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A NEW VARIATION ON ABSOLUTE SUMMABILITY

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Abstract. In [4], Bor has proved a main theorem dealing with absolute weighted arithmetic mean summability factors of infinite series by using a positive non-decreasing sequence. In this paper, we have extended this result to absolute matrix summability method by using an almost increasing sequence in place of a positive non-decreasing sequence. Also, some new and known results are also obtained.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums be denoted by (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [9])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0.$$

A series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

If we take $\alpha = 1$, then we have the $|C,1|_k$ summability. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

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defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [11]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all n, then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.
Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix with nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [15])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability is reduced to the $|A, p_n|_k$ summability (see [17]). If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then $\varphi - |A, p_n|_k$ summability is reduced to the $|\bar{N}, p_n|_k$ summability. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then

$\varphi - |A, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

1. Known Result

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants A and B such that $Az_n \leq b_n \leq Bz_n$ (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. The following theorem concerning on absolute summability factors of infinite series has been obtained.
Theorem 2.1 [4] Let (X_n) be a positive non-decreasing sequence and let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n), \tag{1.1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{1.2}$$

If the sequences (X_n) , (β_n), and (λ_n) satisfy the conditions

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.3}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{1.4}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{1.5}$$

$$|\lambda_n|X_n = O(1), \tag{1.6}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{1.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.
Remark 2.1 It should be noted that, under the conditions on the sequence (λ_n) we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. Main Result

The aim of this paper is to generalize Theorem 2.1 by using an almost increaing sequence for $\varphi - |A, p_n|_k$ summability method, which is more general matrix summability method than $|\bar{N}, p_n|_k$ summability method. Some papers have been done dealing with absolute summability methods (see [5]-[8], [18]-[24]).
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, .. \tag{2.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{2.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{2.3}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{2.4}$$

Let ω be the class of all matrices $A = (a_{nv})$ satisfying

$$A \quad \text{is a positive normal matrix,} \tag{2.5}$$

$$\bar{a}_{n0} = 1, \ n = 0, 1, ... \tag{2.6}$$

$$a_{n-1,v} \geq a_{nv}, \quad n \geq v + 1. \tag{2.7}$$

With this notation we have the following theorem.
Theorem 3.1 Let $A \in \omega$ satisfying

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{2.8}$$

$$1 = O(na_{nn}), \tag{2.9}$$

$$\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}| = O(a_{nn}). \tag{2.10}$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (1.2)-(1.6) of Theorem 2.1, and the condition

$$\sum_{n=1}^{m} (\frac{\varphi_n p_n}{P_n})^{k-1} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty \tag{2.11}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.

Remark 3.1 It is noted that by using the conditions (2.8) and (2.9), we have

$$P_n = O(np_n) \tag{2.12}$$

We need the following lemmas for the proof of Theorem 3.1.
Lemma 3.1 [12] Under the conditions of Theorem 2.1, we have

$$nX_n\beta_n = O(1),$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Lemma 3.2 [14] If the condition (1.2) of Theorem 2.1 and (2.12) are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right)$.
Lemma 3.3 [16] Let $A \in \omega$ and by using (2.1) and (2.2), we have that

$$\sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \leq 1,$$

and by using $\Delta_v(\hat{a}_{nv}) = a_{nv} - a_{n-1,v}$, we get

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \leq a_{nn},$$

and

$$\sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \leq a_{vv}.$$

3. Proof of Theorem 3.1

Let (V_n) denotes the A-transform of the series $\sum a_n \frac{P_n \lambda_n}{np_n}$. Then

$$\bar{\Delta} V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \frac{P_v \lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v$$

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v p_v} \right) s_v + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} s_n,$$

by the formula for the difference of products of sequences (see [11]) we have

$$\bar{\Delta}V_n = \frac{a_{nn} P_n \lambda_n}{n p_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{v p_v} \Delta_v(\hat{a}_{nv}) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) s_v$$
$$+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1) p_{v+1}} \Delta \lambda_v s_v$$
$$\bar{\Delta}V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.$$

To complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \mid V_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{3.1}$$

Firstly, by using condition (2.8), (2.11) and (2.12) and applying Abel's transformation, we have

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid V_{n,1} \mid^k \leq \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k \left(\frac{P_n}{p_n} \right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\frac{P_n}{p_n} \right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\frac{P_n}{p_n} \right)^{k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} n^{k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\lambda_n|^k |s_n|^k \frac{1}{n}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\lambda_n|^{k-1} |\lambda_n| \frac{|s_n|^k}{n}$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \frac{1}{X_n^{k-1}} |\lambda_n| \frac{|s_n|^k}{n}$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{|s_v|^k}{vX_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\frac{|s_n|^k}{nX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. By applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, and as in $V_{n,1}$, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\varphi_n^{k-1}\mid V_{n,2}\mid^k = \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v}{vp_v}\Delta_v(\hat{a}_{nv})s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}a_{vv}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right)\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}v^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{1}{vX_v^{k-1}}|\lambda_v||s_v|^k\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1, Lemma 3.1 and Lemma 3.3. Also, by using the the fact that $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ and Lemma 3.3, again as in $V_{n,1}$, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid V_{n,3} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \left(\frac{P_v}{v p_v} \right) \lambda_v s_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}| |\lambda_v|^k |s_v|^k \frac{1}{v^k} \right\} \times \left\{ \sum_{v=1}^{n-1} a_{vv} |\hat{a}_{n,v+1}| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}| |\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} v^{k-1} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\lambda_v|^{k-1} |\lambda_v| |s_v|^k \frac{1}{v} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \frac{1}{v X_v^{k-1}} |\lambda_v| |s_v|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1. Finally, by virtue of the hypotheses of Theorem 3.1, Lemma 3.1, Lemma 3.3 and considering the fact that $v\beta_v = O(\frac{1}{X_v})$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid V_{n,4} \mid^k &= \sum_{n=2}^{m+1} \varphi_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1)p_{v+1}} \Delta\lambda_v s_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}| (\beta_v)^k |s_v|^k \right\} \times \left\{ \sum_{v=1}^{n-1} a_{vv} |\hat{a}_{n,v+1}| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}| (\beta_v)^k |s_v|^k \\
&= O(1) \sum_{v=1}^{m} a_{vv}^{1-k} (\beta_v)^k |s_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} a_{vv}^{1-k} (\beta_v)^k |s_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} a_{vv}^{1-k} (\beta_v)^k |s_v|^k \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} (v\beta_v)^{k-1} \beta_v |s_v|^k \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} v\beta_v \frac{|s_v|^k}{vX_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \left(\frac{\varphi_r p_r}{P_r} \right)^{k-1} \frac{|s_r|^k}{rX_r^{k-1}} + O(1) m\beta_m \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \frac{|s_v|^k}{vX_v^{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} |(v+1)\Delta\beta_v - \beta_v| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

This completes the proof of Theorem 3.1.

4. Conclusions

1. If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we have Theorem 2.1.
2. If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then we obtain a new result concerning the $|C,1|_k$ summability (see [13]).

3. If we take $\varphi_n = \frac{P_n}{p_n}$, then we have a new result concerning the $|A, p_n|_k$ summability.

References

[1] N. K. Bari, S. B. Stečkin, Best approximation and differential of two conjugate functions, Trudy. Moskov. Mat. Obšč. 5 (1956) 483-522 (in Russian).

[2] H. Bor, On two summability methods, Math. Proc. Camb. Philos. Soc. 97 (1985) 147-149.

[3] H. Bor, A note on $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series, Indian J. Pure Appl. Math. 18 (1987) 330-336.

[4] H. Bor, Factors for absolute weighted arithmetic mean summability of infinite series, Int. J. Anal. Appl. 14 (2017) 175-179.

[5] H. Bor, An application of quasi-monotone sequences to infinite series and Fourier series, Anal. Math. Phys. 8 (2018) 77-83.

[6] H. Bor, On absolute summability of factored infinite series and trigonometric Fourier series, Results Math. 73 (2018) 116.

[7] H. Bor, On absolute Riesz summability factors of infinite series and their application to Fourier series, Georgian Math. J. 26 (2019) 361-366.

[8] H. Bor, Certain new factor theorems for infinite series and trigonometric Fourier series, Quaest. Math. 43 (2020) 441-448

[9] E. Cesàro, Sur la multiplication des séries, Bull. Sci. Math. 14 (1890) 114-120.

[10] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. Lond. Math. Soc. 7 (1957) 113-141.

[11] G. H. Hardy, Divergent Series, Clarendon Press. Oxford, (1949).

[12] K. N. Mishra, On the absolute Nörlund summability factors of infinite series, Indian J. Pure Appl. Math. 14 (1983) 40-43.

[13] K. N. Mishra and R. S. L. Srivastava, On the absolute Cesàro summability factors of infinite series, Portugal Math. 42 (1983/84) 53-61.

[14] K. N. Mishra and R. S. L. Srivastava, On $|\bar{N}, p_n|$ summability factors of infinite series, Indian J. Pure Appl. Math. 15 (1984) 651-656.

[15] H. S. Özarslan, T. Kandefer, On the relative strength of two absolute summability methods, J. Comput. Anal. Appl. 11 (2009) 576–583.

[16] B. E. Rhoades, Inclusion theorems for absolute matrix summability methods, J. Math. Anal. Appl. 238 (1999) 82-90.

[17] W. T. Sulaiman, Inclusion theorems for absolute matrix summability methods of an infinite series, IV. Indian J. Pure Appl. Math. 11 (2003) 1547-1557.

[18] Ş. Yıldız, On application of matrix summability to Fourier series, Math. Methods Appl. Sci. (2018) 664–670.

[19] Ş. Yıldız, On the absolute matrix summability factors of Fourier series, Math. Notes 103 (2018) 297-303.

[20] Ş. Yıldız, A matrix application on absolute weighted arithmetic mean summability factors of infinite series, Tibilisi Math. J. 11 (2018) 59-65.

[21] Ş. Yıldız, On the generalizations of some factors theorems for infinite series and Fourier series, Filomat 33 (2019) 4343-4351.

[22] Ş. Yıldız, Matrix application of power increasing sequences to infinite series and Fourier series, Ukranian Math. J. 72 (2020) 730-740.

[23] Ş. Yıldız, A variation on absolute weighted mean summability factors of Fourier series and its conjugate series, Bol. Soc. Parana. Mat. 38 (2020) 105-113.

[24] Ş. Yıldız, A recent extension of the weighted mean summability of infinite series, J. Appl. Math. Inform. 39 (2021) 117-124.

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