Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi (EFMED)
Cilt 15, Sayı 1, Haziran 2021, sayfa 118-143.ISSN: 1307-6086

Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education
Vol. 15, Issue 1, June 2021, pp.118-143.ISSN: 1307-6086

Araştırma Makalesi /Research Article

Ortaokul Matematik Öğretmen Adaylarının İspat Kavramlarının Fenomenografik Bir İncelemesi

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Gönderme Tarihi: 18.05.2021 Kabul Tarihi: 11.06.2021

Doi: 10.17522/balikesirnef.939068

Özet – Öğretmen adaylarının matematiksel akıl yürütmeyi öğretme yetenekleri sahip oldukları ispat kavramlarının kalitesine bağlıdır. Bu nitel çalışma, ortaokul matematik öğretmen adaylarının ispat kavramlarına odaklanmaktadır. Bu amaçla, bu çalışma öğretmen adaylarının ispat deneyimlerindeki farklılıkları belirlemek için fenomenografik bir yaklaşım kullanmıştır. Yarı yapılandırılmış görüşmelerin analizi, niteliksel olarak farklı beş kategori ortaya çıkarmıştır. Buna göre, ispat (a) bir problem çözme yoludur, (b) anlamanın bir aracıdır, (c) düşünmeyi ikna edici bir şekilde açıklamaktır, d) mantıksal argümanlar kullanarak varsayımları doğrulamaktır ve (e) matematiğin keşfi için bir araçtır. Bu çalışma, ispat kavramlarıyla ilgili pedagojik bilgiye katkıda bulunmaktadır. Sonuçlar, matematik öğretmeni hazırlık programlarının kalitesini artırmak için kullanılabilir.

Anahtar kelimeler: fenomenografi, ispat kavramı, öğretmen adayı

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Geniş Özet

Giriş

Matematiğin daha iyi öğretilebilmesi için tüm sınıf seviyelerinde ispata dayalı düşünmeye önem verilmesi gerekmektedir (Herbst & Balacheff, 2009; Stylianides & Stylianides, 2009). Öğrenciler, bulundukları sınıf seviyelerine uygun öğretim etkinlikleriyle matematiksel ifadelerin doğruluğunu veya yanlışlığını gösterme potansiyeline sahiptir (Stylianides, 2007). Önceki çalışmalar, hem ilkokul hem de ortaokul öğrencilerinin ispata yönelik etkinliklerde bulunabileceklerini ve matematiksel bir iddianın doğruluğunu nasıl tartışacaklarını anlayabileceklerini göstermektedir (Almeida, 2001; Miyazaki vd., 2017). Bununla birlikte, öğretimlerinde ispat etkinliklerine yer vermeleri ve öğrencileri bu etkinlikler için motive edebilmeleri için, matematik öğretmenlerinin etkili bir ispat anlayışına sahip olmaları gerekir (Knuth, 2002a; Stylianou vd., 2015). Ancak yapılan çalışmalar matematik öğretmenlerinin ve öğretmen adaylarının ispatın doğasını anlamamış olabileceklerini göstermektedir (Bansilal vd., 2017; Tanışlı, 2016). Ayrıca, öğretmen yetiştirme programlarının, öğretmenlerin kapsamlı bir ispat anlayışına sahip olmalarına rehberlik etmesini destekleyecek şekilde düzenlenmesine katkıda bulunabilecek olmasına rağmen, az sayıda araştırma matematik öğretmen adaylarının ispat kavramlarına odaklanmıştır (Lesseig vd., 2019; Sears, 2019). Bu nedenle, matematik öğretmen adaylarının ispat kavramlarının daha iyi anlaşılmasını sağlayabilecek çalışmalara ihtiyaç duyulmaktadır.

İspat kavramlarıyla ilgili çalışmalar, matematiksel bilgiye, yani deneysel, tümevarımlı veya tümdengelimli süreçlere (Makowski, 2020; Martin & Harel, 1989; Morris, 2002; Sears, 2019; Zeybek, 2017) veya içerik bilgisine yani ispatın anlamına odaklanmaktadır (Knuth, 2002a; Lesseig vd., 2019; Varghese, 2009). Bu çalışmada da, ispatla ilgili içerik bilgisi araştırılmıştır.

Bu çalışmada, matematik öğretmen adaylarının ispat kavramlarını belirlemek için nitel bir yaklaşım türü olan, fenomenografi kullanılmıştır. Bu bağlamda, çalışmanın araştırma problemi şu şekilde ifade edilmiştir: Ortaokul matematik öğretmen adaylarının ispat kavramlarının doğası ve kapsamı nedir?

Bu çalışma, öğretmen eğitimcilerinin öğretmen adaylarının ispatla ilgili içerik bilgilerini daha iyi anlamalarını sağlayabilir. Elde edilecek sonuçlar, öğretmen adaylarının matematik öğretim pratiklerini olumlu etkileyebilecek bir ispat anlayışına sahip olarak mezun olmalarına katkıda bulunabilir. Ayrıca, son yıllarda yapılan bir çalışma öğretmen adaylarının ispat kavramlarının kültürel olarak farklılaşabileceğini göstermiştir (Lesseig vd., 2019). Farklı ülkelerde çalışmalar yapılması öğretmen adaylarının ispat kavramlarının doğasının daha iyi anlaşılmasını sağlayabilir. Bu nedenle, bu çalışmanın sonuçları öğretmen eğitiminde kültürel farklılıkların anlaşılmasına da katkıda bulunabilir.

Yöntem

Eğitim araştırmalarında fenomenografi, öğrencilerin belirli bir fenomeni (bu çalışmada ispat kavramı) deneyimledikleri, anladıkları, algıladıkları veya kavramsallaştırdıkları niteliksel olarak farklı yolları ve bu fenomeni anlama yollarındaki çeşitliliği tanımlamayı amaçlamaktadır (Marton, 1981). Bu niteliksel olarak farklı anlama yollarının "referans" (veya atfedilen) ve "yapısal" olmak üzere iki bileşeni vardır (Marton & Pong, 2005; Pang, 2003). Referans ve yapısal bileşenlerin bir araya getirilmesi "sonuç alanı" olarak adlandırılır. Sonuç alanı, bir fenomenin çalışılan grup tarafından nasıl anlaşıldığının farklı yollarını ve bu yollar arasındaki ilişkileri temsil eden bir çerçevedir. Bu çerçeve birbiri ile nasıl ilişkili olduğu açıklanan kategorilerden oluşur (Akerlind, 2005).

Bu çalışmadaki katılımcılar bir devlet üniversitesinin matematik eğitimi anabilim dalında öğrenim gören dördüncü sınıf öğretmen adaylarıdır. Öğretmen adaylarının ispatı öğrendikleri ve kullandıkları matematik derslerinin çeşitliliği, ispat ile ilgili daha fazla ve farklı deneyimler yaşamalarını sağlayabilir. Bu farklı deneyimler öğretmen adaylarının ispat anlayışlarındaki çeşitliliğin daha iyi anlaşılması için önemlidir. Bu nedenle, ispatla karşılaşabilecekleri tüm matematik derslerini almış ve genel not ortalamalarına göre farklı başarı seviyelerinde olan öğretmen adaylarının ispat kavramları incelenmiştir. Veriler yarı yapılandırılmış bireysel görüşme yoluyla toplanmış ve analiz edilmiştir.

Bulgular

Veri analizinde araştırmacılar kendi ispat kavramlarının ya da literatürde önceden belirlenmiş ispat kategorilerinin veride olup olmadığına değil, tamamen öğretmen adaylarının yaklaşımlarına ve verdikleri yanıtlardaki çeşitliliğe odaklanmaya çalışmıştır. Fenomenografik analiz, ortaokul matematik öğretmen adaylarının ispat kavramlarının niteliksel olarak beş farklı kategoriyle ifade edilebileceğini göstermiştir. Ayrıca, bu kategorilerdeki farklılıklar "bilişsel süreçler", "kapsam", "ana odak" ve "duygular" olmak üzere dört boyutta tanımlanabilmiştir. Bu boyutlar ve kategoriler öğretmen adaylarının perspektifinden ortaya çıkmış ancak literatürde yer alan ispatla ilgili bazı anlamları da içermektedir. Buna göre, kategoriler ve ifade ettikleri anlamlar arasındaki ilişkiler basitten karmaşığa şu şekildedir: İspat bir problem çözme yoludur (Kategori A), ispat anlamak için bir araçtır (Kategori B), ispat düşüncenin ikna edici bir şekilde açıklanmasıdır (Kategori C), ispat mantıksal argümanlarla varsayımların doğrulanmasıdır (Kategori D) ve ispat matematiğin keşfi için bir araçtır (Kategori E).

İspat problem çözerken gerekli olan bilişsel süreçlerden biridir. Ancak ispatın sadece bir problem çözme yolu olarak görülmesi (Kategori A) farklı bağlamlardaki ispat farkındalıklarını sınırlandırabilir. İspatın bir çözüm yolu olarak görüldüğü bu kategori bu nedenle, ispat anlamak için bir araçtır kategorisinden (Kategori B) daha az anlam içermektedir. Kategori B' de sadece çözüm yolu değil bu çözüm yolunun kavramsal olarak bir konunun anlaşılmasındaki rolü de önemlidir. Bir düşüncenin veya matematikte kavramsal olarak anlaşılanın başkalarına da yazılı veya sözel olarak ikna edici bir şekilde açıklanmasının (Kategori C) bir ispat olduğunu düşünmek de Kategori A ve Kategori B' den daha fazla anlam içermektedir. Benzer şekilde, varsayımların mantıksal argümanlarla doğrulanmasını (Kategori D) bir ispat olarak görmek, önceki kategorilerden daha fazla anlam içerir çünkü bu kategoride öğretmen adayları ispatın matematiksel yapısının farkındadırlar. İspatı matematiğin keşfi için bir araç olarak görmek (Kategori E) ise önceki tüm kategorileri kapsamaktadır. Çünkü bu kategoride ispat sadece var olan matematiksel bilgiyle (problem çözüm yolu, matematiğin anlaşılması, açıklanması veya varsayımların doğrulanması) sınırlı değildir. Var olan bu bilgilere ek olarak yeni matematiksel bilgilerin keşfedilmesi olarak da görülmektedir. Bu nedenle Kategori E öğretmen adaylarındaki en karmaşık anlamı taşıyan kategori olarak nitelendirilmiştir.

Kategorilerin yapısal bileşeninin matematiksel bilgi ve ispat literatürüne paralel olarak tanımlanabileceği görülmüştür. Örneğin, Sfard (2000) matematiksel bilgiyi bir söylem olarak tanımlamakta ve bireysel olarak sonuçları bulma veya bu sonuçların diğerlerine iletilmesi olarak sınıflandırmaktadır. Buna göre, bu çalışmada elde edilen Kategori A ve B ispatın bireysel olarak sonuçların bulunması ve anlaşılması, Kategori C, D ve E ise sonuçların diğerlerine iletilmesi olarak sınıflandırılabilir. Ayrıca, Zaslavsky ve diğerleri (2012) bir açıklamanın matematiksel yapıdan bağımsız kişisel olarak ispat olarak kabul edilebileceğini belirtmektedir. Benzer şekilde, Kategori A, B ve C ispatın kişisel, Kategori D ve E ise ispatın matematiksel kabulü olarak sınıflandırılabilir.

Tartışma ve Sonuç

Bu çalışmada elde edilen kategoriler toplanan veriden ortaya çıkmıştır, ancak literatürde daha önce görülen bazı ispat anlamlandırmalarını da içermektedir. Önceki çalışmalardan farklı olarak kategoriler arasında görülen hiyerarşik ilişki ortaokul matematik öğretmen adaylarının ispat kavramsallaştırmalarındaki artan farkındalıklarla ilgili bir çerçeve sunmaktadır. Bu çerçevenin karşılaştırılabileceği başka bir çalışma bulunmamaktadır. Ancak, elde edilen kategorilerin içerdiği her bir anlam, önceki çalışmalarda belirtilen ispat anlamlandırmalarıyla karşılaştırılabilir. Örneğin, ispatın problem çözme yolu olduğu Uygan ve diğerlerinin (2014) çalışmasının, ispatın matematiği anlamadaki rolü ise Baştürk (2010) tarafından yapılan çalışmanın sonuçları ile benzerlik göstermektedir. Kategoriler kültürel açıdan da karşılaştırılabilir. Örneğin, bu çalışmadaki ispatın anlamak için bir araç olması ve ispatın matematiksel argümanlar kullanarak varsayımların doğruluğunun gösterilmesiyle ilgili anlamlar Lesseig ve diğerlerinin (2019) çalışmasındaki Koreli ve Amerikalı öğretmen adaylarının ispat tanımlarında da görülmüştür. İspatın matematiğin keşfi için bir araç olmasına ise sadece Koreli öğretmen adaylarının yanıtlarında rastlanmıştır.

Sonuç olarak, bu çalışmada bulunan kategoriler, öğretmen eğitimcileri tarafından öğretmen adaylarının ispat anlayışlarını anlamak ve değerlendirmek için bir model olarak kullanılabilir. Ayrıca, bu çerçeve öğretmen adaylarının matematik öğrenimlerinde ve öğretimlerinde ispatın önemli yönlerini göz önünde bulundurmalarına yardımcı olabilir. Ancak, daha detaylı ve genelleştirilebilir bir çerçeve elde etmek için başka çalışmalara da ihtiyaç vardır.

A Phenomenographic Investigation of Middle School Pre-service Mathematics Teachers' Conceptions of Proof

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Received :18.05.2021 Accepted : 11.06.2021

Doi: 10.17522/balikesirnef.939068

Abstract – The capability of pre-service teachers to teach mathematical reasoning depends on the quality of their proof conceptions. This qualitative study focuses on proof conceptions of middle school pre-service mathematics teachers. To this end, this study employed a phenomenographic approach to identify the variation in pre-service teachers' experience of proof. Analysis of semi-structured interviews revealed five qualitatively different categories: proof is (a) a way of problem-solving, (b) a means for understanding, (c) explaining thinking in a convincing way, d) validating conjectures using logical arguments, and (e) a means for discovery of mathematics. This study contributes to the pedagogical knowledge about a framework of proof conceptions. Results may be used to promote the quality of the mathematics teacher preparation programs.

Key words: phenomenography, pre-service teachers, proof conception

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Introduction

Mathematics educators argue that proving activities should become a part of mathematics teaching practices at all grade levels (Hanna, 2000; Herbst & Balacheff, 2009; Stylianides & Stylianides, 2009). The reason for this argument is that the logic of proof is necessary to develop students' mathematical reasoning. Students of all grades have the potential to understand the truth or falsehood of mathematical statements through some grade-level appropriate teaching activities (Stylianides, 2007). For example, previous research shows that students at elementary grades can engage in proof-oriented activities and understand how to argue for the truth of a mathematical claim (Almeida, 2001; Shifter, 2009). Similarly, previous studies report that middle-grade students can construct justifications for their solutions and convince their peers by explaining their reasoning (Aylar & Şahiner, 2014;

Miyazaki et al., 2017; Mueller, 2009). These findings support the view that students should encounter the activities involving reasoning and proving in early grades.

The manifestation of proof in school mathematics largely depends on the nature of teachers' views and understandings about proof (Knuth, 2002a; Stylianou et al., 2015). However, providing opportunities for their students to engage in proof-oriented activities can be challenging for mathematics teachers (Stylianides et al., 2013). Previous research shows that mathematics teachers may not have a proper understanding of proof (Tanışlı, 2016). Similarly, research demonstrates that pre-service teachers have difficulties in understanding and constructing proofs (Bansilal et al., 2017; Baştürk, 2010; Zeybek, 2015). These results suggest that teacher preparation programs should pay a careful attention to the proof understandings of pre-service mathematics teachers. The research on proof conceptions may lead to refining teacher preparation programs in a way to better support the development of teachers' understanding of proof. However, the studies focusing on pre-service mathematics teachers' conceptions of proof are not so abundant (Lesseig et al., 2019; Sears, 2019). Therefore, there is still much work to do to further clarify pre-service mathematics teachers' proof conceptions.

Studies regarding proof conceptions usually focus on mathematical knowledge, i.e., empirical, inductive, or deductive processes (e.g., Makowski, 2020; Martin & Harel, 1989; Morris, 2002; Sears, 2019; Zeybek, 2017), or subject matter content knowledge, i.e., the meaning of proof (e.g., Knuth, 2002a; Lesseig et al., 2019; Varghese, 2009). This present study focuses on the subject matter content knowledge.

Literature Review

Understanding Mathematical Proof

Proof plays a central role in mathematical thinking and it can be defined primarily as an argumentation (i.e., logical process) that justifies the truth of mathematical claims. Besides, researchers indicate that proof is not only whether the mathematical claims are true, but also is about why these claims are true (Hanna, 1995; Hersh, 1993). Rav (1999) states that proofs involve the know-how of mathematics and they are the bearers of mathematical knowledge. According to him, proof should be seen as new methods, strategies, concepts to solve problems, the foundation of interconnections between theories, and the systematization of the mathematical results. Similar to Rav (1999), de Villiers (1999) also emphasizes proof as a systematization (i.e., integrating various mathematical results to establish a deductive system of axioms, definitions, and theorems.). Besides, de Villiers (1999) argues that proof serves as a means of the creation of new mathematical results through a deductive process. He states that proof should also be regarded as a communication tool among members in a community, for example, between mathematicians or between teachers and students. He also states that proof should be understood as an intellectual challenge. Mathematicians may view proving as a challenging activity such as solving puzzles or other creative attempts. In addition, researchers emphasize that proof serves as a means of understanding mathematics (Hersh, 1993; Knuth 2002a; Weber, 2010).

Prior Research on Meanings of Proof

According to de Villiers (1990), teachers should understand that the above discussed aspects of proof are crucial at all grade levels. Otherwise, teachers may think that it is not reasonable to introduce proof in early grade mathematics. Knuth (2002a) finds similar meanings in teachers' explanations regarding the nature and the role of proof. According to him, teachers view proof as logical thinking, displaying thinking, communicating mathematics, explaining mathematical reasons, and creating mathematics knowledge. Similarly, in another research, Knuth (2002b) reveals that mathematics teachers view proof as a means of verification, explanation, communication, knowledge creation, and systematization of results. However, studies show that pre-service teachers may not be aware of some of these meanings of proof. For example, in the work of Mingus and Grassl (1999), the majority of pre-service mathematics teachers state that proof is an explanation for mathematical concepts used in arguments. Varghese (2009) finds that a minority of the pre-service teachers consider the explanatory and discovery meanings of proof. She also finds that secondary school pre-service teachers mostly define proof as verification of a mathematical statement. Pre-service teachers' awareness of proof as a verification tool is the most reported finding in the studies (e.g., Baştürk, 2010; Likando & Ngoepe, 2014; Uygan et al., 2014). In another study, Dickerson and Doerr (2014) state that how high school mathematics teachers think about the role of proof varies widely. In their study, teachers view proof as a tool to build mathematical understanding, develop logical thinking and metacognitive skills, and reify mathematical knowledge.

In addition, Lesseig et al. (2019) demonstrate that a minority of secondary school pre-service teachers are aware of the discovery and communication meanings of proof. Their study also shows that proof conceptions vary across countries. For example, in their research, Korean pre-service teachers mostly mention the verification and understanding functions of proof but do not mention the systematization and communication functions of proof. Unlike the Korean pre-service teachers, American and Australian pre-service teachers consider the systematization function of proof. Also, only American pre-service teachers mention the communication role of proof.

The present study

Given that the pre-service teachers' capability to teach reasoning depends on the quality of their proof conceptions, in this study, we aim to understand the subject matter proof conceptions of middle school pre-service mathematics teachers. Pre-service mathematics teachers' understanding of proof concept mostly depends on their proof experiences in the high school or university mathematics courses. The variety of these experiences can reveal different conceptualizations of proof. A particular focus on these conceptualizations may lead to allow a deeper understanding of the proof conceptions of pre-service teachers. This focus may provide a framework for teacher educators to facilitate the pre-service teachers' education by changing less desirable proof conceptions to the more desirable ones. As pre-service teachers' proof conceptions may vary from culture to culture (Wilder, 1981) addressing proof conceptions of pre-service teachers in different countries might provide some additional information to reveal a more definite picture of proof conceptions. Therefore, the results of this study may also contribute to the understanding of cultural differences in teacher education. As will be described later in the paper, in this study, the approach we chose to determine the proof concepts of pre-service mathematics teachers is phenomenography. In this context, the present study investigates the following research question: What are the nature and range of pre-service mathematics teachers’ conceptions of proof?

Methods

Research Design

In educational research, phenomenography aims to describe the qualitatively different ways that students experience, understand, perceive or conceptualize a particular phenomenon (conception of proof in this study) and the variation in the way of understanding this phenomenon (Marton, 1981). Conceptions, ways of experiencing or ways of understanding are the examples of the terms used to describe the knowledge that phenomenography investigates (Marton & Pong, 2005). Phenomenography assumes that the way of understanding or conceptualizing something is an internal relationship between the experiencer and the experienced phenomenon. The experiencers cannot be aware of all the characteristics of a phenomenon at the same time. Also, individuals may experience different characteristics of a phenomenon. Even if they experience some similar characteristics, they may perceive them in different ways (Marton & Pang, 2008; Pang, 2003; Yates et al., 2012). Therefore, there are qualitatively different ways of understanding a phenomenon. On the other hand, these different ways are not endless but are limited in number (Marton & Booth, 1997; Pang, 2003; Cibangu & Hepworth, 2016). These qualitatively different ways of understanding can be described regarding two aspects: the referential aspect (or attributed component) and the structural aspect (Marton & Pong, 2005; Pang, 2003).

In line with the research question of this study, the referential aspect deals with what proof understandings the pre-service mathematics teachers have. The referential aspect aims to produce description categories to represent such conceptions. Thus, using phenomenography in this research, we aimed to reveal a set of description categories describing the pre-service mathematics teachers' ways of understanding proof.

The structural aspect, on the other hand, deals with the relationship among the description categories. Despite some distinctions among the categories, they are oriented on the same phenomenon. Therefore, it is reasonable to expect the categories to be logically related to each other (Limberg, 2008; Marton, 1981). This relationship can be described hierarchically from least to most complex or from general to specific (Marton, 2000). The nature of this relationship (i.e., how these qualitatively different categories are related to each other and how they together form a whole) is referred to as the structural aspect. The structural aspect can also be considered as the researcher's interpretation of the variation among the description categories (Marton & Pong, 2005; Pang, 2003).

The combination of the referential and the structural aspects forms the outcome space. The outcome space can also be regarded as a framework that represents the different ways of how a phenomenon is understood by the studied group, and the relationships among these ways (Akerlind, 2005).

Finally, it is important to state that this present study does not intend to reveal if pre-service mathematics teachers possess some specific or pre-determined proof conceptions.

Instead, the study intends to provide an outcome space that illustrates the range of proof conceptions present within the group of pre-service mathematics teachers.

Participants

The participants of this study were 16 undergraduate students (11 females and 5 males) majoring in mathematics education (middle school mathematics, grades 5-8) at a state university. The participants were selected using the purposive sampling method to maximize the variation in their proof understandings. The criteria used to this end were the following: First, the participants should be among the fourth-grade level students who took all the mathematics courses in their curriculum (e.g., Abstract Mathematics, Analysis I-II, Algebra, Linear Algebra I-II). The variety of mathematics lessons pre-service teachers took may allow them opportunities to experience different aspects of the concept of proof. Second, the study group should be composed of students from a wide range of achievement levels. Since students' achievement levels may be related to their proof schemes, cumulative grade point average (CGPA) scores were taken into account to obtain the required variation in the sample. CGPA scores of the students sampled in this concern varied from 1.94 to 3.51 (out of 4). Third, the sample should include students of different gender because gender may also be related with the conceptualization of proof.

Overall, with these criteria, we aimed to assure that the participants possessed a wide range of experiences a phenomenographic study requires (Marton & Booth, 1997). The pre-service teachers voluntarily participated in the study. We informed the participants about what we aim, and what we expect for the output in the study. We also informed the participants that their names and interview records would be kept confidential.

Data Collection

The suggestions of Akerlind (2005) on preparing interview questions and conducting interviews guided our data collection process. We used the semi-structured interview procedure that is usually referred as a useful data collection technique in a phenomenographic study. We tried to prepare interview questions that would reveal the variation in proof understandings of participants. To this end, we conducted pilot interviews with a different group of pre-service teachers to test the interview questions we prepared. The final set of interview questions after the revision process, could be specified as follows: a) what proof and proving mean to pre-service teachers, b) how pre-service teachers define proof, c) what features they think an argument should include to be a proof, and d) how they evaluate their own or others' proofs. Also, during the interview process, participants were sometimes asked to further explain "what they mean" and "why they think that way" to elicit their intentional attitude towards the proof concept. Each interview lasted around 30 minutes. Interviews were recorded and then transcribed verbatim for data analysis.

Data Analysis

Although some steps have been suggested to follow, there is no agreed method for conducting phenomenographic data analysis (Han & Ellis, 2019). In this study, we followed similar data analysis steps as in the work of Akerlind's (2005) and Gonzales's (2010). First, we independently read all interviews several times to understand the details and gain a sense of the data. We highlighted the sections related to the question being investigated based on their commonalities and differences. In this process, to minimize subjectivity, we attempted to discern meanings found in the participants' responses and did not attempt to measure predetermined categories in the literature. Also, we refrained imposing what we believe to be the concept of proof.

Second, after the investigations of all meanings between and within the transcripts' sections, we came together to compare and discuss the initial list of meaning statements we had specified. In this step, first, we agreed on the meaning statements. Then we tried to reduce these initial sets of meaning statements by looking for dimensions of variation as experienced by the participants. We grouped the meaning statements which we thought representing a similar understanding of proof. As a result, we obtained initial "categories of description" that represented the variation in pre-service teachers' understanding of proof.

Third, we re-read the transcripts to better distinguish each category from the others and decided on the final version of the description categories. We independently re-examined the accuracy of the categories several times by focusing on critical aspects of the proof understandings of the pre-service teachers. This process enabled us to define the variation among the categories of the proof itself as experienced by the pre-service teachers. After reaching a consensus on the categories of description, we grouped the transcripts by the category they best represent.

Next, we re-examined the dimensions of variation in the categories together with the participants' responses to interpret the structural aspect. We investigated empirical and logical inclusive relationships to reveal the hierarchical relationships between the categories. Concerning the empirical inclusiveness, we saw that pre-service teachers' responses in one category include some meanings that exist in preceding categories. For example, the meanings such as "correctness" and "explaining" in Category C (i.e., proof is explaining thinking in a convincing way) exist in the responses related to Category D (i.e., proof is validating conjectures using logical arguments). Therefore, Category D includes Category C. We explained how we defined the logical inclusiveness among the categories in the Findings section below.

Findings

In this section, we primarily present the categories of description that emerged in the data analysis process. Then, we explain the outcome space that describes the relationships (structural and inclusive) between the categories with their referential aspect.

This study identified five categories of description of proof: a) proof is a way of problem solving, b) proof is a means for understanding, c) proof is explaining thinking in a convincing way, d) proof is validating conjectures using logical arguments, and e) proof is a means for discovery of mathematics. The dimensions of variation and the categories of description of proof are seen in Table 1.

Table 1 Dimensions of Variation and Categories of Description of Conceptions of Proof

	Category A	**Category B**	**Category C**	**Category D**	**Category E**
Key aspects	proof is a way of problem solving	proof is a means for understanding	proof is explaining thinking in a convincing way	proof is validating conjectures using logical arguments	proof is a means for discovery of mathematics
Cognitive process	problem solving, using strategies	deep thinking, not learning by heart, reasoning	transferring, communicating, demonstrating, explaining, accepting	articulating reasons, validating, arguing step by step	improving, discovery, expanding
Main Focus	finding a result	conceptual understanding	explaining thinking, arguments or solutions	extended how to validate arguments mathematically	further extended to construct new mathematical results
Scope	correctness of a solution	grasping the meaning	certainty, reality, convincing arguments	existing conjectures, already known knowledge	new conjectures, discovering new knowledge
Feelings	not clearly seen	positive disposition	feeling of no-doubt, making sure	not clearly seen	appreciation, admire mathematicians work

The dimensions of variation are the main focus, scope, cognitive process, and feelings. After explaining each category of description with quotes from transcripts, we demonstrated the outcome space (referential and structural relationships) and inclusive relationships among the categories of description in Table 2.

Description of Categories

To indicate that the selected quotes are from different transcripts a number and gender of participants were provided at the end of the quotes (e.g., PST1, gender).

Category A: Proof is a way of problem solving

In this category, the proof is seen as being synonymous with a way of problem solving. The responses in this category do not go further than considering proof as a way of solution. In this category, pre-service teachers emphasize the solution or right answer to a problem. The following quote illustrates this understanding.

> I think it [proof] is to be the solution to a problem, and, well, if the term "unfalsifiable" is correct, proof can be a way of unfalsifiable solution. The definition of proof is the best of the ways that are used to solve a problem we encountered. [PST12, female]

Some pre-service teachers in this category expressed the way of problem-solving. In the following quote, one pre-service teacher talks about modeling the problem-solving process and describes the similarity between proving and solving a problem. However, her expressions are limited to follow some steps.

> …when being encountered a problem, first, [we're looking at] what is given in the problem,...it [proof] is also, similar to the steps of problem-solving. For the next, what do I know, what is the thing related to what I know? Following [step], what was in the second step of problem-solving? We can use a model, for example. Similarly, we also need modeling while making proof. [PST3, female]

Certainly proving is one of the important cognitive processes for problem solving, but seeing proof as the best way to solve a problem limits the awareness of proof in other contexts. This understanding may cause future teachers to give importance only to problem-solving practices and therefore not to include other proving practices in their teaching. This category is different from Category B because there is no evidence that proof is associated with reasoning in other contexts in mathematics. Therefore, this category is the least complex category we found in the analysis of data.

Category B: Proof is a means for understanding

In category B, pre-service teachers viewed proof as a means for grasping the meaning or going deep into a subject. Pre-service teachers presenting this understanding think that proof is important for conceptual understanding. According to them, subjects cannot be learned without proof but learned by heart. For example:

> What do I mean by proof? If we interpret [proof] in the context of mathematics, I understand something such as grasping, making sense of a subject, going deeper...For example, while teaching the order of fractions, it is written generally directly as "this is greater than that"... [In class] I used a model to explain why one is bigger and why another is smaller. Well, the proof came into play here. We used models to see which one is closer to 1. How closer to 1? How many pieces do I have to add to both of them to reach 1? [I asked]. Is the piece we add to this bigger or is the piece we add to the other bigger? [I asked]. In this way, [students are] making sense of them. Well, here I put the proof into play. [PST8, male]

> If we want to go deeper into a subject [in mathematics], [in other words], if we want to understand it conceptually, for me, proof is necessary … We should understand its logic; there should not be learning by heart. [PST11, male]

In addition, some pre-service teachers possessing the view in this category have a positive disposition towards the proving process due to its role to better understand and teach mathematics. The following quote is an example of these feelings.

> [With the help of proof] When I will explain a subject [in mathematical content] to the students I can easily make them assimilate it, because I have assimilated it. I can get them to love mathematics more easily. Thus, it will attract more attention from the student. [PST14, male]

Category B is different from Category A. In Category A, the main focus is on finding a solution to a problem, whereas in Category B the main focus is on conceptual understanding. Category B reflects a more complex perception of what proof is in mathematics. Therefore, Category B includes Category A. According to pre-service teachers, proof is a valuable means for learning and teaching mathematics. However, although they mention the role of proof in understanding, the answer of how this understanding is possible is seen more clearly in Category C. In addition, there is no evidence about the details of the cognitive processes in proving as seen in Category D.

Category C: Proof is explaining thinking in a convincing way

The conception in this category represents a shift from understanding an idea to explaining thinking. In Category C, the scope of the pre-service teachers' proof understandings is explaining thinking written or verbally. As seen in the following quotes, according to pre-service teachers, it is important to be convinced or to convince others that a thinking or an idea is correct, and proof is a means that enables a thinking to be understood by others.

> For example, we say that prime numbers are infinite. .. I should think about how this happened. If it's proof in my mind so that I can both explain and convince the others. [PST10, female]

> In my opinion, proof means...to be able to explain thinking to someone clearly and to show them its correctness. [PST14, male]

> For example, when a friend of mine does a proof in class, sometimes, I am getting stuck on something in their explanation. When I get stuck with something like this, this proof is not accepted for me. I do not know if their way of proof is right, but it is not accepted for me when I get stuck at a point when I do not understand. It is important to be descriptive and explanatory... As I said, it is also very important that you can convey and write what you say. This is necessary to have it admitted to being true, in other words, to show that it cannot be falsified and that whoever does will get the same thing. [PST12, female]

In this category, it is also seen that there are feelings that encourage pre- service teachers to prove such as, not doubting and being sure.

> I make sure that something [an idea] is correct, so I feel comfortable. I do a proof to be sure. The proof is done to show how it [the idea] exists. [PST7, female]

Category C differs from Category B in that it focuses not only on grasping the mean of an idea but also on convincingly explaining its reality or rationality. Similarly, Category C different from Category A. In category A, the scope is only a solution of a problem, whereas, in Category C, the scope is an understandable explanation of a solution. Therefore, Category C reflects a more complex understanding of what proof is.

Category D: Proof is validating conjectures using logical arguments

In Category D, proof is described as a means for validation of given conjectures in mathematics. As seen in the following quotes, pre-service teachers are more aware of the processes of mathematical proof.

> When you say proof, the first thing that comes to my mind is to prove something, to show its correctness. Well, it means to show that a conjecture is true by using particular things mathematically, [for example], by using particular axioms. [PST7, female]

> When we go from the least to the most complex feature, we are explaining lots of features and finally, after combining and justifying all of these features, we are getting that complex structure [proof]. [PST5, female]

In the following quotes, pre-service teachers explain logical arguments in the justification process by using the expressions such as “sequence,” “path,” “cause-effect,” “pre-end relationship,” or “steps.”

> When I look at the proofs, they all have a sequence. I realized that one part of "proof" comes before

> another according to a logical sequence... If one comes after, it will not support the other. So, one must come before the other. [PST15, female]

> The proof is a conclusion that being reached after following a particular path by establishing a pre-end relationship or a cause-effect relationship. [PST1, male]

> [In the proof] I must have given a justification for each step. [PST14, male]

> I examine the steps. I examine the meanings of statements...one statement must be the reason for the following statement. I look at these steps. The proof is used to see how these (statements) are connected and related to each other. [PST7, female]

In this category, arguing step by step, justifying, and articulating reasons are the new cognitive processes we found in pre-service teachers' responses. These processes describe how pre-service teachers use proof to validate conjectures. In both Category C and Category D, pre-service teachers focus on demonstrating the truth of an argument. However, in Category D, pre-service teachers ascribed a more specific meaning to proof. The content of this category is more aligned with the formal definition of "proof" in mathematics compared to previous categories. Therefore, Category D is qualitatively different from the other categories.

Category E: Proof is a means for discovery of mathematics

What makes this last category qualitatively different from the prior categories is the understanding of proof as a means for the discovery or invention of new results or knowledge in mathematics. In this category, rather than proof being a means for validating existing conjectures, the new conjectures themselves are discovered or invented by proof. The following quotes illustrate this understanding.

> For example, the proof was required to reveal number theorems…Proof was required to find the existence of natural numbers, the existence of integers, the existence of all of them… So nothing had certainty. I think all emerge from particular proofs and finding out the truth of particular theorems. [PST4, female]

> Because through the use of proof, some concepts in nature also were discovered like Fibonacci. [PST5, female]

As seen in the following quote, some pre-service teachers see the discovery function of proof as expanding the boundaries of mathematics.

> There is different mathematics used by someone [a mathematician] in terms of further development of mathematics we use, and the proof is required to expand the boundaries of this mathematics. [PST1, male]

In addition, admiration or appreciation is the feeling that we found in the responses of pre-service mathematics teachers who present this view.

> The proof is the stages of the emergence of something geometric or anything mathematically... I admire people who can prove, because... I could never think of that. [PST2, female]

Category E is the most complex category we found as the proof concept of pre-service mathematics teachers.

In Table 2, the outcome space that describes the relationships between the categories with their referential aspect is presented. As seen in Table 2, we organized the categories from least to the most complex way of understanding proof according to the depth of meaning we explained above. The outcome space we presented is hierarchically inclusive. In Table 2, we presented this outcome space with referential and structural aspects of each category together.

Table 2 Referential and Structural Aspects of Categories of Description, and Hierarchical Relationships

	Structural (discourse)		**Structural** (acceptance)	
Referential	Oneself	Others	Personal	Mathematical
Proof is a way of problem solving	A		A	
As in (A) and a means for understanding	B		B	
As in (B) and explaining thinking in a convincing way		C	C	
As in (C) and validating conjectures using logical arguments		D		D
As in (D) and a means for discovery of mathematics		E		E

As stated above, Category E reflects the more inclusive and complex understanding of proof in mathematics. It includes the elements of previous categories. In this category, "proof" is not only seen as a solution for problems, a means for understanding or verification of thinking and already known knowledge but it also is considered as a means of discovery of new knowledge in mathematics. Similarly, Category D includes not only some elements of previous categories such as finding solutions, understanding mathematics, and explaining thinking, but it also includes an awareness of how to validate conjectures mathematically. On the other hand, Category A represents the least complex understanding of proof, as it shows no awareness of conceptual understanding and explaining thinking or validating conjectures at all. Proof awareness in Category A is only limited to the solution of a problem.

According to Sfard (2000), mathematical knowledge is a discourse practice with oneself (trying to find or understand results) and with others (trying to communicate results to someone else). Similarly, we considered Categories A and B as the "oneself" component because pre-service mathematics teachers' conceptualizations are mostly related to their own proving and understanding. We considered Categories C, D, and E as the "others" component because the conceptualizations in these categories include meanings of communicating a proof with others. Besides, Zaslavsky et al. (2012) stated that the decisions given in accepting something as proof may not always be mathematical but may also be personal. In Categories A, B, and C, there is no evidence that the proof understandings of pre-service teachers should be mathematical. Therefore, in the acceptance aspect, Categories A, B and C are identified as the "personal" component. In category A, pre-service teachers may think that their solution to a problem is proof although it is not proof. In Category B, the reasoning that helps them to understand mathematics better may not be a mathematical proof. Similarly, in Category C, it seems that the proof understanding is personal because there is no strong evidence that the responses related to "explaining thinking in a convincing way" are consist of the cognitive processes of mathematical proof. On the other hand, in Categories D and E, the cognitive processes better reflect the nature of mathematicians' work. Therefore, we considered Categories D and E as the mathematical component.

Discussion and Conclusion

The phenomenographic analysis of the pre-service teachers' responses regarding the proof conception revealed five qualitatively different categories. These categories emerged from the data but also reflected some of the meanings presented in the introduction. The hierarchical relationship between these categories from least to most complex is; proof is a way of problem solving, proof is a means for understanding, proof is explaining thinking in a convincing way, proof is validating conjectures using logical arguments, and proof is a means for discovery of mathematics. Four dimensions of variation were emerged (cognitive processes, scope, main focus, and feelings) in understanding these proof conceptions. Unlike previous studies, this study reveals expanding awareness of the proof conception that provides a deeper understanding of the proof conceptions of pre-service middle school mathematics teachers. The framework we obtained may be helpful to understand how teachers need to know proof they teach (Ball et al., 2008). Below, we discussed the categories found in this study with a selection of the available literature.

Previous research indicates that pre-service teachers may view proof as a problem-solving process (Uygan et al., 2014). Similarly, in this study, we found a category (Category A) in which pre-service teachers equate proof with a way of problem solving. In recent research, Son and Lee (2021) show that pre-service teachers' problem-solving views do not go further than a means to a solution. In this respect, combined with the literature, pre-service teachers who focus on answer or solution in this proof conception may tend to view problem-solving as finding a solution. However, with this lower-level perspective on problem-solving, the mathematical reasoning feature of proof may not go beyond finding a solution to a problem in school mathematics. Therefore, the relationship between pre-service teachers' proof conceptions and problem-solving conceptions needs to be detailed in further studies.

Also, previous studies indicate that proof should be used to promote the understanding of mathematical concepts (Knuth, 2002a; Weber, 2010). In this study, Category B (proof is a means for understanding) is in line with this view. Similarly, Dickerson and Doerr (2014) report that one of the concepts in the answers of mathematics teachers is "proof provides understanding." Baştürk (2010) finds a similar conception in which proof is seen as deep thinking by pre-service teachers. In his research, first-year secondary school pre-service mathematics teachers who presented this view focused on learning mathematical ideas and criticized learning by heart. This result parallels the cognitive processes we found in Category B.

In this study, Category C denotes that proof is explaining thinking in a convincing way. This finding is in line with the literature that indicates that the notion of convincing argument can be seen in the proof conceptions of undergraduates (Davies, Alcock, & Jones 2021, Knuth, 2002a).

Results of this study might be considered from a cultural perspective as well. For example, in the work of Lesseig et al. (2019), both American and Korean pre-service teachers indicate that proof serves to deepen understanding of mathematical concepts. Similarly, teachers in this study also emphasized the "understanding role" of proof. The work of Lesseig et al. (2019) also states that Korean pre-service teachers focus on the verification meaning of proof. Korean pre-service teachers do not consider the systematization role of proof while American and Australian pre-service teachers consider it. Korean and Turkish pre-service teachers' proof understandings may be very similar. Like Korean pre-service teachers, Turkish pre-service teachers hold conception (Category D, i.e., proof is validating conjectures using logical arguments) related to the verification/validation meaning of proof. Likewise, in this study, an understanding linked to the systematization role of proof is absent in Turkish pre-service teachers' responses.

This similarity also exists concerning the discovery meaning of proof. In the work of Lesseig et al. (2019), few but only Korean pre-service teachers mention the discovery of mathematical theorems. In another study, Varghese (2009) states that only one pre-service teacher is aware of the discovery role of proof in his study group. The most complex level category of description revealed in this study is “proof is a means for discovery of mathematics.” This more advanced understanding of proof might be a reason of why only the minority of pre-service teachers mention this conception.

Before concluding, it is crucial to address some limitations in the study. The study revealed proof conceptions of 16 participants coming from one Turkish university. Further research from different universities with more participants may reveal different proof understandings of middle school pre-service mathematics teachers. Another limitation of the study may be that the participants' lack of appropriate vocabulary in explaining their thinking about proof. This study is the first phenomenographic attempt to examine the pre-service mathematics teachers' conceptions of proof. Despite its limitations, the categories of description found in this study offer a framework for expanding awareness of proof conceptions. The framework in this study may be used by teacher educators as a model to understand and evaluate pre-service teachers' conceptions of proof. Also, this framework may be helpful for pre-service teachers to consider crucial aspects of proof in their mathematics learning and teaching. However, further studies are needed to obtain a more detailed and generalizable framework. In this respect, research associating conceptions of teaching proof in middle school and conceptions of proof may provide more insight into relevant literature.

Notes

Ethical approval for this study was obtained from Ethics Committee for Human Studies in Social Sciences of Bolu Abant İzzet Baysal University (01.02.2021 / 21).

References

Akerlind, G. S. (2005). Learning about phenomenography: Interviewing, data analysis and the qualitative research paradigm. In J.A. Bowden and P. Green (Eds.) *Doing developmental phenomenography* (pp. 63-74). Melbourne: RMIT University Press.

Almeida, D. (2001). Pupils' proof potential. *International Journal of Mathematical Education in Science and Technology, 32*(1), 53-60. https://doi.org/10.1080/00207390119535

Aylar, E., & Şahiner, Y. (2014). A study on teaching proof to 7th grade students. *Procedia - Social and Behavioral Sciences, 116,* 3427-3431. https://doi.org/10.1016/j.sbspro.2014.01.777

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bansilal, S., Brijlall, D., & Trigueros, M. (2017). An APOS study on pre-service teachers' understanding of injections and surjections. *Journal of Mathematical Behavior, 48,* 22-37. https://doi.org/10.1016/j.jmathb.2017.08.002

Baştürk, S. (2010). First-year secondary school mathematics students' conceptions of mathematical proofs and proving. *Educational Studies, 36*(3), 283-298. https://doi.org/10.1080/03055690903424964

Cibangu, S. K., & Hepworth, M. (2016). The uses of phenomenology and phenomenography: A critical review. *Library & Information Science Research, 38*(2), 148-160. https://doi.org/10.1016/j.lisr.2016.05.001

Davies, B., Alcock, L, & Jones, I. (2021). What do mathematicians mean by proof? A comparative-judgement study of students' and mathematicians' views. *Journal of Mathematical Behavior, 61*, 100824. https://doi.org/10.1016/j.jmathb.2020.100824

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24,* 17-24.

de Villiers, M. (1999). *Rethinking Proof with Sketchpad.* Key Curriculum Press.

Dickerson D. D., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. *Mathematics Education Research Journal*, *26*(4), 711–733. https://doi.org/10.1007/s13394-013-0091-6

González, C. (2010). What do university teachers think eLearning is good for in their teaching? *Studies in Higher Education, 35*(1), 61-78. https://doi.org/10.1080/03075070902874632

Han, F., & Ellis, R. A. (2019). Using phenomenography to tackle key challenges in science education. *Frontiers in Psychology, 10,* 1-10. https://doi.org/10.3389/fpsyg.2019.01414

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42-49. https://www.jstor.org/stable/40248188

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*(1-2), 5-23. https://doi.org/10.1023/A:1012737223465

Herbst, P., & Balacheff, N. (2009). Proving and knowing in public: The nature of proof in a classroom. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 40-63). Routledge.

Hersh, R. (1993). Proving is convincing and explaining. *Educational studies in Mathematics, 24 (4)*, 389-399. https://www.jstor.org/stable/3482651

Knuth, E. J. (2002a). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61-88. https://doi.org/10.1023/A:1013838713648

Knuth, E. J. (2002b). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33* (5), 379-405. https://doi.org/10.2307/4149959

Lesseig, K., Hine, G., Na, G., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across pre-service secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal, 31*(4), 393–418. https://doi.org/10.1007/s13394-019-00260-7

Likando, K. M., & Ngoepe, M. G. (2014). Investigating mathematics trainee teachers' conceptions of proof writing in algebra: A case of one college of education in Zambia. *Mediterranean Journal of Social Sciences, 5*(14), 331-338. https://doi.org/10.5901/mjss.2014.v5n14p331

Limberg, L. (2008). Phenomenography. In L. M. Given (Ed.), *The SAGE encyclopedia of qualitative research methods* (pp. 611-614).SAGE Publications.

Makowski, M. B. (2020). The written and oral justifications of mathematical claims of middle school pre-service teachers. *Research in Mathematics Education.* https://doi.org/10.1080/14794802.2020.1777190

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for research in mathematics education*, *20*(1), 41-51. https://doi.org/10.5951/jresematheduc.20.1.0041

Marton, F. (1981). Phenomenography-describing conceptions of the world around us. *Instructional Science, 10*(2), 177-200. https://doi.org/10.1007/BF00132516

Marton, F. (2000). The structure of awareness. In J. A. Bowden, & E. Walsh (Eds.), *Phenomenography* (pp. 102-116). RMIT Publishing.

Marton, F., & Booth, S. (1997). *Learning and awareness.* Lawrence Erlbaum Associates.

Marton, F., & Pang, M. F. (2008). The idea of phenomenography and the pedagogy of conceptual change. In S. Vosniadou (Ed.), *International Handbook of Research on Conceptual Change* (pp. 533-559). Routledge.

Marton, F., & Pong, W. Y. (2005). On the unit of description in phenomenography. *Higher Education Research and Development, 24*(4), 335-348. https://doi.org/10.1080/07294360500284706

Mingus, T. T. Y., & Grassl, R. M. (1999). Preservice teacher beliefs about proofs. *School Science and Mathematics, 99*(8), 438-444. https://doi.org/10.1111/j.1949-8594.1999.tb17506.x

Miyazaki, M., Fujita, T., & Jones, K. (2017). Students' understanding of the structure of deductive proof. *Educational Studies in Mathematics, 94*(2), 223-239. https://doi.org/10.1007/s10649-016-9720-9

Morris, A. (2002). Mathematical reasoning: Adults’ ability to make the inductive-deductive distinction. *Cognition and Instruction, 20*(1), 79–118. https://doi.org/10.1207/S1532690XCI2001_4

Mueller, M. F. (2009). The co-construction of arguments by middle-school students. *The Journal of Mathematical Behavior, 28*(2-3), 138-149. https://doi.org/10.1016/j.jmathb.2009.06.003

Pang, M. F. (2003). Two faces of variation: On continuity in the phenomenographic movement. *Scandinavian Journal of Educational Research, 47*(2), 145–156. https://doi.org/10.1080/00313830308612

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, *7*(1), 5–41. https://doi.org/10.1093/philmat/7.1.5

Sears, R. (2019). Proof schemes of pre-service middle and secondary mathematics teachers. *Investigations in Mathematics Learning, 11*(4), 258-274. https://doi.org/10.1080/19477503.2018.1467106

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2*(3), 157-189.https://doi.org/10.1207/S15327833MTL0203_1

Shifter, D. (2009). Representation-based proof in the elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 71-86). Routledge.

Son, J. W., & Lee, M. Y. (2021). Exploring the relationship between preservice teachers' conceptions of problem solving and their problem-solving performances. *International Journal of Science and Mathematics Education, 19,* 129-150. https://doi.org/10.1007/s10763-019-10045-w

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321. https://doi.org/10.2307/30034869

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314-352. https://doi.org/10.5951/jresematheduc.40.3.0314

Stylianides, G., Stylianides, A., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning and-proving. *International Journal of Science and Mathematics Education, 11(6),* 1463–1490. https://doi.org/10.1007/s10763-013-9409-9

Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. *International Journal of Research in Undergraduate Mathematics Education, 1*(1), 91-134. https://doi.org/10.1007/s40753-015-0003-0

Tanışlı, D. (2016). How do students prove their learning and teachers their teaching? Do teachers make a difference? *Eurasian Journal of Educational Research,* (66), 47-70. http://dx.doi.org/10.14689/ejer.2016.66.3

Uygan, C., Tanışlı, D., & Köse, N. Y. (2014). Research of pre-service elementary mathematics teachers' beliefs in proof, proving processes and proof evaluation processes. *Turkish Journal of Computer and Mathematics Education (TURCOMAT), 5*(2), 137-157. https://doi.org/10.16949/turcomat.33155

Varghese, T. (2009). Secondary-level student teachers' conceptions of mathematical proof. *Issues in the Undergraduate Mathematics Preparation of School Teachers.* http://www.k-12prep.math.ttu.edu

Weber, K. (2010). Proofs that develop insight. *For the Learning of Mathematics, 30*(1), 32–36.

Wilder, R.W. (1981). *Mathematics as a cultural system,* Pergamon, New York.

Yates, C., Partridge, H. L., & Bruce, C. (2012). Exploring information experiences through phenomenography. *Library and Information Research, 36*(112), 96-119. https://doi.org/10.29173/lirg496

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2012). The need for proof and proving: Mathematical and Pedagogical Perspectives. In G. Hanna, & M. de Villiers (Eds.). *Proof and Proving in Mathematics Education. The 19th ICMI study* (pp. 215-229).Springer.

Zeybek, Z. (2015). Prospective teachers' conceptions of proof. *The Journal of Academic Social Sciences*, *3*(10),593-602. https://doi.org/10.16992/ASOS.583

Zeybek, Z. (2017). Pre-service elementary teachers' conceptions of counterexamples. *International Journal of Education in Mathematics, Science and Technology (IJEMST), 5*(4), 295-316. DOI:10.18404/ijemst.70986