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SOME GROUP ACTIONS AND FIBONACCI NUMBERS

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Abstract. The Fibonacci sequence has many interesting properties and studied by many mathematicians. The terms of this sequence appear in nature and is connected with combinatorics and other branches of mathematics. In this paper, we investigate the orbit of a special subgroup of the modular group. Taking

$$T_c := \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix} \in \Gamma_0(c^2), c \in \mathbb{Z}, c \neq 0,$$

we determined the orbit

$$\{T_c^r(\infty) : r \in \mathbb{N}\}.$$

Each rational number of this set is the form $P_r(c)/Q_r(c)$, where $P_r(c)$ and $Q_r(c)$ are the polynomials in $\mathbb{Z}[c]$. It is shown that $P_r(1)$, and $Q_r(1)$ the sum of the coefficients of the polynomials $P_r(c)$ and $Q_r(c)$ respectively, are the Fibonacci numbers, where

$$P_r(c) = \sum_{s=0}^{r} \begin{pmatrix} 2r-s \\ s \end{pmatrix} c^{2r-2s} + \sum_{s=1}^{r} \begin{pmatrix} 2r-s \\ s-1 \end{pmatrix} c^{2r-2s+1}$$

and

$$Q_r(c) = \sum_{s=1}^{r} \begin{pmatrix} 2r-s \\ s-1 \end{pmatrix} c^{2r-2s+2}.$$

1. Introduction

The modular group theory plays an important role in many areas of mathematics, such as number theory, graph theory, automorphic function theory and combinatorics. A natural action of the modular group on extended rationals, yields interesting results. In [4], by using this action, Jones et. al. studied the suborbital graphs known as the Farey graph for the modular group. Kader et al. studied the

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suborbital graphs for the extended modular group in [11]. Değer et. al. investigate some results on continued fractions in suborbital graphs [1]. In [8,9], Keskin searched the suborbital graphs for the normalizer of $\Gamma_0(n)$. Güler et. al. examined relations between elliptic elements and circuits in graphs for the normalizer of $\Gamma_0(n)$ in $PSL(2,\mathbb{R})$ which turns to be a very important group in the studies of moonshine [2].

Some results in these studies are directly related to the number theory. Köroğlu et. al. obtained interesting results about the Fibonacci numbers and the suborbital graphs by means of the action of a special subgroup of the modular group on extended rationals [7]. Güler et. al. studied on solutions of congruence equations that come from the action of the normalizer of $\Gamma_0(n)$ via suborbital graphs [3].

On the other hand, it is known that Pascal and Fibonacci numbers are crucial subjects in combinatorics [5]. In [10], Falcon and Plaza obtained some results about Fibonacci sequence and Pascal's triangle.

The aim of the paper is to examine the action of a special subgroup of the modular group on the extended rationals. With the idea of this group action, some interesting results are obtained about the number theory. Many properties of Fibonacci numbers are deduced and associated with the so-called Pascal's triangle mentioned.

2. Modular Group

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$T : z \to \frac{az+b}{cz+d}$, where a, b, c and d are real and $ad - bc = 1$.

In terms of the matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices

$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

These matrix representations are composed of the special linear group denoted by $SL(2,\mathbb{R})$. The modular group denoted by Γ is the subgroup of $SL(2,\mathbb{R})$ consisting of the 2×2 matrices having integer entries. Furthermore, the modular group is generated by the matrices

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

with defining relationships $x^2 = y^3 = -I$, where I is the identity matrix. Here, x and y are cyclic matrices of order two and three, respectively. And we can write

$$\Gamma = < x, y > .$$

We remark that something very related to the trace $Tr(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := |a + d|$ will be of great use in the classification. Note that, an element of modular group is called elliptic, parabolic or hyperbolic if its trace $Tr(\cdot) < 2$, $Tr(\cdot) = 2$ or $Tr(\cdot) > 2$ respectively. Important subgroups of the modular group Γ, called congruence subgroups, are given by imposing congruence relations on the associated matrices. One of them is

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \; mod \, n \right\}.$$

3. The Action of Γ on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$

Any element of $\hat{\mathbb{Q}}$ (the extended rational numbers set) can be written as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$; since $\frac{x}{y} = \frac{-x}{-y}$, this representation is unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action $z \to \frac{az+b}{cz+d}$ of Γ on $\hat{\mathbb{Q}}$ now becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}.$$

Note that as

$$c(ax + by) - a(cx + dy) = -y$$

and

$$d(ax + by) - b(cx + dy) = x,$$

it follows that $(ax+by, cx+dy) = 1$ and so $(ax+by)/(cx+dy)$ is a reduced fraction.

4. Main Calculations

In this section, we investigate the action of a special subgroup of the congruence subgroup $\Gamma_0(c^2)$ on extended rationals for some integer $c \neq 0$. Here, we use the action of the group generated by the commutator of the elements $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ on $\hat{\mathbb{Q}}$. Let

$$xyx^{-1}y^{-1} = \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}.$$

Since $Tr(T_c) = c^2 + 2 > 2$, we can say the element T_c is hyperbolic element of modular group for $c \neq 0$.

Proposition 1. *The fixed points of the element T_c are,*

$$\frac{c+2}{2c} \pm \frac{\sqrt{c^2+4}}{2c}. \tag{1}$$

Furthermore, T_c generates an infinitely ordered subgroup $< T_c >$ whose elements are in congruence subgroup $\Gamma_0(c^2)$. At the same time, the group $< T_c >$ generated by T_c is a subgroup of commutator subgroup of modular group. Also, $T_c(\infty) = \frac{c^2+c+1}{c^2}$ is an element of $\hat{\mathbb{Q}}$.

Proposition 2. *The group* $\Gamma_0(c^2)$ *acts on the set* $M := \{\frac{x}{c^2 y} : x, y \in \mathbb{Z}, gcd(x, yc) = 1, y \neq 0\} \cup \{\infty\}$ *transitively.*

Note that, if $y = 0$ *and* $x \neq 0$ *then we assumed that,* $\frac{x}{c^2 y} = \frac{x}{0} = \infty$ *such as the definition of extended rationals in [4].*

Proof. For arbitrary $x, y \in \mathbb{Z},\ gcd(x, yc) = 1$, there exists $T = \begin{pmatrix} x & * \\ yc^2 & * \end{pmatrix} \in \Gamma_0(c^2)$. such that $T(\infty) = \frac{x}{yc^2}$. This completes the proof. □

We interested in sequence of natural powers of the number $T_c(\infty)$ denoted by $\{T_c^r(\infty)\}$, where $r \in \mathbb{N}$. Clearly $\{T_c^r(\infty)\} \subset M \cup \{\infty\}$. Hence, there is some element of $\Gamma_0(c^2)$ such that its orbit coincidence the terms of the sequence $\{T_c^r(\infty)\}$. The following theorem show us how $\{T_c^r(\infty)\}$ sequence proceeds.

Theorem 1. *Let* $T_c = \begin{pmatrix} c^2 + c + 1 & -c \\ c^2 & 1 - c \end{pmatrix}$, *with* $c \in \mathbb{Z}$. *Suppose*

$$T_c^r(\infty) := \frac{P_r(c)}{Q_r(c)}.$$

Then

$$P_r := P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}, \tag{2}$$

$$Q_r := Q_r(c) = \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2}. \tag{3}$$

Since Theorem 1 includes the combinatorial identities we frequently use some combinatorial basics such as,

$$\binom{r}{s} = \binom{r-1}{s-1} + \binom{r-1}{s} \tag{4}$$

so-called the Pascal Identity for integers $1 \leq s \leq r$.

Before the proof of the theorem, we give the following lemma.

Lemma 1. *Assume that the identities* (2) *and* (3) *are true for any* $r > 1$. *Then, we have*

$$c^2 P_r - cQ_r = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2}. \tag{5}$$

Proof. By using (4) and other properties of the combinatorial theory we get proof easily, as follow:

$$c^2 P_r - cQ_r = c^2 \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + c^2 \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}$$

$$
\begin{aligned}
&- c\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2}\\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+3}\\
&-\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+3}\\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}.
\end{aligned}
$$

□

Now we give the proof of Theorem 1 by using the mathematical induction method.

Proof. For $r = 1$, it is clear that

$$T_c(\infty) = \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} c^2+c+1 \\ c^2 \end{pmatrix}.$$

So, $P_1 = c^2 + c + 1$ and $Q_1 = c^2$. This shows that (2) and (3) are true for $r = 1$. As

$$T_c^{r+1}(\infty) = \frac{P_{r+1}}{Q_{r+1}} \tag{6}$$

and

$$
\begin{aligned}
T_c^{r+1}(\infty) = T_c(T_c^r(\infty)) &= \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}\begin{pmatrix}P_r\\Q_r\end{pmatrix}\\
&= \begin{pmatrix} c^2P_r + cP_r + P_r - cQ_r \\ c^2P_r - cQ_r + Q_r \end{pmatrix},
\end{aligned} \tag{7}
$$

we get

$$P_{r+1} = c^2aP_r + cP_r + P_r - cQ_r \tag{8}$$

and

$$Q_{r+1} = c^2P_r - cQ_r + Q_r. \tag{9}$$

Now assume that (2) and (3) are true for any $r > 1$. We will show that (2) and (3) are true for $r + 1$. To complete the proof, by using Lemma 1, it can be shown that the following two equations can be obtained from the identities (8) and (9).

$$P_{r+1} = \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} + (c+1)P_r, \tag{10}$$

$$Q_{r+1} = \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} + Q_r. \tag{11}$$

Indeed, if we begin with the right side of the equation (10), then we obtain desired results as follow.

$$\begin{aligned}
&\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+(c+1)P_r\\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}\\
&\quad+(c+1)\left[\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}\right]\\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+c^{2r+1}+\sum_{s=1}^{r}\binom{2r-s}{s}c^{2r-2s+1}\\
&\quad+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2}+\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s}\\
&\quad+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}.
\end{aligned}\tag{12}$$

From the equation (12), we obtain

$$\begin{aligned}
&\sum_{s=0}^{r+1}\binom{2r-s+2}{s}c^{2r-2s+2}\\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2}\\
&\quad+\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s},
\end{aligned}\tag{13}$$

and

$$\begin{aligned}
&\sum_{s=1}^{r+1}\binom{2r-s+2}{s-1}c^{2r-2s+3}\\
&=\sum_{s=1}^{r}\binom{2r-s}{s}c^{2r-2s+1}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}+c^{2r+1}.
\end{aligned}\tag{14}$$

So, by using (13) and (14) we have

$$P_{r+1}=\sum_{s=0}^{r+1}\binom{2r-s+2}{s}c^{2r-2s+2}+\sum_{s=1}^{r+1}\binom{2r-s+2}{s-1}c^{2r-2s+3}.$$

Hence, the equation (2) is true for $r+1$.

By using Lemma 1, we get

$$
\begin{aligned}
Q_{r+1} = c^2 P_r - cQ_r + Q_r &= \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2} + Q_r \\
&= \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2} \\
&= c^{2r+2} + \sum_{s=1}^{r} \binom{2r-s}{s} c^{2r-2s+2} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2} \\
&= c^{2r+2} + \sum_{s=1}^{r} \left[\binom{2r-s}{s} + \binom{2r-s}{s-1} \right] c^{2r-2s+2} \\
&\overset{(4)}{=} c^{2r+2} + \sum_{s=1}^{r} \binom{2r-s+1}{s} c^{2r-2s+2} \\
&= \sum_{s=0}^{r} \binom{2r-s+1}{s} c^{2r-2s+2} \\
&= \sum_{s=1}^{r+1} \binom{2r-s+2}{s-1} c^{2r-2s+4} = Q_{r+1}.
\end{aligned}
$$

This implies that (3) is true for $r+1$.

□

5. Pascal Numbers and Fibonacci Sequence

In this section, we give some useful informations for Fibonacci numbers related to our results in this study. The Fibonacci numbers F_r are given by the recurrence in [6];

$$F_1 = F_2 = 1,\ F_{r+2} = F_{r+1} + F_r,\ r \geq 1.$$

Thus, the first few Fibonacci numbers are

$$1,\ 1,\ 3,\ 5,\ 8,\ ,13,\ 21,\ \ldots$$

Also, the elegant formula is

$$F_{r+1} = \sum_{s=0}^{\lfloor r/2 \rfloor} \binom{r-s}{s} \tag{15}$$

where $\lfloor r/2 \rfloor$ denotes the largest integer less than or equal to $r/2$ [6].

We consider coefficients of the polynomials P_r and Q_r as shown below in first five terms of P_r and Q_r. Furthermore we investigate that these coefficents are related to the Pascal triangle.

$$P_1 = c^2 + c + 1$$

$$P_2 = c^4 + c^3 + 3c^2 + 2c + 1$$

$$P_3 = c^6 + c^5 + 5c^4 + 4c^3 + 6c^2 + 3c + 1$$
$$P_4 = c^8 + c^7 + 7c^6 + 6c^5 + 15c^4 + 10c^3 + 10c^2 + 4c + 1$$
$$P_5 = c^{10} + c^9 + 9c^8 + 8c^7 + 28c^6 + 21c^5 + 35c^4 + 20c^3 + 15c^2 + 5c + 1$$
$$\vdots$$

Each sequences of numerators obtained from this action consists of numbers in Pascal 2-Triangle as shown in Table 1. For example, second sequences of numerators $(1, 1, 3, 2, 1)$ are located by bold numbers in Table 1.

Table 1. The Pascal 2-Triangle

					1					
					1					
				1		1				
				1		**2**				
			↗	↘	↗	↘				
			1		**3**		**1**			
			1		4		3			
		1		5		6		1		
		1		6		10		4		
	1		7		15		10		1	
	1		8		21		20		5	
1		9		28		35		15		1
1		10		36		56		35		6

Proposition 3. *Sum of all coefficients of P_r gives the $(2r+2)-th$ Fibonacci number denoted by F_{2r+2}, i.e. $P_r(1) = F_{2r+2}$.*

Proof. We remark that sum of all coefficients of P_r is $P_r(1)$. So, by using the identities (4) and (15) we have desired result as follows:

$$\begin{aligned} P_r(1) &= 1 + \sum_{s=1}^{r} \left[\binom{2r-s}{s-1} + \binom{2r-s}{s} \right] \\ &\overset{(4)}{=} 1 + \sum_{s=1}^{r} \binom{2r-s+1}{s} \\ &= \sum_{s=0}^{r} \binom{2r-s+1}{s} \\ &= \sum_{s=0}^{\lfloor (2r+1)/2 \rfloor} \binom{2r-s+1}{s} \overset{(15)}{=} F_{2r+2}. \end{aligned}$$

□

Now, we consider the coefficients of the polynomial Q_r. These coefficients are written at the sub rows in Pascal 2-triangle in Table 1. For example, 1, 3, 1 and 1, 6, 10, 4. So the first five terms of Q_r are listed as follow:

$$\begin{aligned} Q_1 &= c^2 \\ Q_2 &= c^4 + 2c^2 \\ Q_3 &= c^6 + 4c^4 + 3c^2 \\ Q_4 &= c^8 + 6c^6 + 10c^4 + 4c^2 \\ Q_5 &= c^{10} + 8c^8 + 21c^6 + 20c^4 + 5c^2 \\ &\vdots \end{aligned}$$

Proposition 4. *Sum of all coefficients of Q_r gives the $2r-th$ Fibonacci number, i.e. $Q_r(1) = F_{2r}$.*

Proof.

$$\begin{aligned} Q_r(1) &= \sum_{s=1}^{r} \binom{2r-s}{s-1} = \sum_{j=0}^{r-1} \binom{2r-(j+1)}{j} \\ &= \sum_{j=1}^{r-1} \binom{2r-1-j}{j} = \sum_{j=0}^{\lfloor (2r-1)/2 \rfloor} \binom{2r-1-j}{j} \\ &\overset{(15)}{=} F_{2r}. \end{aligned}$$

□

Proposition 5. $P_r(-1) = F_{2r-1}$.

Proof.

$$\begin{aligned}
P_r(-1) &= \sum_{s=0}^{r}\binom{2r-s}{s} - \sum_{s=1}^{r}\binom{2r-s}{s-1} = \sum_{s=0}^{r}\binom{2r-s}{s} - \sum_{s=0}^{r-1}\binom{2r-s-1}{s} \\
&= 1 + \sum_{s=0}^{r-1}\binom{2r-s}{s} - \sum_{s=0}^{r-1}\binom{2r-s-1}{s} \\
&= 1 + \sum_{s=0}^{r-1}\left[\binom{2r-s}{s} - \binom{2r-s-1}{s}\right] \\
&\overset{(4)}{=} 1 + \sum_{s=1}^{r-1}\binom{2r-s-1}{s-1} = 1 + \sum_{s=0}^{r-2}\binom{2(r-1)-s}{s} \\
&= 1 + \sum_{s=0}^{u-1}\binom{2u-s}{s} = \sum_{s=0}^{u}\binom{2u-s}{s} \overset{(15)}{=} F_{2u+1} = F_{2r-1}.
\end{aligned}$$

□

Proposition 6. *Sum of the coefficients of odd order terms of P_r is F_{2r}.*

Proof. We remark that the sum of the coefficients of odd order terms of P_r is $\frac{P_r(1)-P_r(-1)}{2}$. Therefore, by using Proposition 3, Proposition 5 and recurrence relations of Fibonacci numbers, we obtain the desired result as follows:

$$\begin{aligned}
\frac{P_r(1)-P_r(-1)}{2} &= \frac{F_{2r+2}-F_{2r-1}}{2} = \frac{F_{2r}+F_{2r+1}-F_{2r-1}}{2} \\
&= \frac{F_{2r-1}+2F_{2r}-F_{2r-1}}{2} = F_{2r}.
\end{aligned}$$

□

Also considering the scope of this study, we can also talk about $k-$Fibonacci numbers. Let $k \neq 0$ be an integer and $F_{k,0} = 0$, $F_{k,1} = 1$, and $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$. The sequence $(F_{k,n})$ is called $k-$Fibonacci sequence. A few terms of this sequence are

$$0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, k^5+4k^3+3k, k^6+5k^4+6k^2+1, \ldots$$

In [10], it is proved that

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1-i}{i} k^{n-1-2i} \text{ for } n \geq 2.$$

That is,

$$F_{k,n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-i}{i} k^{n-2i} \text{ for } n \geq 1.$$

Considering this result for $k = c$ and $n = r$, we can give the following two conjectures:

$$Q_r = Q_r(c) = cF_{c,2r}$$

and

$$P_r = P_r(c) = F_{c,2r} + F_{c,2r+1}.$$

6. Conclusion

In this paper, we examined the action of a special subgroup of the congruence subgroup on $\hat{\mathbb{Q}}$. Using this action we obtained some results on Pascal and Fibonacci numbers via the modular group. The results obtained are important for the fields of number theory and combinatorics. Further, it has also been observed that

$$\infty \to T_c(\infty) \to T_c^2(\infty) \to \cdots \to T_c^r(\infty) \to T_c^{r+1}(\infty) \to \cdots$$

is an infinitely long path in the suborbital graph $G(\infty, \frac{c^2+c+1}{c^2})$. Hence, this action is related to suborbital graphs theory which firstly studied by Jones et. al. in the reference [4]. This relationship can be examined in the future studies.

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