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Linear Convex Combination Estimators and Comparisons

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Abstract

In this paper, we introduce two linear convex combination estimators by using known estimators such as ordinary least squares, ridge and Liu estimators and examine the predictive performance of these estimators. Furthermore, a numerical example is examined to compare these estimators under the prediction mean squared error criterion.

***Keywords*:** Biased estimation; Ridge estimator; Linear convex combination; Liu estimator; Prediction mean square error.

Lineer Konveks Kombinasyon Tahmin Ediciler ve Karşılaştırmalar

Öz

Bu makalede, en küçük kareler, ridge ve Liu tahmin ediciler gibi bilinen tahmin edicilerle öngörü performansını karşılaştırmak için iki lineer konveks kombinasyon tahmin edicisi

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tanımlanmıştır. Ayrıca, öngörü hata kareleri ortalaması kriterine göre bu tahmin edicilerin karşılaştırılmaları bir sayısal örnek ile incelenmiştir.

***Anahtar Kelimeler*:** Yanlı tahmin; Ridge tahmin edici; Lineer konveks kombinasyon; Liu tahmin edici; Öngörü hata kareleri ortalaması.

1. Introduction

Consider the following multiple linear regression model:

$$y = X\beta + \varepsilon, \quad (1)$$

where y is an $nx1$ vector of responses, X is an nxp full column rank matrix of explanatory variables, β is a $px1$ vector of unknown parameters, and ε is an $nx1$ vector of random errors with $iid(0, \sigma^2)$.

The ordinary least squares (OLS) estimator is given by

$$\hat{\beta} = (X'X)^{-1}X'y. \quad (2)$$

In the presence of multicollinearity, the OLS estimator is unstable and gives unreliable information. As biased alternatives, ridge, Liu, and two-parameter estimators can be handled in this context.

Hoerl and Kennard [1] proposed the ordinary ridge regression (ORR) estimator which is given by

$$\hat{\beta}(k) = (X'X + kI)^{-1}X'y, \qquad k \geq 0\ , \quad (3)$$

where k is the biasing parameter. The ORR estimator was commonly used in applied researches. For example; Askin [2] suggested several approaches for extending estimation results to forecasting with multicollinearity, Montgomery and Friedman [3] examined several biased estimation methods for forecasting and prediction with multicollinearity.

Liu [4] defined the following alternative biased estimator dealing with multicollinearity

$$\begin{aligned}\hat{\beta}(d) &= (X'X + I)^{-1}(X'y + d\hat{\beta}) \\ &= (X'X + I)^{-1}(X'X + dI)\hat{\beta},\ 0 < d < 1, \end{aligned} \quad (4)$$

where d is the biasing parameter. $\hat{\beta}(d)$ is called the Liu estimator by Akdeniz and Kaçıranlar [5]. Liu estimator has an advantage over the ORR estimator because it is a linear function of d and it has smaller mean square error (MSE) than the OLS estimator. Sakallıoğlu et al. [6] compared the performance of Liu estimator with the ORR and the iterative estimators using the matrix MSE

(MMSE) criterion. In the literature, Liu and Liu-type estimators were widely used in linear models.

Furthermore, Özkale and Kaçıranlar [7] introduced a new two-parameter estimator (TPE) by grafting the contraction estimator into the modified ridge estimator proposed by Swindel [8]. This estimator is given by

$$\hat{\beta}(k,d) = (X'X + kI)^{-1}\left(X'y + kd\hat{\beta}\right),\ k \geq 0, 0 < d < 1. \tag{5}$$

$\hat{\beta}(k,d)$ is a two-parameter variation of the Liu estimator. Özkale [9] has also noted that $\hat{\beta}(k,d)$ can also be demonstrated as

$$\hat{\beta}(k,d) = d\hat{\beta} + (1-d)\hat{\beta}(k). \tag{6}$$

The TPE is a convex combination of the OLS and the ORR estimator. It is also called the 'affine combination type' estimator by Özkale [9]. Using the mixed estimation method suggested by Theil [10] and Theil and Goldberger [11], we also derive $\hat{\beta}(k,d)$. Similar to the ORR and Liu estimator, $\hat{\beta}(k,d)$ was used both theoretically and practically by researchers in various fields. Özbay and Kaçıranlar [12] introduced Almon TPE based on the TPE procedure for the distiributed lag models. Özbay and Kaçıranlar [13] introduced a new two-parameter-weighted mixed estimator (TPWME) by unifying the weighted mixed estimator of Schaffrin and Toutenburg [14] and the TPE. Tekeli et al. [15] introduced new algorithms using genetic algorithm (GA) for estimating the biasing parameters of TPE. Çetinkaya and Kaçıranlar [16] introduced new TPE for negative binomial regression (NBR) and Poisson regression (PR) models by unifying the TPE.

Gruber [17, 18] demonstrated that $\hat{\beta}(k,d)$ is a special case of the linear Bayes, mixed and minimax estimators. This new estimator is a general estimator which includes the OLS, the ORR, the Liu, and the contraction estimators as special cases. We have the following properties:

1. $\lim_{d\to 1}\hat{\beta}(k,d) = \hat{\beta}$ and $\lim_{k\to 0}\hat{\beta}(k,d) = \hat{\beta}$

2. $\lim_{d\to 0}\hat{\beta}(k,d) = \hat{\beta}(k)$

3. For $k = 1$, we get the Liu estimator, $\hat{\beta}(1,d) = \hat{\beta}(d)$

4. $\hat{\beta}(k,d)$ has the following alternative forms

$$\begin{aligned}\hat{\beta}(k,d) &= [I + k(X'X)^{-1}]^{-1}(\hat{\beta} - d\hat{\beta}) + d\hat{\beta}\\ &= (X'X + kI)^{-1}(X'X + kdI)\hat{\beta}.\end{aligned}$$

From this representation, it is clear that $\lim_{k\to\infty}\hat{\beta}(k,d) = d\hat{\beta}$, which is the contraction estimator [19]. In this sense, $\hat{\beta}(k,d)$ overcomes the disadvantage of the contraction estimator.

Then, Gruber [18] demonstrated that the Liu-type estimator can be given as follows:

$$\hat{\beta}_{LOB} = d\hat{\beta} + (1-d)\hat{\beta}_b, \tag{7}$$

where d is a biasing parameter, $0 < d < 1$ and $\hat{\beta}_b$ is the linear Bayes estimator (see in details, p. 3741, Eqn. (3.7), Eqn. (3.8) for $\hat{\beta}_b$ and p. 3742, Eqn. (3.12) for $\hat{\beta}_{LOB}$).

Gruber [18] showed how the Liu-type estimator is optimal according to the Zellner's balanced loss function (ZBLF) criterion and compared the efficiency of the Liu-type estimator to the OLS estimator in terms of the MSE and the ZBLF criteria. A convex combination of two estimators can be useful when both estimators appear to be appropriate in a specific situation. Following the Liu-type estimator in Eqn. (7), we consider linear convex combination estimators taking the ORR and the Liu estimators as the special cases of $\hat{\beta}_b$. Then, the linear convex combination of the OLS estimator and the ORR estimator (LOR) can be given as follows:

$$\hat{\beta}_{LOR} = \hat{\beta}(k,d) = d\hat{\beta} + (1-d)\hat{\beta}(k)\ ,\ k \geq 0,\ 0 < d < 1. \tag{8}$$

Similarly, we can define another linear convex combination of the OLS estimator and the Liu estimator as follows:

$$\hat{\beta}_{LOL} = \hat{\beta}(d,\gamma) = \gamma\hat{\beta} + (1-\gamma)\hat{\beta}(d), \tag{9}$$

where γ is an arbitrary scalar and $0 \leq \gamma \leq 1$. Also, $\hat{\beta}_{LOL} = \hat{\beta}(d,\gamma)$ in Eqn. (9) includes $\hat{\beta}$ and $\hat{\beta}(d)$ as special cases:

1. $\lim_{\gamma\to 1}\hat{\beta}(d,\gamma) = \hat{\beta}$,

2. $\lim_{\gamma\to 0}\hat{\beta}(d,\gamma) = \hat{\beta}(d)$.

Friedman and Montgomery [20] compared the predictive performance (PP) of the ORR, OLS and the principal component (PC) estimators according to the prediction mean square error (PMSE) criterion. Later, Özbey and Kaçıranlar [21] compared the Liu estimator with the OLS, PC and ORR estimators. Dawoud and Kaçıranlar [22] examined the PP of biased regression predictors with correlated errors. Dawoud and Kaçıranlar [23, 24] evaluated the PP of the r-k and r-d class estimators and they also focused on evaluating the PP of the Liu-type estimator which is defined by Liu [25]. This estimator is different from Gruber's Liu-type estimator which is given in Eqn. (7). Following Özbey and Kaçıranlar [21] and Dawoud and Kaçıranlar [22], Li et al. [26] evaluated the PP of the principal component two-parameter estimator which is defined by Chang and Yang [27].

As a consequence, since $\hat{\beta}_{LOR} = \hat{\beta}(k,d)$ and $\hat{\beta}_{LOL} = \hat{\beta}(d,\gamma)$ are more general than the ORR and the Liu estimators, respectively. Therefore, the PP of the LOR and the LOL estimators are examined in the sense of the PMSE criterion. To examine the theoretical results, a numerical example study is conducted.

2. Comparisons of the Prediction Mean Squared Errors

We can obtain the PMSE of the LOR and the LOL estimators. The PMSE of a predictor $\hat{y}_0$ is given by

$$PMSE = E(y_0 - \hat{y}_0)^2, \tag{10}$$

where y_0 is the value to be predicted. Let J represents the PMSE. J is the sum of the variance (V) and the squared bias (B):

$$J = V + B. \tag{11}$$

The variance and the bias can be given as follows:

$$V(y_0 - \hat{y}_0) = V(y_0) + V(\hat{y}_0), \tag{12}$$

and

$$Bias = E(y_0 - \hat{y}_0). \tag{13}$$

Now, we consider the following canonical form of the model (1)

$$y = Z\alpha + \varepsilon, \tag{14}$$

where $\alpha = U'\beta$ and $Z = XU$. Then the OLS estimator of α is

$$\hat{\alpha} = (Z'Z)^{-1}Z'y = \Lambda^{-1}Z'y, \tag{15}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is the matrix of the eigenvalues of $Z'Z$ and for $i = 1,2,\ldots,p$ λ_i's are in descending order. Its PMSE is given by

$$J_{OLS} = \sigma^2 \left(1 + \sum_{i=1}^{p} \frac{z_{0i}^2}{\lambda_i}\right), \tag{16}$$

where z_0 is the orthonormalized point for $\hat{y}_0$. Since $\hat{\alpha}$ is unbiased, we have

$$J_{OLS} = V_{OLS}. \tag{17}$$

The ridge estimator of α is

$$\hat{\alpha}_k = (Z'Z + kI)^{-1}Z'y = (\Lambda + kI)^{-1}Z'y,\ \ k \geq 0, \tag{18}$$

and its PMSE is

$$J_k = \sigma^2 \left(1 + \sum_{i=1}^{p} \frac{z_{0i}^2 \lambda_i}{a_i^2}\right) + k^2 \left(\sum_{i=1}^{p} \frac{z_{oi}\alpha_i}{a_i}\right)^2, \tag{19}$$

where $a_i = \lambda_i + k$. The Liu estimator of α is

$$\begin{aligned} \hat{\alpha}_d &= (Z'Z + I)^{-1}(Z'y + d\hat{\alpha}) \\ &= (\Lambda + I)^{-1}(\Lambda + dI)\hat{\alpha},\ \ 0 < d < 1, \end{aligned} \tag{20}$$

and its PMSE is

$$J_d = \sigma^2 \left(1 + \sum_{i=1}^{p} \frac{z_{0i}^2 c_i^2}{\lambda_i b_i^2}\right) + (1-d)^2 \left(\sum_{i=1}^{p} \frac{z_{oi}\alpha_i}{b_i}\right)^2, \tag{21}$$

where $b_i = \lambda_i + 1$ and $c_i = \lambda_i + d$. The LOR estimator or TPE of α is

$$\begin{aligned} \hat{\alpha}_{LOR} &= \left[d(Z'Z)^{-1} + (1-d)(Z'Z + kI)^{-1}\right]Z'y \\ &= [d\Lambda^{-1} + (1-d)(\Lambda + kI)^{-1}]Z'y, k \geq 0. \end{aligned} \tag{22}$$

The variance and bias of the prediction error of the LOR estimator are given by respectively

$$\begin{aligned} V_{\mathrm{LOR}}(y_0 - \hat{y}_0) &= V(y_0) + V_{\mathrm{LOR}}(\hat{y}_0) \\ &= \sigma^2 + V(z_0'\hat{\alpha}_{\mathrm{LOR}}) \\ &= \sigma^2 \left(1 + \sum_{i=1}^{p} \frac{[(1-d)\lambda_i + da_i]^2 z_{0i}^2}{\lambda_i a_i^2}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} Bias_{\mathrm{LOR}} &= E(y_0 - \hat{y}_0) = z_0'\alpha - z_0'E(\hat{\alpha}_{\mathrm{LOR}}) \\ &= k(1-d)\sum_{i=1}^{p} \frac{z_{oi}\alpha_i}{a_i}. \end{aligned} \tag{24}$$

So, the squared bias is

$$B_{\mathrm{LOR}} = Bias_{\mathrm{LOR}}^2 = k^2(1-d)^2 \left(\sum_{i=1}^{p} \frac{z_{oi}\alpha_i}{a_i}\right)^2. \tag{25}$$

By summing up the variance and the squared bias of the LOR estimator we obtain

$$\begin{aligned} J_{\mathrm{LOR}} &= V_{\mathrm{LOR}} + B_{\mathrm{LOR}} \\ &= \sigma^2 \left(1 + \sum_{i=1}^{p} \frac{[(1-d)\lambda_i + da_i]^2 z_{0i}^2}{\lambda_i a_i^2}\right) + k^2(1-d)^2 \left(\sum_{i=1}^{p} \frac{z_{oi}\alpha_i}{a_i}\right)^2. \end{aligned} \tag{26}$$

The LOL estimator of α is

$$\begin{aligned}\hat{\alpha}_{\mathrm{LOL}} &= [\gamma I + (1-\gamma)(Z'Z+I)^{-1}(Z'Z+dI)]\hat{\alpha} \\ &= [\gamma I + (1-\gamma)(\Lambda+I)^{-1}(\Lambda+dI)]\Lambda^{-1}Z'y,\ 0<d<1. \end{aligned} \quad (27)$$

The variance of the prediction error of the LOL estimator is

$$\begin{aligned}V_{\mathrm{LOL}}(y_0-\hat{y}_0) &= V(y_0)+V_{\mathrm{LOL}}(\hat{y}_0) \\ &= \sigma^2 + V(z_0'\hat{\alpha}_{\mathrm{LOL}}) \\ &= \sigma^2\left(1+\sum_{i=1}^{p}\frac{[\gamma b_i+(1-\gamma)c_i]^2 z_{0i}^2}{\lambda_i b_i^2}\right). \end{aligned} \quad (28)$$

Similarly, the bias, the squared bias and PMSE of the prediction error of the LOL estimator are given by respectively

$$\begin{aligned}Bias_{\mathrm{LOL}} &= E(y_0-\hat{y}_0) = z_0'\alpha - z_0'E(\hat{\alpha}_{\mathrm{LOL}}) \\ &= (1-\gamma)(1-d)\sum_{i=1}^{p}\frac{z_{oi}\alpha_i}{b_i}, \end{aligned} \quad (29)$$

$$B_{\mathrm{LOL}} = Bias_{\mathrm{LOL}}^2 = (1-\gamma)^2(1-d)^2\left(\sum_{i=1}^{p}\frac{z_{oi}\alpha_i}{b_i}\right)^2, \quad (30)$$

and

$$\begin{aligned}J_{\mathrm{LOL}} &= V_{\mathrm{LOL}} + B_{\mathrm{LOL}} \\ &= \sigma^2\left(1+\sum_{i=1}^{p}\frac{[\gamma b_i+(1-\gamma)c_i]^2 z_{0i}^2}{\lambda_i b_i^2}\right) + (1-\gamma)^2(1-d)^2\left(\sum_{i=1}^{p}\frac{z_{oi}\alpha_i}{b_i}\right)^2. \end{aligned} \quad (31)$$

3. Comparisons of Prediction Mean Squared Errors in Two Dimensional Space

We will study the PP of the LOR and the LOL estimators. Considering a two-dimensional space, a single prediction point (z_{01}, z_{02}) is to be predicted, the ratio z_{02}^2/z_{01}^2 can be obtained and used for a reference point in their comparisons. α_1^2 will be set to zero because non-zero values of α_1^2 increase only the intercept values for J_k, J_d, J_{LOR} and J_{LOL} but leave the curve for J_{OLS} unchanged. So, comparisons of J_{LOR} with J_{OLS} and J_k and J_{LOL} with J_{OLS} and J_d will be made.

Theorem 1.

a) If $\alpha_2^2 > \frac{\sigma^2(a_2^2 - ((1-d)\lambda_2 + da_2)^2)}{\lambda_2 k^2 (1-d)^2}$, then

-$J_{LOR} < J_{OLS}$ for $a_1^2 < ((1-d)\lambda_1 + da_1)^2$,

-$J_{LOR} < J_{OLS} \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2)$ for $a_1^2 > ((1-d)\lambda_1 + da_1)^2$.

b) If $\alpha_2^2 < \frac{\sigma^2\left(a_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2 (1-d)^2}$, then

-$J_{LOR} < J_{OLS}$ for $a_1^2 > ((1\text{-}d)\lambda_1 + da_1)^2$,

-$J_{LOR} < J_{OLS} \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2)$ for $a_1^2 < ((1-d)\lambda_1 + da_1)^2$,

where

$$f_1(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1}-\frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right)}{\left(\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2}+\frac{k^2(1-d)^2\alpha_2^2}{a_2^2}-\frac{\sigma^2}{\lambda_2}\right)}. \tag{32}$$

Proof. If the LOR estimator is better than $\hat{\alpha}$, we have $J_{LOR} < J_{OLS}$. That is,

$$\sigma^2 + \sigma^2\left[\frac{((1-d)\lambda_1 + da_1)^2 z_{01}^2}{\lambda_1 a_1^2} + \frac{((1-d)\lambda_2 + da_2)^2 z_{02}^2}{\lambda_2 a_2^2}\right] + \frac{k^2(1-d)^2\alpha_2^2 z_{02}^2}{a_2^2} <$$
$$\sigma^2 + \sigma^2\left(\frac{z_{01}^2}{\lambda_1} + \frac{z_{02}^2}{\lambda_2}\right).$$

Rearranging this inequality, we will obtain

$$z_{02}^2\left(\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2} + \frac{k^2(1-d)^2\alpha_2^2}{a_2^2} - \frac{\sigma^2}{\lambda_2}\right) < z_{01}^2\sigma^2\left(\frac{1}{\lambda_1} - \frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right).$$

If both

$$\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2} + \frac{k^2(1-d)^2\alpha_2^2}{a_2^2} - \frac{\sigma^2}{\lambda_2} \tag{33}$$

and

$$\sigma^2\left(\frac{1}{\lambda_1} - \frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right) \tag{34}$$

have the same signs, the superiority condition of the LOR estimator over $\hat{\alpha}$ is

$$\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2). \tag{35}$$

If Eqn. (33) and Eqn. (34) have opposite signs, we have

$$\frac{z_{02}^2}{z_{01}^2} > f_1(\alpha_2^2). \tag{36}$$

If Eqn. (33) and Eqn. (34) have different signs, the right-hand side of Eqn. (36) is smaller than zero, thus, Eqn. (36) always holds. That is, in this region the LOR estimator is superior to $\hat{\alpha}$. The condition for the positiveness of Eqn. (33) can be easily written as

$$\alpha_2^2 > \frac{\sigma^2\left(a_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2(1-d)^2} \tag{37}$$

and the condition for the positiveness of Eqn. (34) can be given as

$$a_1^2 > ((1-d)\lambda_1 + da_1)^2. \tag{38}$$

The contrary conditions are required for the negativeness of Eqn. (33) and Eqn. (34). The vertical asymptote of the hyperbola $f_1(\alpha_2^2)$ is at the point

$$\alpha_2^2 = \frac{\sigma^2\left(a_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2(1-d)^2}. \tag{39}$$

Corollary 1. If $d = 0$ in Theorem 1, we get Friedman and Montgomery's [20] results.

Corollary 2. If $k = 1$ in Theorem 1, we get Özbey and Kaçıranlar's [21] results.

Theorem 2.

a) If $\alpha_2^2 > \frac{\sigma^2\left(\lambda_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2[(1-d)^2-1]}$, then

-$J_{LOR} < J_k$ for $\lambda_1^2 < ((1-d)\lambda_1 + da_1)^2$,

-$J_{LOR} < J_k \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$ for $\lambda_1^2 > ((1-d)\lambda_1 + da_1)^2$.

b. If $\alpha_2^2 < \frac{\sigma^2\left(\lambda_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2[(1-d)^2-1]}$, then

-$J_{LOR} < J_k$ for $\lambda_1^2 > ((1-d)\lambda_1 + da_1)^2$,

-$J_{LOR} < J_k \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$ for $\lambda_1^2 < ((1-d)\lambda_1 + da_1)^2$.

where

$$f_2(\alpha_2^2) = \frac{\sigma^2\left(\frac{\lambda_1}{a_1^2}-\frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right)}{\left(\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2}+\frac{k^2(1-d)^2\alpha_2^2}{a_2^2}-\frac{\sigma^2\lambda_2}{a_2^2}-\frac{k^2\alpha_2^2}{a_2^2}\right)}. \tag{40}$$

Proof. Suppose LOR estimator is better than $\hat{\alpha}_k$, then, $J_{LOR} < J_k$. That is,

$$\sigma^2 + \sigma^2\left[\frac{((1-d)\lambda_1 + da_1)^2 z_{01}^2}{\lambda_1 a_1^2} + \frac{((1-d)\lambda_2 + da_2)^2 z_{02}^2}{\lambda_2 a_2^2}\right] + \frac{k^2(1-d)^2\alpha_2^2 z_{02}^2}{a_2^2} <$$

$$\sigma^2 + \sigma^2\left(\frac{\lambda_1 z_{01}^2}{a_1^2} + \frac{\lambda_2 z_{02}^2}{a_2^2}\right) + \frac{k^2\alpha_2^2 z_{02}^2}{a_2^2}.$$

Rearranging this inequality, we get

$$z_{02}^2\left(\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2}+\frac{k^2(1-d)^2\alpha_2^2}{a_2^2}-\frac{\sigma^2\lambda_2}{a_2^2}-\frac{k^2\alpha_2^2}{a_2^2}\right)<$$
$$z_{01}^2\sigma^2\left(\frac{\lambda_1}{a_1^2}-\frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right).$$

If both

$$\left(\frac{\sigma^2((1-d)\lambda_2+da_2)^2}{\lambda_2 a_2^2}+\frac{k^2(1-d)^2\alpha_2^2}{a_2^2}-\frac{\sigma^2\lambda_2}{a_2^2}-\frac{k^2\alpha_2^2}{a_2^2}\right) \quad (41)$$

and

$$\sigma^2\left(\frac{\lambda_1}{a_1^2}-\frac{((1-d)\lambda_1+da_1)^2}{\lambda_1 a_1^2}\right) \quad (42)$$

have the same signs, we have

$$\frac{z_{02}^2}{z_{01}^2}<f_2(\alpha_2^2). \quad (43)$$

If Eqn. (41) and Eqn. (42) have opposite signs, we have

$$\frac{z_{02}^2}{z_{01}^2}>f_2(\alpha_2^2). \quad (44)$$

If Eqn. (41) and Eqn. (42) have opposite signs, the right-hand side of Eqn. (44) is negative, so, Eqn. (44) always holds. The condition for the positiveness of Eqn. (41) can be written as

$$\alpha_2^2>\frac{\sigma^2\left(\lambda_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2[(1-d)^2-1]}. \quad (45)$$

The condition for the positiveness of Eqn. (42) can be given as

$$\lambda_1^2>((1-d)\lambda_1+da_1)^2. \quad (46)$$

The contrary conditions are required for the negativeness of Eqn. (41) and Eqn. (42). The vertical asymptote of the hyperbola $f_2(\alpha_2^2)$ is

$$\alpha_2^2=\frac{\sigma^2\left(\lambda_2^2-((1-d)\lambda_2+da_2)^2\right)}{\lambda_2 k^2[(1-d)^2-1]}. \quad (47)$$

Theorem 3.

a) If $\alpha_2^2>\frac{\sigma^2\left(b_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-\gamma)^2(1-d)^2}$, then

-$J_{LOL}<J_{OLS}$ for $b_1^2<(\gamma b_1+(1-\gamma)c_1)^2$,

-$J_{LOL}<J_{OLS}\Leftrightarrow\frac{z_{02}^2}{z_{01}^2}<f_3(\alpha_2^2)$ for $b_1^2>(\gamma b_1+(1-\gamma)c_1)^2$.

b) If $\alpha_2^2 < \frac{\sigma^2(b_2^2 - (\gamma b_2 + (1-\gamma)c_2)^2)}{\lambda_2(1-\gamma)^2(1-d)^2}$, then

-$J_{LOL} < J_{OLS}$ for $b_1^2 > (\gamma b_1 + (1-\gamma)c_1)^2$,

-$J_{LOL} < J_{OLS} \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$ for $b_1^2 < (\gamma b_1 + (1-\gamma)c_1)^2$,

where

$$f_3(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1} - \frac{(\gamma b_1 + (1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right)}{\left(\frac{\sigma^2(\gamma b_2 + (1-\gamma)c_2)^2}{\lambda_2 b_2^2} + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2} - \frac{\sigma^2}{\lambda_2}\right)}. \quad (48)$$

Proof. If the LOL estimator is superior to $\hat{\alpha}$, we have $J_{LOL} < J_{OLS}$. That is,

$$\sigma^2 + \sigma^2\left[\frac{(\gamma b_1 + (1-\gamma)c_1)^2 z_{01}^2}{\lambda_1 b_1^2} + \frac{(\gamma b_2 + (1-\gamma)c_2)^2 z_{02}^2}{\lambda_2 b_2^2}\right] + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2 z_{02}^2}{b_2^2} < \sigma^2 + \sigma^2\left(\frac{z_{01}^2}{\lambda_1} + \frac{z_{02}^2}{\lambda_2}\right).$$

Rearranging this inequality, we get

$$z_{02}^2\left(\frac{\sigma^2(\gamma b_2 + (1-\gamma)c_2)^2}{\lambda_2 b_2^2} + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2} - \frac{\sigma^2}{\lambda_2}\right) < z_{01}^2\sigma^2\left(\frac{1}{\lambda_1} - \frac{(\gamma b_1 + (1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right).$$

If both

$$\frac{\sigma^2(\gamma b_2 + (1-\gamma)c_2)^2}{\lambda_2 b_2^2} + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2} - \frac{\sigma^2}{\lambda_2} \quad (49)$$

and

$$\sigma^2\left(\frac{1}{\lambda_1} - \frac{(\gamma b_1 + (1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right) \quad (50)$$

have the same signs, we have

$$\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2). \quad (51)$$

If Eqn. (49) and Eqn. (50) have opposite signs, we have

$$\frac{z_{02}^2}{z_{01}^2} > f_3(\alpha_2^2). \quad (52)$$

If Eqn. (49) and Eqn. (50) have opposite signs, the right-hand side of Eqn. (52) is negative, thus Eqn. (52) always holds. The condition for the positiveness of Eqn. (49) can be written as

$$\alpha_2^2 > \frac{\sigma^2\left(b_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-\gamma)^2(1-d)^2}\,. \tag{53}$$

Similarly, the condition for the positiveness of Eqn. (50) can be given as

$$b_1^2 > (\gamma b_1 + (1-\gamma)c_1)^2. \tag{54}$$

The contrary conditions are required for the negativeness of Eqn. (49) and Eqn. (50). The vertical asymptote of the hyperbola $f_3(\alpha_2^2)$ is at the point

$$\alpha_2^2 = \frac{\sigma^2\left(b_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-\gamma)^2(1-d)^2}. \tag{55}$$

Corollary 3: If $\gamma = 0$ in Theorem 3, we get Özbey and Kaçıranlar's [21] results.

Theorem 4.

a) If $\alpha_2^2 > \frac{\sigma^2\left(c_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-d)^2[(1-\gamma)^2-1]}$, then

-$J_{LOL} < J_d$ for $c_1^2 < (\gamma b_1 + (1-\gamma)c_1)^2$,

-$J_{LOL} < J_d \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2)$ for $c_1^2 > (\gamma b_1 + (1-\gamma)c_1)^2$.

b) If $\alpha_2^2 < \frac{\sigma^2\left(c_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-d)^2[(1-\gamma)^2-1]}$, then

-$J_{LOL} < J_d$ for $c_1^2 > (\gamma b_1 + (1-\gamma)c_1)^2$,

-$J_{LOL} < J_d \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2)$ for $c_1^2 < (\gamma b_1 + (1-\gamma)c_1)^2$,

where

$$f_4(\alpha_2^2) = \frac{\sigma^2\left(\frac{c_1^2}{\lambda_1 b_1^2}-\frac{(\gamma b_1+(1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right)}{\left(\frac{\sigma^2(\gamma b_2+(1-\gamma)c_2)^2}{\lambda_2 b_2^2}+\frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2}-\frac{\sigma^2 c_2^2}{\lambda_2 b_2^2}-\frac{(1-d)^2\alpha_2^2}{b_2^2}\right)}. \tag{56}$$

Proof. If the LOL estimator is superior to $\hat{\alpha}_d$, we have $J_{LOL} < J_d$. That is,

$$\sigma^2 + \sigma^2\left[\frac{(\gamma b_1 + (1-\gamma)c_1)^2 z_{01}^2}{\lambda_1 b_1^2} + \frac{(\gamma b_2 + (1-\gamma)c_2)^2 z_{02}^2}{\lambda_2 b_2^2}\right] + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2 z_{02}^2}{b_2^2} <$$
$$\sigma^2 + \sigma^2\left(\frac{c_1^2 z_{01}^2}{\lambda_1 b_1^2} + \frac{c_2^2 z_{02}^2}{\lambda_2 b_2^2}\right) + \frac{(1-d)^2\alpha_2^2 z_{02}^2}{b_2^2}.$$

Rearranging this inequality, we get

$$z_{02}^2\left(\frac{\sigma^2(\gamma b_2 + (1-\gamma)c_2)^2}{\lambda_2 b_2^2} + \frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2} - \frac{\sigma^2 c_2^2}{\lambda_2 b_2^2} - \frac{(1-d)^2\alpha_2^2}{b_2^2}\right) <$$

$$z_{01}^2\sigma^2\left(\frac{c_1^2}{\lambda_1 b_1^2}-\frac{(\gamma b_1+(1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right).$$

If both

$$\left(\frac{\sigma^2(\gamma b_2+(1-\gamma)c_2)^2}{\lambda_2 b_2^2}+\frac{(1-\gamma)^2(1-d)^2\alpha_2^2}{b_2^2}-\frac{\sigma^2 c_2^2}{\lambda_2 b_2^2}-\frac{(1-d)^2\alpha_2^2}{b_2^2}\right) \tag{57}$$

and

$$\sigma^2\left(\frac{c_1^2}{\lambda_1 b_1^2}-\frac{(\gamma b_1+(1-\gamma)c_1)^2}{\lambda_1 b_1^2}\right) \tag{58}$$

have the same signs, we have

$$\frac{z_{02}^2}{z_{01}^2}<f_4(\alpha_2^2). \tag{59}$$

If Eqn. (55) and Eqn. (56) have opposite signs, we have

$$\frac{z_{02}^2}{z_{01}^2}>f_4(\alpha_2^2). \tag{60}$$

If Eqn. (55) and Eqn. (56) have opposite signs, the right-hand side of Eqn. (58) is negative, thus Eqn. (58) holds. The condition for the positiveness of Eqn. (55) can be given as follows

$$\alpha_2^2>\frac{\sigma^2\left(c_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-d)^2[(1-\gamma)^2-1]}. \tag{61}$$

Similarly, the condition for the positiveness of Eqn. (56) can be given as

$$c_1^2>(\gamma b_1+(1-\gamma)c_1)^2. \tag{62}$$

The contrary conditions are required for the negativeness of Eqn. (55) and Eqn. (56). The vertical asymptote of the hyperbola $f_4(\alpha_2^2)$ is

$$\alpha_2^2=\frac{\sigma^2\left(c_2^2-(\gamma b_2+(1-\gamma)c_2)^2\right)}{\lambda_2(1-d)^2[(1-\gamma)^2-1]}. \tag{63}$$

The estimation of the parameters k and d is an important issue. We have not made any attempt to estimate them. However, we refer our readers to Hoerl and Kennard [1], Kibria [28], Khalaf and Shukur [29], Muniz and Kibria [30] and Liu [4] among others.

4. Numerical Example

In this section, we will illustrate theoretical results using the example given by Friedman and Montgomery [20] (i.e., $\sigma^2=1$, $k=0.1$ and $r_{12}=0.95$) and Özbey and Kaçıranlar [21] (i.e., $d=0.9$) as well as we let $\gamma=0.5$.

Let us consider the LOR and the OLS estimators. From Eqn. (32), we get

$$f_1(\alpha_2^2) = \frac{0.004991}{0.004444\alpha_2^2 - 2.57778}, \tag{64}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 = 580. \tag{65}$$

Because of both z_{02}^2/z_{01}^2 and α_2^2 are positive, we are interested only in the points which lie in the first quadrant. Figure 1 illustrates this situation. For values of α_2^2 smaller than 580, the LOR estimator is better than the OLS estimator. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_1(\alpha_2^2)$, then the LOR estimator is superior to the OLS estimator; otherwise, the OLS estimator is better than the LOR estimator. We take different values of d as $0.1, 0.2, \ldots, 0.9$ to determine the effect of d on the predictive performance of the LOR estimator and the OLS estimator. Table 1 shows that if d increases, the value of α_2^2 increases. That means, when α_2^2 increases, the region where the LOR estimator is uniformly superior to the OLS estimator increases.

In this part, we get the same results of the example given by Friedman and Montgomery [20] if $d = 0$. Also, we get the same results of the example given by Özbey and Kaçıranlar [21] if $k = 1$.

Let us consider the ORR and the LOR estimators. From Eqn. (40) and Eqn. (47), we get

$$f_2(\alpha_2^2) = \frac{0.04382}{0.44\alpha_2^2 - 15.2}, \tag{66}$$

and

$$\alpha_2^2 = 34.54. \tag{67}$$

Figure 2 shows this case. For values of $\alpha_2^2 < 34.54$, the LOR estimator is better than $\hat{\alpha}_k$. For great values of α_2^2 there is a trade-off between these estimators. If $(z_{02}^2/z_{01}^2) < f_2(\alpha_2^2)$, then the LOR estimator is superior to $\hat{\alpha}_k$, otherwise $\hat{\alpha}_k$ is better than the LOR estimator.

The effect of d on the PP of the LOR estimator and $\hat{\alpha}_k$ is described in Table 2. Table 2 shows that if d increases, the value of α_2^2 increases. That means, when α_2^2 increases, the region where the LOR estimator is better than $\hat{\alpha}_k$increases.

Let us take into account the PP of the OLS and the LOL estimators. From Eqn. (48) and Eqn. (55), we have

$$f_3(\alpha_2^2) = \frac{0.017236}{0.00226\alpha_2^2 - 1.8595}, \tag{68}$$

and

$$\alpha_2^2 = 820. \tag{69}$$

Figure 3 shows this situation. For values of $\alpha_2^2 < 820$, the LOL estimator is uniformly superior to the $\hat{\alpha}$. If $(z_{02}^2/z_{01}^2) < f_3(\alpha_2^2)$, then the LOL estimator is better than $\hat{\alpha}$. Otherwise, $\hat{\alpha}$ is better than the LOL estimator.

The effect of γ on the PP of the LOL estimator and $\hat{\alpha}$ is described in Table 3. Table 3 shows that if γ increases, the value of α_2^2 increases. That means, when α_2^2 increases, the region where the LOL estimator is uniformly superior to $\hat{\alpha}$ increases.

In this part, we get the same results of the example given by Özbey and Kaçıranlar [21] if $\gamma = 0$.

Let us examine the PP of the LOL and $\hat{\alpha}_d$. From Eqn. (56) and Eqn. (63), we have

$$f_4(\alpha_2^2) = \frac{0.01694}{0.00681\alpha_2^2 - 1.7687}, \tag{70}$$

and

$$\alpha_2^2 = 260. \tag{71}$$

Figure 4 shows this case. For values of $\alpha_2^2 < 260$, the LOL estimator is superior to $\hat{\alpha}_d$. If $(z_{02}^2/z_{01}^2) < f_4(\alpha_2^2)$, then the LOL estimator is superior to $\hat{\alpha}_d$; otherwise, $\hat{\alpha}_d$ is superior to the LOL estimator.

The effect of γ on the PP of the LOL estimator and $\hat{\alpha}_d$ is described in Table 4. Table 4 shows that if γ increases, the value of α_2^2 increases. That means, when α_2^2 increases, the region where the LOL estimator is uniformly superior to $\hat{\alpha}_d$ increases.

Table 1. d and α_2^2 values for the LOR vs. the OLS

d	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
α_2^2	46.67	55.00	65.71	80.00	100.00	130.00	180.00	280.00	580.00

Table 2. d and α_2^2 values for the LOR vs. the ORR

d	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
α_2^2	11.58	13.33	15.29	17.50	20.00	22.86	26.15	30.00	34.54

Table 3. γ and α_2^2 values for the LOL vs. the OLS

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
α_2^2	446.667	505.000	580.000	680.000	820.000	1030.000	1380.000	2080.00	4180.00

Table 4. γ and α_2^2 values for the LOL vs. the Liu

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
α_2^2	201.053	213.333	227.059	242.500	260.000	280.000	303.077	330.000	361.818

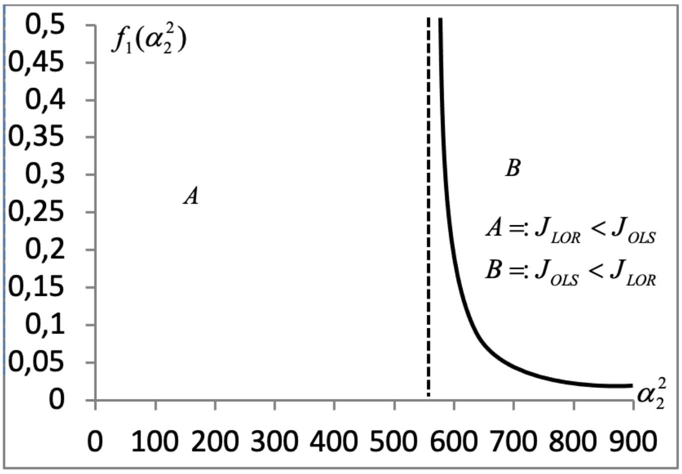

Figure 1: Comparison of the PMSE for LOR and OLS estimators

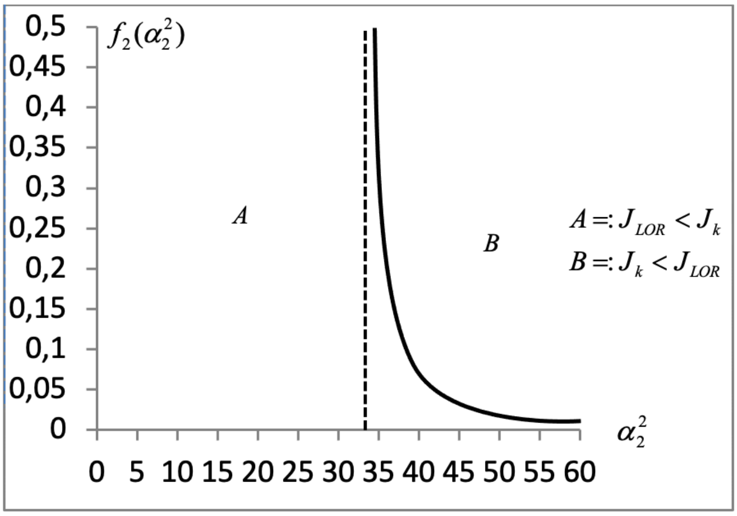

Figure 2: Comparison of the PMSE for LOR and ORR estimators

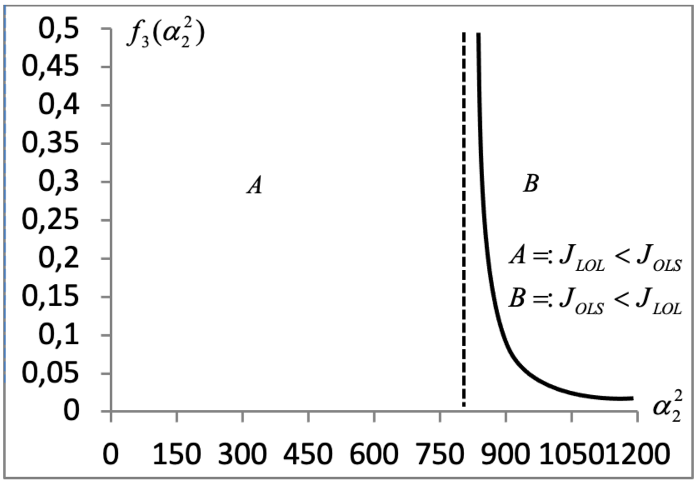

Figure 3: Comparison of the PMSE for LOL and OLS estimators

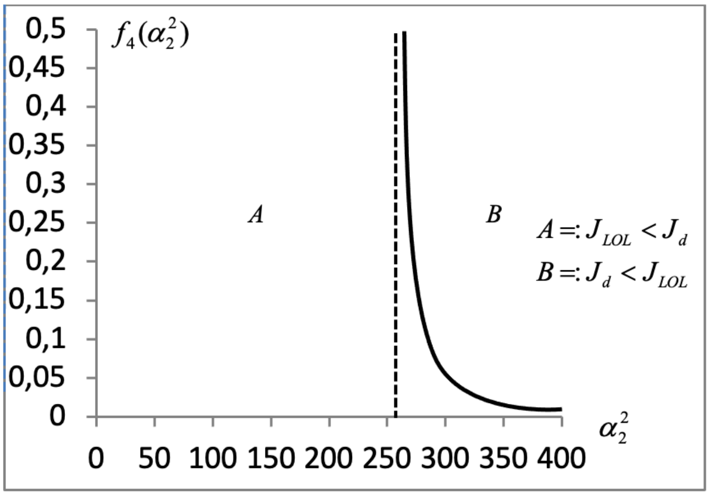

Figure 4: Comparison of the PMSE for LOL and Liu estimators

5. Conclusion

The predictive performance of the LOR estimator over the OLS and the ORR estimators is evaluated. Similarly, the predictive performance of the proposed LOL estimator over the OLS and the Liu estimators is examined in the sense of the PMSE. The comparisons of these estimators are in terms of the PMSE criterion at a specific point in the two-dimensional regressor variable spaces. In this context, the PMSE of the LOR and the LOL estimators are developed and four theorems are given. In addition, three corollaries are given here examining that the theorems given by Friedman and Montgomery [20] and Özbey and Kaçıranlar [21] are just special cases of the Theorems 1 and 3.

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