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Numerical Range of Left Invariant Lorentzian Metrics on the Heisenberg Group H3 of Dimension Three

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Abstract

The study of eigenvalues and numerical range appears in diffrent scientific fields. We can cite for example the domain of physics, spectral theory, the stability of dynamics electricity, the quantum mechanics. In this paper, we find the spectrum, pseudo-spectrum and numerical range of left invariant Lorentzian metrics on the Heisenberg group H_3 of dimension three. An example is given for metrics g_1 and g_2 while the second example is provided to support g_3.

1. Introduction

Varah published an article entitled "On the separation of two matrices" in which he defined with standard 2 the pseudospectrum using the smallest singular value $\sigma_{\min}(zI-A)$ under the notion $\Lambda_\varepsilon(A)$ see [1]. In the 1960s the pseudospectrum was studied in several by L. N. Trefethen [2], [3].

The pseudospectrum of a normal matrix A consists of circles of radius ε around each eigenvalue. For non-nirmal matrices, the pseudospectrum takes different forms in the complex plane. in [2] The pseudospectrum of thirteen highly non-normal matrices is presented.

Let T be an operator in $B(H)$ (i.e bounded linear operator on a Hilbert space H), the numerical range of T is the set $W(T)$ of complex numbers defined by

$$W(T)=\{\langle Tu,u\rangle \ : u\in H,\ \|u\|=1\}.$$

The numerical range has a wide history and there is a lot of new and exclusive research on this concept and its generalizations.

In particular, the subject has connections and applications to various areas including $\mathbb{C}^*-$algebras, iterations methods, several operator theory, dilation theory, Krein space operators, factorizations of matrix polynomials, unitary similarity,etc. (See [4], [5], [6], [7] and [8]).

M. M. Khorami, F. Ershad and B. Yousefi published an article entitled "On the Numerical Range of some Bounded Operators". In this article, They gave conditions under which the numerical range of a weighted composition operator, acting on a Hilbert space, contains zero as an interoir point and they investigated exterme points of the numerical range of an operator acting on an arbitrary Banach space. Also, they gave necessary and sufficient conditions under which the numerical range of an operator on some Banach spaces, to be closed. Finally, they characterized the structure of the numerical range of an operator acting on Banach weighted Hardy spaces. (See [9]).

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We recall that the Heisenberg group H_3 of dimension three is the subgroup of $GL(3,\mathbb{R})$ formed by the matrices of the type

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

it is a connected, simply connected, 2-nilpotent Lie group.

Its Lie algebra noted $\mathfrak{h}_3$ is defined by the matrices of $M(3,\mathbb{R})$ of the type

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

S. Rahmani (See [10])showed that there are three classes of left-invariant Lorentz metrics on H_3, non-isometric, one of which is flat . This is in contrast to the Riemannian case, because it is well known that the Heisenberg Riemannian group does not have a flat Riemannian metric flat metric, nor is it even conformably flat. More precisely, S. Rahmani has classified, up to automorphism of the Lie algebra $\mathfrak{h}_3$, all the left-invariant Lorentzian metrics on the Heisenberg group H_3:

Theorem 1 *[10], [11] Each left-invariant Lorentz metric on the Heisenberg group H_3 is isometric to one of the following metrics :*

$$\begin{aligned} g_1 &= -\tfrac{1}{\lambda^2}dx^2 + dy^2 + (xdy+dz)^2, \\ g_2 &= \tfrac{1}{\lambda^2}dx^2 + dy^2 - (xdy+dz)^2, \qquad \lambda > 0 \\ g_3 &= dx^2 + (xdy+dz)^2 - ((1-x)dy - dz)^2. \end{aligned}$$

Further more, the Lorentz metrics are nonisometric and the Lorentz metric g_3 is flat.

Researcher S. Rahmani extracted Killing vectors feilds from the Heisenberg group H_3 [10],

2. Preliminaries

The matrices associated to the metrics g_1, g_2 and g_3 respectively are given by

$$\begin{aligned} A &= \begin{pmatrix} -\frac{1}{\lambda^2} & 0 & 0 \\ 0 & 1+x^2 & x \\ 0 & x & 1 \end{pmatrix}, \\ B &= \begin{pmatrix} \frac{1}{\lambda^2} & 0 & 0 \\ 0 & 1-x^2 & -x \\ 0 & -x & -1 \end{pmatrix}, \ \lambda > 0 \end{aligned}$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x-1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 1 *[12] Let T be an operator in $B(H)$ (i.e bounded linear operator on a Hilbert space H), the numerical range of T is the set $W(T)$ of complex numbers defined by*

$$W(T) = \{\langle Tu, u\rangle \ : u \in H, \ \|u\| = 1\}.$$

Definition 2 *[13] Let M be an $n\times n$ complex matrix. Then the numerical range of M, $W(M)$, is defined to be*

$$W(M) = \left\{\frac{t^*Mt}{t^*t}, \ t \in \mathbb{C}^n, \, t \neq 0\right\}.$$

where t^* denotes the conjugate transpose of the vector t.

Proposition 1 *For all operators M and N in $B(H)$, μ_1, $\mu_2 \in \mathbb{C}$. We have the following properties:*
1) $W(\mu_1 M + \mu_2 I) = \mu_1 W(M) + \mu_2$.
2) $W(M+N) \subset W(M) + W(N)$.
3) $W(M^*) = \{\overline{\alpha},\ \alpha \in W(M)\}$, *where M^* is the adjunct of M.*
4) $W(U^*MU) = W(M)$, *where U is a unit operator.*
5) *Let $A \in M_n(\mathbb{C})$, if A is* ***Hermitian*** *(i.e., $A^* = A$), then $W(A) \subset \mathbb{R}$.*

3. Eigenvalues and Pseudo-spectrum

The eigenvalues of matrices A, B and C are:

$$\begin{aligned} \sigma(A) &= \left\{a = -\frac{1}{\lambda^2}\right\}, \\ \sigma(B) &= \left\{b = \frac{1}{\lambda^2}\right\} \end{aligned}$$

and

$$\sigma(C) = \left\{c_1 = 1, c_2 = x + \frac{1}{2}\sqrt{4x^2-4x+5} - \frac{1}{2}, c_3 = x - \frac{1}{2}\sqrt{4x^2-4x+5} - \frac{1}{2}\right\}.$$

Pseudo-spectrum

Sinc A, B and C are symmetrical therefore are normal and pseudo-spectrum given by:

$$\begin{aligned} \Lambda_\varepsilon(A) &= \{z \in \mathbb{C} : |z-a| \le \varepsilon\}\ , \\ \Lambda_\varepsilon(B) &= \{z \in \mathbb{C} : |z-b| \le \varepsilon\} \end{aligned}$$

and

$$\Lambda_\varepsilon(C) = \{z \in \mathbb{C} : |z-c_i| \le \varepsilon\},\ \text{with } i \in \{1,2,3\}.$$

4. The main result

In engineering, numerical ranges are used as a rough estimate of eigenvalues. Recently, generalizations of the numerical range are used to study quantum computig. In the section, we find the numerical rang of matrice A, B and C.

In our work, we find numerical range of left invariant Lorentzian metrics on the Heisenberg group H_3 of dimension three, as shown in the following theorem.

Theorem 2 *Let A, B and C the associated matrices to the metrics g_1, g_2 and g_3 respectively, such that the numerical range of each matrices defined as follow:*

1.

$$W(A) \subseteq \left[-\frac{1}{\lambda^2} + x\ , x^2 - x + 1\right],\ \textit{if } x \le 0,$$
$$W(A) \subseteq \left[-\frac{1}{\lambda^2} - x\ , x^2 + x + 1\right],\ \textit{if } x \ge 0.$$

2.

$$W(B) \subseteq \left[-(x^2 + |x| + 1)\ , \frac{1}{\lambda^2} + |x|\right],\ \textit{if } |x| > 1,$$
$$W(B) \subseteq \left[-2\ , \frac{1}{\lambda^2} + 2\right],\ \textit{if } |x| \le 1.$$

3.

$$W(C) \subseteq [-2(1-x)\ , 1],\ \textit{if } x \le 0,$$
$$W(C) \subseteq [-1\ , 1+2x],\ \textit{if } x > 0.$$

Proof 1 *Let $t \in \mathbb{C}^3$ such that $t \neq 0$, we put* $t = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, *with* $z_i = r_i e^{i\theta_i}, i \in \{1,2,3\}$.

1. We have

$$t^*At = -\frac{1}{\lambda^2}|z_1|^2 + (1+x^2)|z_2|^2 + x(z_2\overline{z_3} + z_3\overline{z_2}) + |z_3|^2,$$

so,

$$\frac{t^*At}{t^*t} = \frac{-\frac{1}{\lambda^2}|z_1|^2}{\sum\limits_{i=1}^{3}|z_i|^2} + \frac{x^2|z_2|^2 + x(z_2\overline{z_3} + z_3\overline{z_2}) + |z_2|^2 + |z_3|^2}{\sum\limits_{i=1}^{3}|z_i|^2}.$$

Moreover we have

$$\frac{|z_j|^2}{\sum\limits_{i=1}^{3}|z_i|^2} \leq 1,\ \forall j \in \{1,2,3\}, \tag{1}$$

and

$$-1 \leq \frac{z_i\overline{z_j} + z_j\overline{z_i}}{\sum\limits_{i=1}^{3}|z_i|^2} \leq 1, \quad \forall i,j \in \{1,2,3\}, \tag{2}$$

So from (1) and (2) we find

$$-\frac{1}{\lambda^2} + x \leq \frac{t^*At}{t^*t} \leq x^2 - x + 1, \ \text{if } x \leq 0.$$

If $x \geq 0$ *then*

$$-\frac{1}{\lambda^2} - x \leq \frac{t^*At}{t^*t} \leq x^2 + x + 1.$$

2. We have

$$\frac{t^*Bt}{t^*t} = \frac{\frac{1}{\lambda^2}|z_1|^2}{\sum\limits_{i=1}^{4}|z_i|^2} + \frac{(1-x^2)|z_2|^2 - x(z_2\overline{z_3} + z_3\overline{z_2}) - |z_3|^2}{\sum\limits_{i=1}^{3}|z_i|^2},$$

if $|x| > 1$ *then* $1 - x^2 < 0$. *So*

$$-x^2 - |x| - 1 \leq \frac{t^*Bt}{t^*t} = \frac{\frac{1}{\lambda^2}|z_1|^2}{\sum\limits_{i=1}^{3}|z_i|^2} + \frac{(1-x^2)|z_2|^2 - x(z_2\overline{z_3} + z_3\overline{z_2}) - |z_3|^2}{\sum\limits_{i=1}^{3}|z_i|^2} \leq \frac{1}{\lambda^2} + |x|,$$

if we have the opposite $|x| \leq 1$ *then*

$$-2 \leq \frac{t^*Bt}{t^*t} = \frac{\frac{1}{\lambda^2}|z_1|^2}{\sum\limits_{i=1}^{4}|z_i|^2} + \frac{|z_2|^2 - x(z_2\overline{z_3} + z_3\overline{z_2})}{\sum\limits_{i=1}^{3}|z_i|^2} - \frac{x^2|z_2|^2 + |z_3|^2}{\sum\limits_{i=1}^{3}|z_i|^2} \leq \frac{1}{\lambda^2} + 2.$$

3. Now shows us that 3, we have

$$\begin{aligned} \frac{t^*Ct}{t^*t} &= \frac{|z_1|^2 + (2x-1)|z_2|^2 + z_2\overline{z_3} + z_3\overline{z_2}}{\sum\limits_{i=1}^{3}|z_i|^2} \\ &= 1 - \frac{2|z_2|^2 + |z_3|^2 - 2x|z_2|^2 - 2r_2r_3\cos(\theta_2 - \theta_3)}{\sum\limits_{i=1}^{3}|z_i|^2} \end{aligned}$$

if $x \leq 0$ *then*

$$-2 + 2x \leq \frac{t^*Ct}{t^*t} \leq 1,$$

if $x > 0$ *then*

$$-1 \leq \frac{t^*Ct}{t^*t} = -1 + \frac{2\left|z_2\right|^2 + \left|z_3\right|^2 + 2x\left|z_2\right|^2 + 2r_2r_3\cos(\theta_2 - \theta_3)}{\sum_{i=1}^{3}\left|z_i\right|^2} \leq -1 + 2x + 1 + 1 = 2x + 1.$$

Example 4.1 *For* $\lambda = 1$ *and* $x = 0$,

$$A_1^0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$\frac{g_1(t^*,t)}{t^*t} = 1 - \frac{2r_1^2}{r_1^2 + r_2^2 + r_3^2} \leq 1,$$

moreover $1 \in W(A_1^0)$

On the other hand, we have

$$\frac{g_1(t^*,t)}{t^*t} = -1 + \frac{2r_2^2 + 2r_3^2}{r_1^2 + r_2^2 + r_3^2} \geq -1,$$

moreover $-1 \in W(A_1^0)$. *So*

$$W(A_1^0) = [-1, 1].$$

For matrix

$$B_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so

$$-1 \leq \frac{g_2(t^*,t)}{t^*t} = -1 + 2\frac{r_1^2 + r_2^2}{r_1^2 + r_2^2 + r_3^2} \leq 1,$$

moreover $\{1, -1\} \subset W(B_1^0)$. *So*

$$W(B_1^0) = [-1, 1].$$

Example 4.2 *For* $x = \frac{1}{2}$, *matrix*

$$C_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

so

$$\begin{aligned} \frac{g_3(t^*,t)}{t^*t} &= \frac{r_1^2 + 2r_2r_3\cos(\theta_2 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} \\ &= 1 - \frac{r_2^2 + r_3^2 - 2r_2r_3\cos(\theta_2 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} \leq 1 \end{aligned}$$

on the other hand

$$\frac{g_3(t^*,t)}{t^*t} = -1 + \frac{2r_1^2 + r_2^2 + r_3^2 + 2r_2r_3\cos(\theta_2 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} \geq -1,$$

moreover $1, -1 \in W(B_1^0)$. *So*

$$W(C_{\frac{1}{2}}) = [-1, 1].$$

5. Conclusion

Through to the heart of our study "the numerical range of operator" and exactly we kept our touch on the numerical range of matrices. This latter contributed to many mathemathical discplines such as the theory of operators, matrix polynomials, applications to various areas including C-algebras. And the subject of numerical range of matrices is still open for scientific research in all aspects algebric, geometric. We touched in this paper for a generalization spectrum, pseudo-spectrum and numerical range of left invariant Lorentzian metrics (g_1, g_2 and g_3) on H_3 of dimension three.

Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Rafik Derkaoui: Writing, Reviewing.
Abderrahmane Smail: Methodology, Supervision.

References

[1] C. Van Loan, "A study of the matrix exponential, Numerical Analysis Report No. 10," University of Manchester, UK, August, 1975, Reissued as MIMS EPrint, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, November 2006.

[2] L. Reichel, L. N .Trefethen, "Eigenvalues and pseudo-eigenvalues of Toeplitz Matrice," Lin. Alg. Applics. 162-164, pp. 153-185, 1992.

[3] L. Trefethen, M. Embree, "Spectra and Pseudospectra :The Behavior of Non-Normal Matrices and Operators," Princeton University Press, Princeton, 2005.

[4] T. Ando and C-K. Li (Special editors), "The Numerical Range and Numerical Radius," Linear and Multilinear Algebra, vol. 37, no. 1–3, 1994.

[5] T. Bayasgalan, "The numerical range of linear operators in spaces with an in-definite metric (Russian)," Acta Math. Hungar. vol. 57, pp. 157-168, 1991.

[6] F. F. Bonsall and Duncan, "Numerical ranges, Vol. I and II," Cambridge University Press, 1971 and 1973.

[7] F. F. Bonsall and Duncan, "Studies in Functional Analysis – Numerical ranges," Studies in Mathematics vol. 21, The Mathematical Association of America, 1980.

[8] D. Farenick, "Matricial Extensions of the numerical range: A brief survey," Linear and Multilinear Algebra, vol. 34, pp. 197-211, 1993

[9] M.M. Khorami, F. Ershad and B. Yousefi, "On the Numerical Range of some Bounded Operators," Journal of Mathematical Extension, vol. 15, 2020

[10] S. Rahmani, "Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension 3," J. Geom. Phys., vol. 9, pp. 295-302, 1992

[11] N. Rahmani, S. Rahmani, "Lorentzian Geometry of the Heisenberg group," Geom. Dedicata, pp. 133-140, 2006.

[12] K. E. Gustafson and K. M. R. Duggirala, "Numerical Range, The Field of Values of Linear Operators and Matrices," Springer, New York, 1997.

[13] M. Fiedler, "Geometry of the numerical range of matrices," Linear Algebra and Its Applications, vol. 37, pp. 81-96, 1981.