamma}(\tau) c(\tau) d\tau};$$

and

$$h(\zeta, D_k, D_k) \leq \sigma_1^{-1} D_k, k = 1, 2, \cdots, l.$$

Then boundary value problem (2) *has at least* l *positive solutions* (u_k, v_k, ω_k) *which satisfy*

$$d_k < \|(u_k, v_k, \omega_k)\| < D_k, \quad k = 1, 2, \cdots, l.$$

Now the nonexistence of positive solutions for boundary value problem (2).

Theorem 1.18. *Suppose* $(B_1) - (B_3)$ *hold,* $f(\zeta, u, u) < \sigma_1^1 u$, $g(\zeta, v, v) < \sigma_2^1 v$ *and* $h(\zeta, \omega, \omega) < \sigma_3^1 \omega$ *for all* $\zeta \in [0, 1]$, $u > 0$, $v > 0$ *and* $\omega > 0$ *then boundary value problem* (2) *has no positive solution.*

Proof. Assume to the contrary that (u, v, ω) is a positive solution of the boundary value problem (2). Then $(u, v, \omega) \in K$, $u > 0$, $v > 0$ and $\omega > 0$ for $\zeta \in [0, 1]$, and

$$\begin{aligned}
\|u\| = \max_{\zeta\in[0,1]} |u(\zeta)| &= \max_{\zeta\in[0,1]} \int_0^1 G_{1\alpha}(\zeta, \tau) a(\tau) f(\tau, v(\tau), v(\tau)) d\tau \\
&\leq \int_0^1 G_{1\alpha}(\tau) a(\tau) f(\tau, v(\tau), v(\tau)) d\tau \\
&< \int_0^1 G_{1\alpha}(\tau) a(\tau) \frac{\|v\|}{\sigma_1} d\tau \\
&= \frac{1}{\sigma_1} \int_0^1 G_{1\alpha}(\tau) a(\tau) d\tau \|v\| \\
&= \|v\|.
\end{aligned}$$

Similarly, $\|v\| < \|u\|, \|v\| < \|\omega\|$ and $\|\omega\| < \|v\|$, which is a contradiction, and Theorem is received. □

Theorem 1.19. *Assume that* $(B_1) - (B_3)$ *hold, and*

$$\begin{aligned}
f(\zeta, u, u) &> \frac{u}{\Upsilon_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(\tau) a(\tau) d\tau}, \\
g(\zeta, v, v) &> \frac{v}{\Upsilon_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(\tau) b(\tau) d\tau}, \\
h(\zeta, \omega, \omega) &> \frac{\omega}{\Upsilon_\gamma^2 \int_\theta^{1-\theta} G_{1\gamma}(\tau) c(\tau) d\tau},
\end{aligned}$$

for all $t \in [0, 1]$, $u > 0, v > 0, \omega > 0$, *then boundary value problem* (2) *has no positive solution.*

Example 1.20. *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases}
D^{\frac{5}{3}} u(\tau) + \frac{\tau}{1+\tau} |sinv(\tau)| = 0, D^{\frac{3}{2}} v(\tau) + \frac{\tau}{1+\tau} |sin\omega(\tau)| = 0, D^{\frac{4}{3}} \omega(\tau) \\
+\frac{\tau}{1+\tau} |sinu(\tau)| = 0, \quad 0 < \tau < 1, \\
\\
u(0) = 0, u(1) = \int_0^1 \tau u(\tau) d\tau, v(0) = 0, v(1) = \int_0^1 \tau v(\tau) d\tau, \omega(0) = 0, \\
\omega(1) = \int_0^1 \tau \omega(\tau) d\tau.
\end{cases} \tag{19}$$

Set $e(\tau), f(\tau), g(\tau) \in [0, +\infty)$ *and* $\tau \in [0, 1]$, *then we have*

$$\begin{aligned}
\Big|\frac{\tau}{1+\tau} |sine(\tau)| - \frac{\tau}{1+\tau} |sinf(\tau)|\Big| &\leq \frac{\tau}{1+\tau} \big|e(\tau) - f(\tau)\big|, \\
\Big|\frac{\tau}{1+\tau} |sinf(\tau)| - \frac{\tau}{1+\tau} |sing(\tau)|\Big| &\leq \frac{\tau}{1+\tau} \big|f(\tau) - g(\tau)\big|, \\
\Big|\frac{\tau}{1+\tau} |sing(\tau)| - \frac{\tau}{1+\tau} |sine(\tau)|\Big| &\leq \frac{\tau}{1+\tau} \big|g(\tau) - e(\tau)\big|.
\end{aligned}$$

Therefore,

$$\rho = \int_0^1 G_{1\alpha}(\tau)a(\tau)m(\tau)d\tau \leq \int_0^1 G_{1\alpha}(\tau)d\tau,$$
$$\theta = \int_0^1 G_{1\beta}(\tau)b(\tau)k(\tau)d\tau \leq \int_0^1 G_{1\beta}(\tau)d\tau,$$
$$\kappa = \int_0^1 G_{1\gamma}(\tau)c(\tau)n(\tau)d\tau \leq \int_0^1 G_{1\gamma}(\tau)d\tau.$$

With the use of Theorem 1.4*, B.V.P* (19) *has a unique positive solution.*

Example 1.21. *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases} D^{\frac{5}{3}}u(\tau) + [v(\tau)]^a = 0, D^{\frac{5}{3}}v(\tau) + [\omega(\tau)]^b = 0, D^{\frac{5}{3}}\omega(\tau) + [u(\tau)]^c = 0, \\ 0 < \tau < 1, \\ \\ u(0) = 0, u(1) = \int_0^1 \tau u(\tau)d\tau, v(0) = 0, v(1) = \int_0^1 \tau v(\tau)d\tau, \omega(0) = 0, \\ \omega(1) = \int_0^1 \tau\omega(\tau)d\tau. \end{cases} \tag{20}$$

Let $f(\tau, v, v) = va$*,* $g(\tau, u, u) = ub$ *and* $h(\tau, \omega, \omega) = \omega c$*,* $0 < a, b, c < 1$*. It is easy to see that* $(B_1) - (B_3)$ *hold. By simple computation, we have* $f_0 = g_0 = h_0 = \infty$ *and* $f^\infty = g^\infty = h^\infty = 0$*. Thus it follows that problem* (20) *has a positive solution by* (H_1)*.*

Example 1.22. *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases} D^{\frac{3}{2}}u(\tau) + [v(\tau)]^{a'} = 0, D^{\frac{3}{2}}v(\tau) + [\omega(\tau)]^{b'} = 0, D^{\frac{3}{2}}\omega(\tau) + [u(\tau)]^{c'} = 0, \\ 0 < \tau < 1, \\ \\ u(0) = 0, u(1) = \int_0^1 \tau u(\tau)d\tau, v(0) = 0, v(1) = \int_0^1 \tau v(\tau)d\tau, \omega(0) = 0, \\ \omega(1) = \int_0^1 \tau\omega(\tau)d\tau. \end{cases} \tag{21}$$

Let $f(\tau, v, v) = va'$*,* $g(\tau, u, u) = ub'$*and* $h(\tau, \omega, \omega) = \omega c'$*,* $0 < a', b', c' < 1$*. It is easy to see that* $(B_1) - (B_3)$ *hold. By simple computation, we have* $f^0 = g^0 = h^0 = 0$ *and* $f_\infty = g_\infty = h_\infty = \infty$*. Thus it follows that problem* (21) *has a positive solution by* (H_3)*.*

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