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Approximately Gamma-Near Rings

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Abstract. The focus of this work is to introduce approximately Γ-near ring, approximately Γ-ideal and approximately Γ-near ring of all descriptive approximately cosets. Also, some properties of approximately Γ-near ring and approximately Γ-ideal are given.

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1. Introduction

Let X be a nonempty set and $\mathcal{R}_\delta$ be a set of proximity relations on X. $(X, \mathcal{R}_\delta)$ is a proximal relator space. Efremovič proximity, descriptive proximity and Lodato proximity are different types of proximity relations [3, 17]. Nonabstract points have location and features. In proximal relator space, the sets consist of these points.

The aim of this work is to obtain algebraic structures in proximal relator spaces using descriptively upper approximations of the subsets of X. Between 2017 and 2019, approximately semigroup and approximately ideal, approximately group, approximately subgroup, approximately ring were introduced by İnan [4–6, 8]. Approximately Γ-semigroup and approximately near ring were also defined [7, 9]. In these works, some examples of these approximately algebraic structures in digital images endowed with proximity relations were given as in this work. Approximately algebraic structures satisfy a framework for further applied areas such as image analysis or classification problems.

In 1983, Pilz introduced the near-ring as a generalization of ring. In near rings, the addition operation does not need to be commutative as only one distributive law is sufficient [20].

Nobusawa [14] introduced the Γ-ring, as a generalization of ring. Barnes [1] weakened the conditions in the definition of the Nobusawa Γ-ring. Barnes [1], Kyuno [11] and Luh [12] worked on Γ-rings and obtained some generalizations of ring theory.

Satyanarayana defined the Γ-near ring as a generalization of near-ring and Γ-ring [21].

Essentially, the aim of this work is to introduce approximately Γ-near ring, approximately Γ-ideal and approximately Γ-near ring of all descriptive approximately cosets. Furthermore, some properties of approximately Γ-near ring and approximately Γ-ideal are given.

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2. Preliminaries

Let X be a nonempty set. Family of relations $\mathcal{R}$ on X is called a relator. The pair $(X,\mathcal{R})$ (or $X(\mathcal{R})$) is a relator space which results from natural generalizations of uniform spaces [22]. If we consider a family of proximity relations on X, we have a proximal relator space $(X,\mathcal{R}_\delta)$ (also denoted by $X(\mathcal{R}_\delta)$). As in [17], $\mathcal{R}_\delta$ contains proximity relations, namely, Efremovič proximity δ_E [2, 3], Wallman proximity δ_W, Lodato proximity $\delta_{\mathcal{L}}$, descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [15, 19].

In this work, we consider the Efremovič proximity δ_E and the descriptive proximity δ_Φ in defining a descriptive proximal relator space (denoted by $(X,\mathcal{R}_{\delta_\Phi})$).

An Efremovič proximity δ_E is a relation on 2^X that satisfies

1^o $A\ \delta_E\ B \Rightarrow B\ \delta_E\ A$,
2^o $A\ \delta_E\ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$,
3^o $A \cap B \neq \emptyset \Rightarrow A\ \delta_E\ B$,
4^o $A\ \delta_E\ (B \cup C) \Leftrightarrow A\ \delta_E\ B$ or $A\ \delta_E\ C$,
5^o $\{x\}\ \delta_E\ \{y\} \Leftrightarrow x = y$,
6^o $A\ \underline{\delta_E}\ B \Rightarrow \exists E \subseteq X$ such that $A\ \underline{\delta_E}\ E$ and $E^c\ \underline{\delta_E}\ B$ EF axiom.

In a discrete space, a nonabstract point has a location and has features that can be measured [10]. Let X be a nonempty set of nonabstract points in a proximal relator space $(X,\mathcal{R}_{\delta_\Phi})$.

Probe functions $\varphi_i : X \to \mathbb{R}$ represent a feature of a sample point in a picture. Let $\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ $(n \in \mathbb{N})$ be an object description, which is a feature vector of x, which provides a description of each $x \in X$. After the choosing a set of probe functions, one obtain a descriptive proximity relation δ_Φ.

Definition 2.1 ([13]). Let X be a nonempty set of nonabstract points, Φ be an object description and A be a subset of X. Then, the set description of A is defined as

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.2 ([13, 16]). Let X be a nonempty set of nonabstract points, A and B be two subsets of X. Then, the descriptive (set) intersection of A and B is defined as

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A)\ \ and\ \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 2.3 ([15]). Let X be a nonempty set of nonabstract points, A and B any two subsets of X. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$, then $A\ \underline{\delta}_\Phi\ B$ reads A is descriptively far from B.

Definition 2.4 ([18]). Let X be a descriptive proximal relator space and $A \subseteq X$. Let $(A, \circ)$ and $(\mathcal{Q}(A), \cdot)$ be groupoids. Lets consider the object description Φ by means of a function

$$\Phi : A \subseteq X \longrightarrow \mathcal{Q}(A) \subset \mathbb{R}^n,\ x \mapsto \Phi(x),\ x \in A.$$

The object description Φ of A into $\mathcal{Q}(A)$ is an object descriptive homomorphism if $\Phi(x \circ y) = \Phi(x) \cdot \Phi(y)$ for all $x, y \in A$.

Definition 2.5 ([4]). Let X be a descriptive proximal relator space and $A \subseteq X$. A descriptively upper approximation of A is defined as

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

Obviously, $A \subseteq \Phi^* A$ for all $A \subset X$.

Definition 2.6 ([4]). Let $(X,\mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and let "$\cdot$" be a binary operation on X. $G \subseteq X$ is called an approximately groupoid in descriptive proximal relator space if $x \cdot y \in \Phi^* G$ for all $x, y \in G$.

Definition 2.7 ([6]). Let X be a descriptive proximal relator space and let "$+$" be a binary operation on X. $G \subseteq X$ is called an approximately group in descriptive proximal relator space or shortly approximately group if the followings are true:

$(\mathcal{A}G_1)$ For all $x, y \in G$, $x + y \in \Phi^* G$,
$(\mathcal{A}G_2)$ For all $x, y, z \in G$, $(x + y) + z = x + (y + z)$ property holds in $\Phi^* G$,

($\mathcal{A}G_3$) There exists $e \in \Phi^* G$ such that $x + e = e + x = x$ for all $x \in G$ (e is called the approximately identity element of G),

($\mathcal{A}G_4$) There exists $y \in G$ such that $x + y = y + x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as $-x$).

A subset G of the set of X is called an approximately semigroup in descriptive proximal relator space if ($\mathcal{A}G_1$–$\mathcal{A}G_2$) properties are satisfied.

Theorem 2.8 ([6]). *Let G be an approximately group, H be a nonempty subset of G and $\Phi^* H$ be a groupoid. H is an approximately subgroup of G iff $-x \in H$ for all $x \in H$.*

Theorem 2.9 ([6]). *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $G \subseteq X$ be an approximately group. Then,*

(*i*) *There is one and only one approximately identity element in G.*

(*ii*) *There is one and only one $y \in G$ such that $x + y = e$ and $y + x = e$ for all $x \in G$; we denote it by $-x$.*

Suppose that G is an approximately groupoid with the binary operation "$\cdot$" in $(X, \mathcal{R}_{\delta_\Phi})$, $g \in G$ and $A, B \subseteq G$. The subsets $g \cdot A$, $A \cdot g$, $A \cdot B \subseteq \Phi^* G \subseteq X$ are defined as follows:

$$g \cdot A = gA = \{ga \mid a \in A\},$$
$$A \cdot g = Ag = \{ag \mid a \in A\},$$
$$A \cdot B = AB = \{ab \mid a \in A, b \in B\}.$$

Theorem 2.10 ([5]). *Let G be an additive approximately group, H be an approximately subgroup of G and $G/_{\rho_l}$ be a set of all approximately left cosets of G by H. If $(\Phi^* G)/_{\rho_l} \subseteq \Phi^* \left(G/_{\rho_l}\right)$, then $G/_{\rho_l}$ is an approximately group under the operation given by $xH \oplus yH = (x + y) H$ for all $x, y \in G$.*

Definition 2.11 ([8]). Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and "+" , "$\cdot$" be binary operations defined on X. A $R \subseteq X$ is called an approximately ring in descriptive proximal relator space if the following properties are satisfied:

($\mathcal{A}R_1$) R is an abelian approximately group with the binary operation "+",

($\mathcal{A}R_2$) R is an approximately semigroup with the binary operation "$\cdot$",

($\mathcal{A}R_3$) For all $x, y, z \in R$,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z), \ (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $\Phi^* R$.

In addition,

($\mathcal{A}R_4$) R is said to be a commutative approximately ring if $x \cdot y = y \cdot x$ for all $x, y \in R$,

($\mathcal{A}R_5$) R is said to be an approximately ring with identity if $\Phi^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$.

Definition 2.12 ([1]). A Γ-ring (in the sense of Barnes) is a pair (M, Γ), where M and Γ are (additive) abelian groups for which exists a $M \times \Gamma \times M \to M$, the image of (a, α, b) being denoted by $a\alpha b$ for $a, b \in M$ and $\alpha \in \Gamma$, satisfying for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$:

- $(a + b)\alpha c = a\alpha c + b\alpha c$,
- $a(\alpha + \beta)b = a\alpha b + a\beta b$,
- $a\alpha(b + c) = a\alpha b + a\alpha c$,
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.13 ([1]). Let M be a Γ-ring. A left (right) ideal of M is an additive subgroup U of M such that $M\Gamma U \subseteq U$ ($U\Gamma M \subseteq U$). If U is both a left and a right ideal, then we say that U is an ideal of M.

Definition 2.14 ([23]). Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $M, \Gamma \subseteq X$ be additive abelian approximately groups in $(X, \mathcal{R}_{\delta_\Phi})$. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the conditions

($\mathcal{A}\Gamma_1$) $a\alpha b \in \Phi^* M$,

($\mathcal{A}\Gamma_2$) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ properties verify on $\Phi^* M$,

($\mathcal{A}\Gamma_3$) $(a\alpha b)\beta c = a\alpha(b\beta c)$ property verify on $\Phi^* M$

are satisfied, then M is called an approximately Γ-ring in descriptive proximal relator space or shortly approximately Γ-ring.

In addition, if $a\alpha b = b\alpha a$ for all $a, b \in M$ and $\alpha \in \Gamma$, then M is called a commutative approximately Γ-ring.

3. An Example of Approximately Γ-Ring

In [23], Example 1 is not an approximately Γ-ring due to some typos. Therefore, another example of approximately Γ-ring is given in Example 3.1.

Example 3.1. Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 25 pixels as in Figure 1.

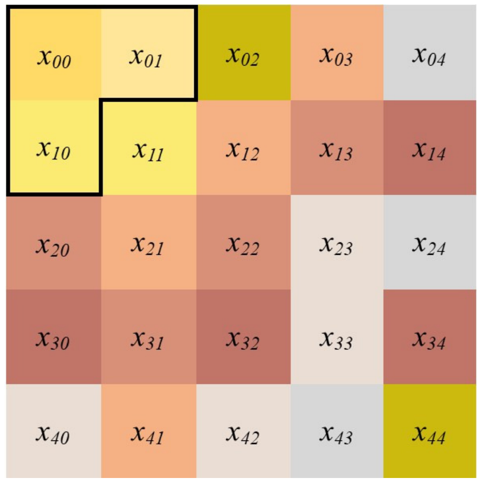

Figure 1. Digital image X and subimage M

A pixel x_{ij} is an element at position (i, j) (row and column) in digital image X. Let φ be a probe function that represent RGB colour of each pixel are given in Table 1.

	x_{00}	x_{01}	x_{02}	x_{03}	x_{04}	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{20}	x_{21}
Red	255	255	204	245	215	251	251	245	217	192	217	245
Green	218	230	186	177	215	235	235	177	144	117	144	177
Blue	102	153	14	131	215	115	115	131	121	104	121	131

	x_{22}	x_{23}	x_{24}	x_{30}	x_{31}	x_{32}	x_{33}	x_{34}	x_{40}	x_{41}	x_{42}	x_{43}	x_{44}
Red	217	234	215	192	217	192	234	192	234	245	234	215	204
Green	144	221	215	117	144	117	221	117	221	177	221	215	186
Blue	121	212	215	104	121	104	212	104	212	131	212	215	14

Table 1. RGB colour of each pixel

Let

$$+_1 \; : \; \begin{array}{ll} X \times X & \longrightarrow X \\ \left(x_{ij}, x_{mn}\right) & \longmapsto x_{ij} +_1 x_{mn} \end{array},$$

$$x_{ij} +_1 x_{mn} = x_{pr}, \quad i + m \equiv p \; (mod\ 2) \text{ and } j + n \equiv r \; (mod\ 2)$$

be a binary operation (first addition) on X. Let $M = \{x_{00}, x_{01}, x_{10}\}$ be a subimage (subset) of X.

We can compute the descriptively upper approximation of M, that is, $\Phi^* M = \{x_{ij} \in X \mid x_{ij}\delta_\Phi M\}$ by using the Definition 2.5. Then, $Q\left(\{x_{ij}\}\right) \cap Q(M) \neq \varnothing$ such that $x_{ij} \in X$, where $Q(M) = \{\varphi(x_{ij}) \mid x_{ij} \in M\}$. From Table 1, we obtain

$$\begin{aligned} Q(M) &= \{\varphi(x_{00}), \varphi(x_{01}), \varphi(x_{10})\} \\ &= \{(255, 218, 102), (255, 230, 153), (251, 235, 115)\}. \end{aligned}$$

Hence, we get $\Phi^* M = \{x_{00}, x_{01}, x_{10}, x_{11}\}$. Consequently, M is an additive abelian approximately group in $(X, \mathcal{R}_{\delta_\Phi})$ from Definition 2.7. Furthermore, let

$$+_2 \quad : \quad \begin{array}{ll} X \times X & \longrightarrow X \\ \left(x_{ij}, x_{mn}\right) & \longmapsto x_{ij} +_2 x_{mn} \end{array} ,$$

$$x_{ij} +_2 x_{mn} = x_{st}, \quad i + m \equiv s \ (mod\ 4) \text{ and } j + n \equiv t \ (mod\ 4)$$

be a binary operation (second addition) on X. Let $\Gamma = \{x_{00}, x_{02}\}$ be a subimage (subset) of X.

We can calculate the descriptively upper approximation of Γ, that is, $\Phi^*\Gamma = \{x_{ij} \in X \mid x_{ij}\delta_\Phi \Gamma\}$ by using the Definition 2.5. Then, $Q\left(\{x_{ij}\}\right) \cap Q(\Gamma) \neq \emptyset$ such that $x_{ij} \in X$, where $Q(\Gamma) = \{\varphi(x_{ij}) \mid x_{ij} \in \Gamma\}$. From Table 1, we have

$$\begin{aligned} Q(\Gamma) &= \{\varphi(x_{00}), \varphi(x_{02})\} \\ &= \{(255, 218, 102), (204, 186, 14)\}. \end{aligned}$$

Hence, we get $\Phi^*\Gamma = \{x_{00}, x_{02}, x_{44}\}$. As a result, Γ is an additive abelian approximately group in $(X, \mathcal{R}_{\delta_\Phi})$ from Definition 2.7.

Also, let

$$\begin{array}{ll} X \times \Gamma \times X & \longrightarrow X \\ \left(x_{ij}, x_{kl}, x_{mn}\right) & \longmapsto x_{ij}x_{kl}x_{mn} \end{array} ,$$

$$x_{ij}x_{kl}x_{mn} = x_{uv}, \quad u = \min\{i, k, m\} \text{ and } v = \min\{j, l, n\}$$

be an operation on X. In this case, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, since

$(\mathcal{A}\Gamma_1)$ $a\alpha b \in \Phi^* M$,
$(\mathcal{A}\Gamma_2)$ $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ properties verify on $\Phi^* M$,
$(\mathcal{A}\Gamma_3)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$ property verify on $\Phi^* M$,

M is an approximately Γ-ring from Definition 2.14.

4. Approximately Γ-Near Rings

Throughout this section, $(X, \mathcal{R}_{\delta_\Phi})$ or shortly X is considered as descriptive proximal relator space, unless otherwise stated.

Definition 4.1. Let $A, \Gamma \subseteq X$ be additive approximately groups in $(X, \mathcal{R}_{\delta_\Phi})$. If for all $a, b, c \in A$ and all $\alpha, \beta \in \Gamma$ the conditions

$(\mathcal{A}\Gamma N_1)$ $a\alpha b \in \Phi^* A$,
$(\mathcal{A}\Gamma N_2)$ $(a + b)\alpha c = a\alpha c + b\alpha c$ property provides on $\Phi^* A$,
$(\mathcal{A}\Gamma N_3)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$ property provides on $\Phi^* A$

are satisfied, then A is called an approximately Γ-near ring in descriptive proximal relator space or shortly approximately Γ-near ring.

In addition, if $a\alpha b = b\alpha a$ for all $a, b \in A$ and all $\alpha \in \Gamma$, then A is called a commutative approximately Γ-near ring.

Example 4.2. Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 25 pixels as in Figure 1 from Example 3.1.

A pixel x_{ij} is an element at position (i, j) (row and column) in digital image X. Let φ be a probe function that represent RGB colour of each pixel are given in Table 1 from Example 3.1.

Let

$$+_1 \quad : \quad \begin{array}{ll} X \times X & \longrightarrow X \\ \left(x_{ij}, x_{mn}\right) & \longmapsto x_{ij} +_1 x_{mn} \end{array} ,$$

$$x_{ij} +_1 x_{mn} = x_{pr}, \quad i + m \equiv p \ (mod\ 2) \text{ and } j + n \equiv r \ (mod\ 2)$$

be a binary operation (first addition) on X. Let $A = \{x_{00}, x_{01}, x_{10}\}$ be a subimage of X.

We get $\Phi^*A = \{x_{00}, x_{01}, x_{10}, x_{11}\}$, as in Example 3.1. Consequently, A is an additive approximately group in X from Definition 2.7. Furthermore, let

$$+_2 \ : \ \begin{array}{ll} X \times X & \longrightarrow X \\ \left(x_{ij}, x_{mn}\right) & \longmapsto x_{ij} +_2 x_{mn} \end{array} \ ,$$

$$x_{ij} +_2 x_{mn} = x_{st}, \quad i + m \equiv s \ (mod\ 4) \text{ and } j + n \equiv t \ (mod\ 4)$$

be a binary operation (second addition) on X. Let $\Gamma = \{x_{00}, x_{02}\}$ be a subimage (subset) of X.

We get $\Phi^*\Gamma = \{x_{00}, x_{02}, x_{44}\}$, as in Example 3.1. As a result, Γ is an additive abelian approximately group in X from Definition 2.7.

Also, let

$$\begin{array}{ll} X \times \Gamma \times X & \longrightarrow X \\ \left(x_{ij}, x_{kl}, x_{mn}\right) & \longmapsto x_{ij}x_{kl}x_{mn} = x_{ij} \end{array} \ ,$$

be an operation on X. In this case, for all $a, b, c \in A$ and all $\alpha, \beta \in \Gamma$, since

($\mathcal{A}\Gamma N_1$) $a\alpha b \in \Phi^*A$,
($\mathcal{A}\Gamma N_2$) $(a + b)\alpha c = a\alpha c + b\alpha c$ property holds on Φ^*A,
($\mathcal{A}\Gamma N_3$) $(a\alpha b)\beta c = a\alpha\,(b\beta c)$ property holds on Φ^*A,

A is an approximately Γ-near ring from Definition 4.1

But since $x_{01}x_{02}\,(x_{01} + x_{01}) \neq x_{01}x_{02}x_{01} + x_{01}x_{02}x_{01}$, so $a\alpha(b + c) = a\alpha b + a\alpha c$ property does not hold in Φ^*A. Consequently, A is an right approximately Γ-near ring.

Theorem 4.3. *Every Γ-near ring in a proximal relator space is an approximately Γ-near ring.*

Proof. Let $A \subseteq X$ be a Γ-near ring. Since $A \subseteq \Phi^*A$, then the properties $(\mathcal{A}\Gamma N_1) - (\mathcal{A}\Gamma N_3)$ hold in Φ^*A. Therefore, A is an approximately Γ-near ring. □

Theorem 4.4. *Every approximately Γ-ring is an approximately Γ-near ring.*

Proof. Let $A \subseteq X$ be an approximately Γ-ring. From definition of approximately Γ-ring, it is easily shown that A is an approximately Γ-near ring. □

Lemma 4.5. *Let $A \subseteq X$ be an approximately Γ-near ring and $0_A \in A$ be an additive approximately identity element of A. If $0_A\gamma a \in A$ and $(-a)\,\gamma b \in A$, then*
(i) $0_A\gamma a = 0_A$,
(ii) $(-a)\,\gamma b = -(a\gamma b)$
for all $a, b \in A$ and all $\gamma \in \Gamma$.

Proof. (i) For all $a \in A$ and all $\gamma \in \Gamma$, from Definition 4.1 ($\mathcal{A}\Gamma N_2$)

$$0_A\gamma a = (0_A + 0_A)\,\gamma a = 0_A\gamma a + 0_A\gamma a.$$

From Theorem 2.9 (i), since the approximately identity element is unique, $0_A\gamma a = 0_A$.
(ii) From (i), $0_A\gamma b = 0_A$ for all $b \in A$ and all $\gamma \in \Gamma$. Then,

$$0_A = 0_A\gamma b = ((-a) + a)\,\gamma b = (-a)\,\gamma b + a\gamma b.$$

From Theorem 2.9 (ii), since the inverse element is unique, $(-a)\,\gamma b = -(a\gamma b)$. □

Definition 4.6. Let $A, \Gamma \subseteq X$, A be an approximately Γ-near ring and $K \subseteq A$. If K is an additive approximately group and satisfy the conditions $(\mathcal{A}\Gamma N_1 - \mathcal{A}\Gamma N_3)$, K is called an approximately Γ-subnear ring of A.

Theorem 4.7. *Let $A, \Gamma \subseteq X$, A be an approximately Γ-near ring, $K \subseteq A$ and Φ^*K be an additive groupoid and a Γ-groupoid. Then, K is an approximately Γ-subnear ring of A iff $-k \in K$ for all $k \in K$.*

Proof. It obvious from Theorem 2.8. □

Definition 4.8. Let A be an approximately Γ-near ring and $I \subseteq A$. I is an approximately Γ-ideal of A if I is an additive approximately subgroup of A and following properties are satisfied:

(1) $I\Gamma A = \{x\gamma a \mid x \in I, \gamma \in \Gamma, a \in A\} \subseteq \Phi^* I$,

(2) $a\gamma(b + x) - a\gamma b \in \Phi^* I$

for all $a, b \in A$, all $x \in I$ and all $\gamma \in \Gamma$.

Furthermore, I is called right approximately Γ-ideal of A if only it satisfies the condition (1). Also, I is called left approximately Γ-ideal of A if only it satisfies the condition (2).

Example 4.9. From Example 4.2, let we consider approximately Γ-near ring $A = \{x_{00}, x_{01}, x_{10}\}$ and $I = A$. Since I is an additive approximately subgroup of A, $I\Gamma A = I$ by definition of the operation $X \times \Gamma \times X \longrightarrow X$ from Example 4.2 and $I \subseteq \Phi^* I$, I is a right approximately Γ-ideal of A. Also, since $a\gamma(b + x) - a\gamma b \in \Phi^* I$ for all $a, b \in A$, all $x \in I$ and all $\gamma \in \Gamma$, I is a left approximately Γ-ideal of A. Hence, I is an approximately Γ-ideal of A.

Remark 4.10. Every approximately Γ-ideal of A is also an approximately Γ-subnear ring of A in (X, δ_Φ).

Let I and J be both left approximately Γ-ideals of A. Then,

$$I + J = \{x + y \mid x \in I, y \in J\}$$

is called the sum of I and J.

The following results are obvious for the approximately Γ-near ring as well as for the approximately Γ-ring [23].

Lemma 4.11. *Let $A \subseteq X$ be an approximately Γ-near ring and $K, L \subseteq A$. If Φ is an object descriptive homomorphism, then*

(1) $cl_\Phi(k) + cl_\Phi(l) = cl_\Phi(k + l)$ *for all $k \in K$ and all $l \in L$,*

(2) $\mathcal{Q}(K + L) = \mathcal{Q}(K) + \mathcal{Q}(L)$.

Lemma 4.12. *Let $A \subseteq X$ be an approximately Γ-near ring and $K, L \subseteq A$. If Φ is an object descriptive monomorphism, then $\Phi^* K + \Phi^* L = \Phi^*(K + L)$.*

Theorem 4.13. *Let $A \subseteq X$ be an approximately Γ-near ring and $K, L \subseteq A$. If Φ is an object descriptive homomorphism, then*

(1) $\Phi_* K + \Phi_* L \subseteq \Phi_*(K + L)$,

(2) $\Phi^* K + \Phi^* L \subseteq \Phi^*(K + L)$.

Theorem 4.14. *Let $A \subseteq X$ be an approximately Γ-near ring, $I, J \subseteq A$ and $\Phi^* I$, $\Phi^* J$ be additive groupoids and Γ-groupoids. If I, J are both right approximately Γ-ideals of A and Φ is an object descriptive homomorphism, then $I + J$ is also a right approximately Γ-ideal of A.*

Proof. Since I and J are both right approximately Γ-ideals of A, $I\Gamma A \subseteq \Phi^* I$ and $J\Gamma A \subseteq \Phi^* J$. Then, from Theorem 4.13 (2),

$$\begin{aligned}
(I + J)\Gamma A &= \{(x + y)\gamma a \mid a \in A, \gamma \in \Gamma, x \in I, y \in J\} \\
&= \{x\gamma a + y\gamma a \mid a \in A, \gamma \in \Gamma, x \in I, y \in J\} \\
&= \{x\gamma a \mid a \in A, \gamma \in \Gamma, x \in I\} + \{y\gamma a \mid a \in A, \gamma \in \Gamma, y \in J\} \\
&= I\Gamma A + J\Gamma A \\
&\subseteq \Phi^* I + \Phi^* J \\
&\subseteq \Phi^*(I + J).
\end{aligned}$$

Therefore, $(I + J)\Gamma A \subseteq \Phi^*(I + J)$, that is, $I + J$ is a right approximately Γ-ideal of A. □

Corollary 4.15. *Let $A \subseteq X$ be an approximately Γ-near ring and $I_i \subseteq A$ ($1 \le i \le n$, $n \geqslant 2$). If I_i are right approximately Γ-ideals of A, Φ is an object descriptive homomorphism and $\Phi^* I_i$ are additive groupoids and $\Phi^* I_i$ are Γ-groupoids, then $\sum_{1 \le i \le n} I_i$ is also a right approximately Γ-ideal of A.*

Theorem 4.16. *Let $A \subseteq X$ be an approximately Γ-near ring, $I, J \subseteq A$ and $\Phi^* I$, $\Phi^* J$ be additive groupoids and Γ-groupoids. If I, J are both right approximately Γ-ideals of A and $\Phi^* I \cap \Phi^* J = \Phi^*(I \cap J)$, then $I \cap J$ is also a right approximately Γ-ideal of A.*

Proof. Since I and J are both right approximately Γ-ideals of A, $I\Gamma A \subseteq \Phi^* I$ and $J\Gamma A \subseteq \Phi^* J$,

$$\begin{aligned}(I\cap J)\,\Gamma A &= \{\, x\gamma a |\, a\in A, \gamma\in\Gamma, x\in I\cap J\}\\ &= \{\, x\gamma a |\, a\in A, \gamma\in\Gamma, x\in I \;\; and \;\; x\in J\}\\ &= \{\, x\gamma a |\, a\in A, \gamma\in\Gamma, x\in I\}\cap\{x\gamma a | a\in A, \gamma\in\Gamma, x\in J\}\\ &= I\Gamma A\cap J\Gamma A\\ &\subseteq \Phi^* I\cap\Phi^* J\\ &= \Phi^*\,(I\cap J)\,.\end{aligned}$$

Therefore, $(I\cap J)\,\Gamma A \subseteq \Phi^*\,(I\cap J)$, that is, $I\cap J$ is a right approximately Γ-ideal of A. □

Corollary 4.17. *Let $A \subseteq X$ be an approximately Γ-near ring, $I_i \subseteq A$ $(1\le i\le n, n\geqslant 2)$ and $\Phi^* I_i$ be additive groupoids and Γ-groupoids. If I_i are right approximately Γ-ideals of A and $\bigcap_{1\le i\le n}\Phi^* I_i = \Phi^*\left(\bigcap_{1\le i\le n} I_i\right)$, then $\bigcap_{1\le i\le n} I_i$ is also a right approximately Γ-ideal of A.*

Definition 4.18. Let K be an approximately Γ-subnear ring of approximately Γ-near ring A. The relation "c_r" defined as

$$ac_r b :\Leftrightarrow a + (-b) \in K\cup\{0_A\}\,,$$

where $a, b\in A$.

Theorem 4.19. *Let A be an approximately Γ-near ring. The relation "c_r" is a right compatible relation on A.*

Proof. Since $(A,+)$ is an approximately group, $-a\in A$ for all $a\in A$. Due to $a+(-a)=0_A\in K\cup\{0_A\}$, $ac_r a$. Let $ac_r b$ for all $a, b\in A$. Then, $a+(-b)\in K\cup\{0_A\}$, that is, $a+(-b)\in K$ or $a+(-b)\in\{0_A\}$. If $a+(-b)\in K$, since $(K,+)$ is an approximately group, then $-(a+(-b)) = b+(-a)\in K$. Hence, $bc_r a$. Also if $a+(-b)\in\{0_A\}$, then $a+(-b)=0_A$. Therefore, $b+(-a) = -(a+(-b)) = -0_A = 0_A$ and so $bc_r a$. Consequently, "c_r" is a right compatible relation on A. □

A class that contains the element $a\in A$, determined by relation "c_r" is

$$\tilde{a}_r = \{k + a \mid k\in K,\ a\in A,\ k+a\in A\}\cup\{a\}\,.$$

Definition 4.20. Let A be an approximately Γ-near ring. A weak class determined by right compatible relation "c_r" is called near right weak coset.

Definition 4.21. Let K be an approximately Γ-subnear ring of approximately Γ-near ring A. The relation "c_ℓ" defined as

$$ac_\ell b :\Leftrightarrow (-a) + b \in K\cup\{0_A\}\,,$$

where $a, b\in A$.

The proof of Theorem 4.22 is similar to proof of Theorem 4.19.

Theorem 4.22. *Let A be an approximately Γ-near ring. The relation "c_ℓ" is a left compatible relation on A.*

A class that contains the element $a\in A$, determined by relation "c_ℓ" is

$$\tilde{a}_\ell = \{a + k \mid k\in K,\ a\in A,\ a+k\in A\}\cup\{a\}\,.$$

Definition 4.23. Let A be an approximately Γ-near ring. A class determined by left compatible relation "c_ℓ" is called near left weak coset.

We can easily show that $\tilde{a}_r = K + a$ and $\tilde{a}_\ell = a + K$. Approximately group $(M,+)$ may not always abelian. If $(M,+)$ is an abelian approximately group, $\tilde{a}_r = \tilde{a}_\ell$. Otherwise $\tilde{a}_r \neq \tilde{a}_\ell$.

Let A be an approximately Γ-near ring and K be an approximately Γ-subnear ring of A. Then,

$$A/_{c_\ell} = \{a + K \mid a\in A\}$$

is a set of all near left weak cosets of A determined by K. If we consider $\Phi^* A$ instead of approximately Γ-near ring A

$$(\Phi^* A)\,/_{c_\ell} = \{a + K \mid a\in \Phi^* A\}\,.$$

Hence,

$$a + K = \{a + k \mid k\in K, a\in\Phi^* A, a+k\in A\}\cup\{a\}\,.$$

Definition 4.24. Let $A \subseteq X$ be an approximately Γ-near ring and K be an approximately Γ-subnear ring of A. For $a, b \in A$, let $a + K$ and $b + K$ be two near left weak cosets that determined by the elements a and b, respectively. Then, sum of two near left weak cosets that determined by $a + b \in \Phi^* A$ can be defined as

$$(a + b) + K = \{(a + b) + k \mid k \in K,\ a + b \in \Phi^* A,\ \ (a + b) + k \in A\} \cup \{a + b\}$$

and denoted by

$$(a + K) \oplus (b + K) = (a + b) + K.$$

Definition 4.25. Let $A \subseteq X$ be an approximately Γ-near ring and K be an approximately Γ-subnear ring of A. For $a, b \in A$, let $a + K$ and $b + K$ be two near left weak cosets that determined by the elements a and b, respectively. Then, product of two near left weak cosets that determined by $a\gamma b \in \Phi^* A$ can be defined as

$$(a\gamma b) + K = \{(a\gamma b) + k \mid k \in K,\ a\gamma b \in \Phi^* A,\ \ (a\gamma b) + k \in A\} \cup \{a\gamma b\}$$

and denoted by

$$(a + K)\,\gamma\,(b + K) = (a\gamma b) + K,$$

where $\gamma \in \Gamma$.

Definition 4.26. Let $A/_{c_\ell}$ be a set of all near left weak cosets of A determined by K and $\xi_\Phi(S)$ be a descriptive approximately collection of $S \in P(X)$. Then,

$$\Phi^*\left(A/_{c_\ell}\right) = \bigcup_{\xi_\Phi(S) \underset{\Phi}{\cap} A/_{c_\ell} \neq \emptyset} \xi_\Phi(S)$$

is called upper approximation of $A/_{c_\ell}$.

Theorem 4.27. *Let A be an approximately Γ-near ring, K be an approximately Γ-subnear ring of A and $A/_{c_\ell}$ be a set of all near left weak cosets of A determined by K. If*

$$(\Phi^* A)/_{c_\ell} \subseteq \Phi^*\left(A/_{c_\ell}\right),$$

then $A/_{c_\ell}$ is an approximately Γ-near ring with the operations given by

$$(a + K) \oplus (b + K) = (a + b) + K$$

and

$$(a + K)\,\gamma\,(b + K) = (a\gamma b) + K$$

for all $a, b \in A$ and all $\gamma \in \Gamma$.

Proof. Let $(\Phi^* A)/_{c_\ell} \subseteq \Phi^*\left(A/_{c_\ell}\right)$. Since A is an approximately Γ-near ring and by Theorem 2.10, $(A/_{c_\ell}, \oplus)$ is an approximately group of all near left weak cosets of A by K. Furthermore,

($\mathcal{A}\Gamma N_1$) Since A is an approximately Γ-near ring, $a\gamma b \in \Phi^* A$ and then $(a + K)\,\gamma\,(b + K) = (a\gamma b) + K \in (\Phi^* A)/_{c_\ell}$. From the hypothesis, $(a + K)\,\gamma\,(b + K) \in \Phi^*\left(A/_{c_\ell}\right)$.

($\mathcal{A}\Gamma N_2$) Since A is an approximately Γ-near ring, right distributive property holds in $\Phi^* A$ for all $a, b, c \in A$ and all $\gamma \in \Gamma$. From the Definitions 4.24 and 4.25, for all $(a + K), (b + K), (c + K) \in A/_{c_\ell}$

$$\begin{aligned} ((a + K) \oplus (b + K))\,\gamma\,(c + K) &= ((a + b) + K)\,\gamma\,(c + K) \\ &= ((a + b)\,\gamma c) + K \end{aligned}$$

and

$$\begin{aligned} ((a + K)\,\gamma\,(c + K)) \oplus ((b + K)\,\gamma\,(c + K)) &= ((a\gamma c) + K) \oplus ((b\gamma c) + K) \\ &= ((a\gamma c) + (b\gamma c)) + K \\ &= ((a + b)\,\gamma c) + K, \end{aligned}$$

where $((a + b)\,\gamma c) + K \in (\Phi^* A)/_{c_\ell}$. From the hypothesis,

$$((a + K) \oplus (b + K))\,\gamma\,(c + K) = ((a + K)\,\gamma\,(c + K)) \oplus ((b + K)\,\gamma\,(c + K))$$

holds in $\Phi^*\left(A/_{c_\ell}\right)$.

($\mathcal{A}\Gamma N_3$) Since A is an approximately Γ-near ring, associative property holds in $\Phi^* A$ for all $a, b, c \in A$ and all $\beta, \gamma \in \Gamma$. Then,

$$\begin{aligned}((a+K)\beta(b+K))\gamma(c+K) &= ((a\beta b)+K)\gamma(c+K)\\ &= ((a\beta b)\gamma c)+K\end{aligned}$$

and

$$\begin{aligned}(a+K)\beta((b+K)\gamma(c+K)) &= (a+K)\beta((b\gamma c)+K)\\ &= (a\beta(b\gamma c))+K\\ &= ((a\beta b)\gamma c)+K,\end{aligned}$$

where $((a\beta b)\gamma c)+K \in (\Phi^* A)/_{c_\ell}$. From the hypothesis,

$$((a+K)\beta(b+K))\gamma(c+K) = (a+K)\beta((b+K)\gamma(c+K))$$

holds in $\Phi^*\left(A/_{c_\ell}\right)$.

Consequently, $A/_{c_\ell}$ is an approximately Γ-near ring. □

Definition 4.28. Let A be an approximately Γ-near ring and K be an approximately Γ-subnear ring of A. The approximately Γ-near ring $A/_{c_\ell}$ is called an approximately Γ-near ring of all near left weak cosets of A determined by K and denoted by $A/_\omega K$.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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