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Research Article

An optimized continuous fractional grey model for forecasting of the time dependent real world cases

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Abstract

The new priority in the grey modelling is to build new models that have more accurate forecasting power than the previous ones. This paper aims to develop the prediction performance of the existing continuous grey models. Therefore, a novel continuous grey model (OCCFGM(1,1)) is proposed with conformable fractional derivative. The numerical results of three case studies show that the novel model's prediction accuracy is higher than other competitive models, and the proposed model is more reasonable for practical cases.

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1. Introduction

The grey systems theory as a systems analysis, which includes control and decision-making, modeling, prediction, was proposed by Prof. Deng Julong in 1982 [6]. The grey modelling in the prediction of time series has been one of the interesting study fields of recent times due to its efficiently and convenience. According to this theory, known, unknown, and semi-known information of the system are represented in white, black and grey, respectively. The term semi-known here means the shortage or lack of information. Therefore the grey prediction models have been a useful tool in processing uncertain or excursive systems with small samples and limited data set as distinct from the machine learning models, the artificial neural network (ANN), hybrid models, the empirical models, and support vector regression models (SVR), etc. [1, 9, 21, 24, 27]. In addition, the small sample usually has more accuracy than the large sample size. However, if the raw data sequence satisfies the conditions of non-negative, quasi-smooth and quasi-exponential then the standard grey model can perform well predictions even with large number of samples [31]. Today, the grey prediction models have been widely and successfully applied to various fields, such as science and technology, agriculture, energy, environmental problems, economy, health, meteorology and other fields [2, 3, 7, 10, 25, 30, 41, 43, 44].

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Recent research on grey modeling focuses on two main purposes: practicality and prediction accuracy. For these purposes, important studies have been carried out in the last years. Ma and Lui [17] proposed a time-delayed polynomial grey prediction model called TDPGM (1,1), a nonhomogeneous grey forecasting model NGM (1,1,k,c) and its optimized (NGMO (1,1,k,c)) is proposed by [4], the grey polynomial model with a tuned background coefficient is proposed by [29], Cui et al. [5] developed a parameter optimization method to improve the ONGM (1,1,k) model, Bilgil [3] proposed an exponential grey model named EXGM(1,1), Ma et al. [18] developed a novel nonlinear multivariate forecasting grey model based on the Bernoulli equation named NGBMC (1, n), Wang et al. [28] introduced a seasonal grey model called SGM (1,1), Wu et al. [33] proposed a new grey model called BNGM (1,1,t^2) model, Liu and Wu [15] proposed ANDGM model, kernel-based KARGM(1,1) model is proposed by [16], Li et al. [13] developed structure-adaptive intelligent grey forecasting model, Wu et al. [34] developed a novel grey Riccati model (GRM), the modified grey prediction model with damping trend factor is proposed by [14], the nonlinear grey Bernoulli model with improved parameters, INGBM (1, 1), is proposed by [11].

Traditional grey models are defined with first order whitening differential equations and they are characterized by their simplicity. However, if the case data is a disordered sequence, the characteristic features of the sequence may not exactly find out by first order accumulative generation operation (AGO) [40]. In addition, first order derivative models are ideal memory models, which are not suitable for describing irregular phenomena. Fractional grey models, on the other hand, have become preferable, despite the difficulty in calculations, since they give more effective results than standard models. Difficulties in fractional accumulation and difference calculations have begun to be overcome thanks to new definitions and theorems made in recent years. The new trend in grey modelling is to compose models that are more useful than the previous ones and give results with less error. As a result of this, the parameters of the model may not be compatible according to the data characteristics of the actual problem for a sequence with large data fluctuation. Therefore, fractional accumulation generating operation and fractional derivative should be introduced into the grey model to overcome this problem [40].

Wu et al. [32] first proposed the fractional accumulation GM(1,1) model known as the FAGM(1,1) by extending the integer accumulated generating operation into the fractional accumulated generating operation. The computational results shows that the FAGM(1,1) has outstanding performance than the traditional GM(1,1) model. Some researchers optimized the FAGM(1,1) model and achieved better prediction accuracy in recent years. A fractional grey model FANGM(1,1,k,c) is proposed by [35]. Mao et al. introduced the fractional grey model FGM(q,1) [20], a power-driven fractional accumulated grey model named PFAGM is proposed by [42], Yuxiao et al. [40] proposed the multivariable Caputo fractional derivative grey model with convolution integral named CFGMC(q,N).

Lately, a limit-based fractional derivative called conformable fractional derivative was introduced by [12]. The structure of conformable fractional derivative is more simpler than other popular fractional derivatives, such as the Caputo derivative, Atangana-Baleanu derivative and Riemann-Liouville derivative. Then, the new definitions of the fractional accumulation and fractional difference based on the conformable fractional derivative in which the computational complexity of accumulation is lower than that of the traditional fractional accumulated operator and they firstly proposed an improved fractional order grey model named CFGM(1,1). A conformable fractional non-homogeneous grey model CFNGM (1,1,k,c) is proposed by [36]. Xie et al. [38] developed a conformable fractional grey model in opposite direction CFGOM(1,1). Xie et al. [39] introduced an improved conformable fractional non-homogeneous grey model named CFONGM(1,1,k,c). Recently, a continuous grey model with conformable fractional derivative is proposed by [37].

1.1. Contribution

In summary, the innovation points of this paper are summarized as follows:

(1) A novel fractional grey model (OCCFGM(1,1)) with conformable fractional difference and accumulation is proposed.
(2) Generally all grey models in the literature are suitable when the given data sequence satisfies exponential growth. However, the abnormal increasing in the given data will have negatively effects on the prediction accuracy. An other innovation of this article is to improve predictive accuracy by adding a decreasing term, $\left(e^{-t^r}\right)$, in the whitenization differential equation. Therefore, the monotone decreasing term $\left(e^{-t^r}\right)$ will suppress the growth of the prediction error.
(3) The Brute Force algorithm is employed to find the optimal value of fractional order.
(4) Three case studies are provided for verifying the validation of OCCFGM(1,1).
(5) The numerical results of three case studies show that the novel model's prediction accuracy is higher than other competitive models.
(6) The proposed model is more suitable for all practical cases.

The rest of this paper is organized below: Section 1 includes relevant literature. A brief overview of the conformable fractional derivative and the definitions of the conformable fractional difference and accumulation are given in Section 2. The presentations and modelling procedures of the OCCFGM(1,1) are introduced in Section 3. In Section 4, the mechanism of obtaining optimal parameters is given. We present a series of case studies to confirm the effect of OCCFGM(1,1) in Section 5. Finally, main conclusions of this article are given in Sections 6.

2. Definitions and properties of conformable fractional derivative, fractional accumulation and fractional difference

In this section, a brief overview of the conformable fractional derivative with some useful properties is given. In addition, the definitions of the conformable fractional difference and the conformable fractional accumulation by [19] are summarized.

2.1. The conformable fractional derivative

Definition 2.1. (See [19]). Given a differentiable function $f:[0,\infty)\to R$. Then the conformable fractional derivative of f with $\alpha\in(n,n+1]$ order is defined as

$$T_\alpha\left(f\right)\left(t\right)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{\lceil\alpha\rceil-\alpha})-f\left(t\right)}{\varepsilon}=t^{\lceil\alpha\rceil-\alpha}\frac{df\left(t\right)}{dt}, \tag{2.1}$$

where $\lceil.\rceil$ is the ceil function, i. e. the $\lceil\alpha\rceil$ is the smallest integer no larger than α. It is clear that $\lceil\alpha\rceil=1$ for $\alpha\in(0,1]$. Hence, Equation (2.1) can be written as

$$T_\alpha\left(f\right)\left(t\right)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f\left(t\right)}{\varepsilon}=t^{1-\alpha}\frac{df\left(t\right)}{dt},$$

for all $t>0$.

The properties of this new definition (2.1) are given below.

Remark 2.2. (See [12]). The α order conformable derivative is defined as follows:

$$T_\alpha\left(f\right)\left(t\right)=\lim_{\varepsilon\to 0}\frac{f^{(n)}(t+\varepsilon t^{\lceil\alpha\rceil-\alpha})-f^{(n)}\left(t\right)}{\varepsilon}=t^{\lceil\alpha\rceil-\alpha}\frac{d^{n+1}f\left(t\right)}{dt^{n+1}}, \tag{2.2}$$

for all $t>0,\alpha\in(n,n+1]$, $n\in N$, and f is $(n+1)-$differentiable. if $\alpha\in(0,1]$ then, Equation (2.2) convert to Definition (2.1).

The following theorem gives the properties of the conformable derivation [12].

Theorem 2.3. *(See [12]). If the function f and g are differentiable, $\alpha \in (0,1]$, then we have*

1. $T_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$,
2. $T_\alpha(mf+ng) = mT_\alpha(f) + nT_\alpha(g)$ *for all* $m,n \in \mathbb{R}$,
3. $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$,
4. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$,
5. $T_\alpha(c) = 0$ *for all constant* c,
6. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$,
7. $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$ *for all* $c \in \mathbb{R}$.

2.2. The conformable fractional accumulation and difference

New definitions to calculate the conformable fractional accumulation (CFA) and the conformable fractional difference (CFD) are given by [19] as follows:

Definition 2.4. (See [19]). The conformable fractional difference (CFD) of f with α order is defined as

$$\Delta^\alpha f(k) = k^{1-\alpha}\Delta f(k) = k^{1-\alpha}\left[f(k) - f(k-1)\right],$$

for all $k \in N^+$, $\alpha \in (0,1]$ and

$$\Delta^\alpha f(k) = k^{\lceil\alpha\rceil-\alpha}\Delta^{n+1} f(k), \tag{2.3}$$

for all $k \in N^+$, $\alpha \in (n, n+1]$.

Definition 2.5. (See [19]). The conformable fractional accumulation (CFA) of f with α order is defined as

$$\nabla^\alpha f(k) = \nabla\left(\frac{f(k)}{k^{1-\alpha}}\right) = \sum_{j=1}^{k}\frac{f(j)}{j^{1-\alpha}}, \tag{2.4}$$

for all $k \in N^+$, $\alpha \in (0,1]$ and

$$\nabla^\alpha f(k) = \nabla^{n+1}\left(\frac{f(k)}{k^{\lceil\alpha\rceil-\alpha}}\right), \tag{2.5}$$

for all $k \in N^+$, $\alpha \in (n, n+1]$.

Definition 2.6. (See [8]). For the original series $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\right)$, r-order fractional order inverse accumulation $X^{(-r)} = \left(x^{(-r)}(1), x^{(-r)}(2), ..., x^{(-r)}(n)\right)$ is defined as

$$x^{(-r)}(k) = \sum_{i=1}^{k}\binom{k-i-r-1}{k-i}x^{(0)}(i),\ \ k = 1, 2, \dots, n, \tag{2.6}$$

where

$$\binom{k-i-r-1}{k-i} = \frac{(k-i-r-1)!}{(k-i)!(-r-i)!}.$$

In addition, the fractional order reducing generation sequence of $X^{(0)}(k)$ was defined by [22] and a negative fractional order accumulated operation was introduced by [26].

3. Presentation of exponential conformable fractional grey model

In this section, a novel conformable fractional grey model OCCFGM(1,1) is introduced, which optimizes the CCFGM(1,1) model [37] with an exponential grey action quantity.

3.1. Formulation of proposed fractional grey model

With the original series $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$ we firstly denote the α order CFA as

$$X^{(\alpha)} = \left(x^{(\alpha)}(1), x^{(\alpha)}(2), dots, x^{(\alpha)}(n)\right), \tag{3.1}$$

where

$$x^{(\alpha)}(k) = \nabla^{\alpha} x^{(0)}(k) = \sum_{i=1}^{k} \begin{bmatrix} \lceil\alpha\rceil \\ k-i \end{bmatrix} \frac{x^{(0)}(i)}{i^{\lceil\alpha\rceil-\alpha}}, \quad \alpha \in \mathbb{R}^{+} \tag{3.2}$$

where $\lceil . \rceil$ is the ceil function and $\left[{\lceil\alpha\rceil \atop k-i}\right] = \frac{\Gamma(k-i+\lceil\alpha\rceil)}{\Gamma(k-i+1)\Gamma(\lceil\alpha\rceil)} = \frac{(k-i+\lceil\alpha\rceil-1)!}{(k-i)!(\lceil\alpha\rceil-1)!}$ (see [36]).

Definition 3.1. The r-order whitening differential equation of the OCCFGM(1,1) is defined as

$$\frac{d^r x^{(\alpha)}(t)}{dt^r} + a x^{(\alpha)}(t) = b + c e^{-t^r}, \tag{3.3}$$

where $r \in [0, 1]$ and a is a development coefficient, b is called driving coefficient and ce^{-t^r} is an exponential grey action quantity. So that, the monotone decreasing term ce^{-t^r} will suppress the growth of the prediction error.

When $c = 0$, it yields the CCFGM(1,1) [37]. In addition, the OCCFGM(1,1) model can be translated to the conventional CFGM(1,1) model for $r = 1$ and $c = 0$ [19].

3.2. Parameters estimation

Theorem 3.2. *With given data and the value of fractional orders (α and r), the system parameters a, b and c of the OCCFGM(1,1) model are*

$$\left[\hat{a}, \hat{b}, \hat{c}\right]^T = \left(B^T B\right)^{-1} B^T Y, \tag{3.4}$$

where the matrix B and Y are

$$B = \begin{bmatrix} -0.5(x^{(\alpha)}(2) + x^{(\alpha)}(1)) & 1 & 0.5(e^{-2^r} + e^{-1^r}) \\ -0.5(x^{(\alpha)}(3) + x^{(\alpha)}(2)) & 1 & 0.5(e^{-3^r} + e^{-2^r}) \\ . & . & . \\ . & . & . \\ -0.5(x^{(\alpha)}(n) + x^{(\alpha)}(n-1)) & 1 & 0.5(e^{-n^r} + e^{-(n-1)^r}) \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(\alpha-r)}(2) \\ x^{(\alpha-r)}(3) \\ . \\ . \\ x^{(\alpha-r)}(n) \end{bmatrix}, \quad \hat{\rho} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}. \tag{3.5}$$

Proof. The discrete form of the whitening Equation (3.3) can be obtained by integrated all the terms from $k-1$ to k and using the trapezoid formula for the second integral as

$$\int\int \ldots \int_{k-1}^{k} \frac{d^r x^{(\alpha)}(t)}{dt^r} dt^r + a \int\int \ldots \int_{k-1}^{k} x^{(\alpha)}(t) dt^r = \int\int \ldots \int_{k-1}^{k} (b + ce^{-t}) dt^r. \tag{3.6}$$

The first item of the left-hand side is easily yielded that

$$\int\int \ldots \int_{k-1}^{k} \frac{d^r x^{(\alpha)}(t)}{dt^r} dt^r \approx \Delta^r x^{(\alpha)}(k) = x^{(\alpha-r)}(k). \tag{3.7}$$

Then, using the generalized trapezoid formula [37], the second item of the left-hand side can be obtained as

$$a \int\int \ldots \int_{k-1}^{k} x^{(\alpha)}(t) dt^r \approx \frac{a}{2}\left(x^{(\alpha)}(k) + x^{(\alpha)}(k-1)\right). \tag{3.8}$$

The right-hand side can be obtained by reference [39], which

$$\int\int ... \int_{k-1}^{k} (b + ce^{-t})dt^r = b + \frac{c}{2}(e^{-k^r} + e^{-(k-1)^r}), \tag{3.9}$$

so

$$x^{(\alpha-r)}(k) + \frac{a}{2}\left(x^{(\alpha)}(k) + x^{(\alpha)}(k-1)\right) = b + \frac{c}{2}(e^{-k^r} + e^{-(k-1)^r}), \tag{3.10}$$

where $k = 2, 3, \ldots, n$.

The linear equations system (3.10) can be written as follows:

$$\begin{aligned} x^{(\alpha-r)}(2) &= -0.5a(x^{(\alpha)}(2) + x^{(\alpha)}(1)) + b + 0.5\ c\ (e^{-2^r} + e^{-1^r}) \\ x^{(\alpha-r)}(3) &= -0.5a(x^{(\alpha)}(3) + x^{(\alpha)}(2)) + b + 0.5\ c\ (e^{-3^r} + e^{-2^r}) \\ &. \\ &. \\ &. \\ x^{(\alpha-r)}(n) &= -0.5a(x^{(\alpha)}(n) + x^{(\alpha)}(n-1)) + b + 0.5\ c\ (e^{-n^r} + e^{-(n-1)^r}) \end{aligned} \tag{3.11}$$

and system in Equation (3.11) can be written as

$$Y = B\ \widehat{\rho}. \tag{3.12}$$

According to Equation (3.12), using the least square method to calculate the parameters $\left[\hat{a}, \hat{b}, \hat{c}\right]^T$ with given samples and α as follow:

$$\left[\hat{a}, \hat{b}, \hat{c}\right]^T = \left(B^T B\right)^{-1} B^T Y, \tag{3.13}$$

where

$$B = \begin{bmatrix} -0.5(x^{(\alpha)}(2) + x^{(\alpha)}(1)) & 1 & 0.5(e^{-2^r} + e^{-1^r}) \\ -0.5(x^{(\alpha)}(3) + x^{(\alpha)}(2)) & 1 & 0.5(e^{-3^r} + e^{-2^r}) \\ . & . & . \\ . & . & . \\ -0.5(x^{(\alpha)}(n) + x^{(\alpha)}(n-1)) & 1 & 0.5(e^{-n^r} + e^{-(n-1)^r}) \end{bmatrix}, Y = \begin{bmatrix} x^{(\alpha-r)}(2) \\ x^{(\alpha-r)}(3) \\ . \\ . \\ x^{(\alpha-r)}(n) \end{bmatrix}, \quad \rho = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.14}$$

in which n is the number of samples used to construct the model. Thence the proof is completed by using the least square estimation method. □

3.3. Response function and restored values

Theorem 3.3. *The discrete form of the response function of OCCFGM(1,1) model is given as*

$$\hat{x}^{(\alpha)}(k) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-r}e^{-1}\right) e^{\frac{a}{r}(1-k^r)} + \frac{b}{a} + \frac{c}{a-r}e^{-k^r}, \tag{3.15}$$

where $k = 2, 3, \ldots, n$.

Proof. It is clear that, by using Teorem 1 and item 1, the r order whitening differential equation (3.3) can be written as

$$t^{1-r}\frac{dx^{(\alpha)}(t)}{dt} + ax^{(\alpha)}(t) = b + ce^{-t^r}. \tag{3.16}$$

Then, the solution of the first order linear differential equation (3.16) can be obtained as

$$x^{(\alpha)}(t) = \frac{b}{a} + \frac{c}{a-r}e^{-t^r} + de^{-\frac{a}{r}t^r}, \tag{3.17}$$

where d is integral constant. By using the initial condition $x^{(\alpha)}(1) = x^{(0)}(1)$, the constant d can be found as

$$d = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-r}e^{-1}\right) e^{\frac{a}{r}}.$$

Therefore the grey prediction model Equation (3.17) can be obtained as following

$$\hat{x}^{(\alpha)}(t) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-r}e^{-1}\right) e^{\frac{a}{r}(1-t^r)} + \frac{b}{a} + \frac{c}{a-r}e^{-t^r}$$

and the discrete form of the response function can be write as

$$\hat{x}^{(\alpha)}(k) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-r}e^{-1}\right) e^{\frac{a}{r}(1-k^r)} + \frac{b}{a} + \frac{c}{a-r}e^{-k^r}.$$

The proof is completed. □

Theorem 3.4. *The restored values can be given as*

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k) = k^{\lceil\alpha\rceil-\alpha}\Delta^{n+1}\hat{x}^{(\alpha)}(k), \quad \alpha \in (n, n+1], \tag{3.18}$$

where $k = 2, 3, \ldots, n$.

Proof. From Equation (3.2) it can be seen that $\hat{x}^{(\alpha)}(k) = \nabla^{\alpha}\hat{x}^{(0)}(k)$. If we apply the inverse operator Δ^{α}, it is obtained as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k)$$

and from Definition (2.4), it can be written as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k) = k^{\lceil\alpha\rceil-\alpha}\Delta^{n+1}\hat{x}^{(\alpha)}(k), \quad \alpha \in (n, n+1].$$

Particularly, when $\alpha \in (0, 1]$ the restored values can be written as

$$\hat{x}^{(0)}(k) = k^{1-\alpha}(\hat{x}^{(\alpha)}(k) - \hat{x}^{(\alpha)}(k-1)). \tag{3.19}$$

Thence the proof is completed. □

3.4. Evaluative accuracy of the forecasting model

To evaluate the prediction accuracy of OCCFGM(1,1), relative percentage error (RPE) and the mean absolute percentage error (MAPE) are used to evaluate the level of prediction performance of the prediction models. They are defined as follows:

$$RPE(k) = \left|\frac{\hat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)}\right| \times 100, \tag{3.20}$$

$$MAPE = \frac{1}{n}\sum_{k=1}^{n} RPE(k), \tag{3.21}$$

where $x^{(0)}(k)$ is the original series, and $\hat{x}^{(0)}(k)$ is the predicted series.

4. Brute Force algorithm for optimal α and r

In this section, our purpose is to find the optimum parameters α and r that minimizes the model's mean absolute percentage error (MAPE).

Brute Force is a straightforward approach, which is also known as the Naive algorithm, for solving optimization problems that rely on sheer computing power and trying every possibility rather than advanced techniques to improve efficiency. Unlike some of the other popular swarm intelligence algorithms, Brute Force is applicable to a very wide variety of problems and it is effective and easy method to find the optimum value in the solution domain.

Despite the convergence speed advantages of other algorithms, it is a disadvantage that they may focus on local extremum point rather than the global extremum. However, Brute Force algorithm scans the whole domain, evaluates each point, then calculates the MAPE's on these points and reaches optimum parameters without any delusion.

The optimum (α, r) value is searched within the rectangular domain bounded by $\alpha \in [0, 1.4]$ and $r \in [0, 1.5]$ (see Figure 1). First by fixing $\alpha = 0$, r takes all the values from 0 to 1.5 with $h = 0.01$ step width. Then by fixing $\alpha = \alpha + h$, r takes all the values from 0 to 1.5 with $h = 0.01$ step width (see Figure 2). This procedure is performed until the entire region has been scanned. In this way, all the (α, r) points in the whole region and the MAPE's at these points are calculated. Hence α and r, which give the minimum of the calculated MAPE's, are determined as the optimum parameters.

All these processes are completed in about 10 seconds by writing a simple FORTRAN code.

Figure 1. The search region and (α, r) points for Brute Force algorithm.

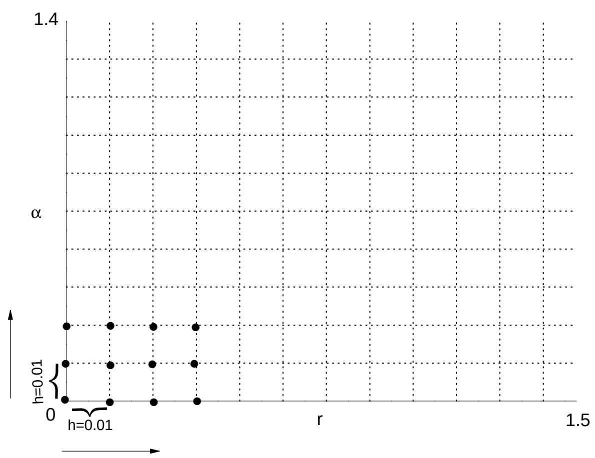

Figure 2. The Brute Force mechanism.

5. Validation of the OCCFGM(1,1)

This section provides three case studies to demonstrate the efficacy of the proposed model.

5.1. Case 1: Domestic energy consumption of China

In this subsection, we use the novel OCCFGM(1,1) model to predict domestic energy consumption of China (10^4 tons of standard coal). The raw data set and some columns from paper [37] are used to test for efficacy and applicability of the proposed fractional grey model.

By using the Brute Force algorithm, the optimum α and r values are obtained as $\alpha = 0.92$ and $r = 1.02$ with 0.9583 Mape. Mapes of all the points within the $\alpha - r$ region are calculated and the MAPE's are shown in Figure 3.

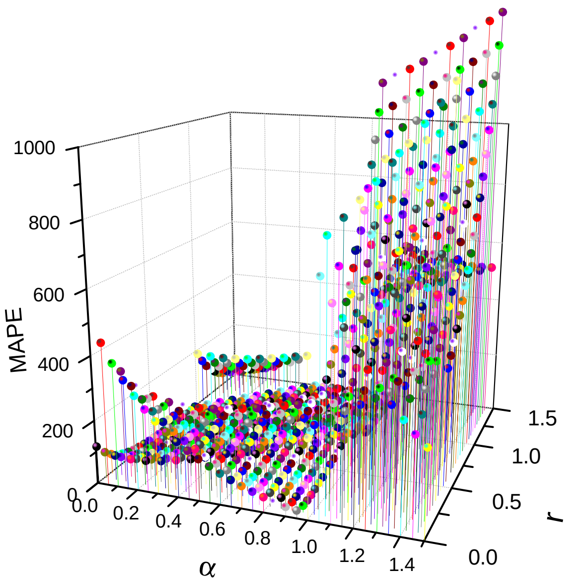

Figure 3. The region of $\alpha - r$ and MAPE values of the all (α, r) points for case 1.

Table 1. Numerical results for domestic energy consumption of China obtained by FAGM, ANN, TDPGM, ARIMA, EXGM, SVR, CCFGM and OCCFGM.

Year	RAW	FAGM	ANN	TDPGM	ARIMA	EXGM	SVR	CCFGM	OCCFGM
2005	27573	27573.00	27576.24	27573.00	27573.00	27573.00	27572.90	27573.00	27573.00
2006	27765	28776.69	28801.47	27990.63	29893,26	28238.42	29574.73	27207.68	27763.98
2007	30814	30529.07	30403.47	30011.65	29728,94	30145.47	31576.57	28992.48	29705.97
2008	31898	32510.82	32409.59	32146.03	33080,51	32181.29	33578.40	31373.76	31919.92
2009	33843	34650.97	34790.27	34398.83	33991,82	34354.57	35580.23	33965.07	34275.81
2010	36470	36925.23	37441.64	36775.47	35987,78	36674.52	37582.07	36671.07	36730.58
2011	39584	39324.40	40193.56	39281.80	38770,78	39150.89	39583.90	39459.34	39280.02
2012	42306	41845.55	42848.65	41924.09	42069,85	41793.80	41585.73	42317.23	41933.19
2013	45531	44488.93	45235.79	44709.09	44834,20	44613.27	43587.57	45239.62	44702.68
2014	47212	47256.57	47249.65	47644.06	48177,20	47617.93	45589.40	48224.63	47601.47
2015	50099	50151.64	48859.33	50736.80	49648,76	50811.34	47591.23	51271.90	50642.18
2016	54209	52721.73	50091.35	53995.70	52627,12	54181.88	49593.07	54381.79	53837.07
2017	57620	55350.93	51003.38	57429.77	57016,85	57675.30	51594.90	57555.00	57198.20
MAPE		1.73	3.00	1.00	2.13	1.57	4.58	1.38	0.96

Furthermore, the OCCFGM(1,1) model compared with the six effective models, including the FAGM, ANN, TDPGM, ARIMA, EXGM, SVR and CCFGM. As can be seen from Table 1, OCCFGM(1,1) yielded the lower MAPE compared with other powerful models. The simulation of these models for the raw data of China's domestic energy consumption is given in Figure 4. The MAPEs of other competing models and the novel model are calculated as 1.73%, 3.00%, 1.00%, 2.13%, 1.57%, 4.58%, 1.38% and 0.96%, respectively (see Figure 5). From a short-term forecasting viewpoint, OCCFGM(1,1) produced the lowest Mape compared with the other models, which means that the proposed model, OCCFGM(1,1), has highly accurate forecasting power.

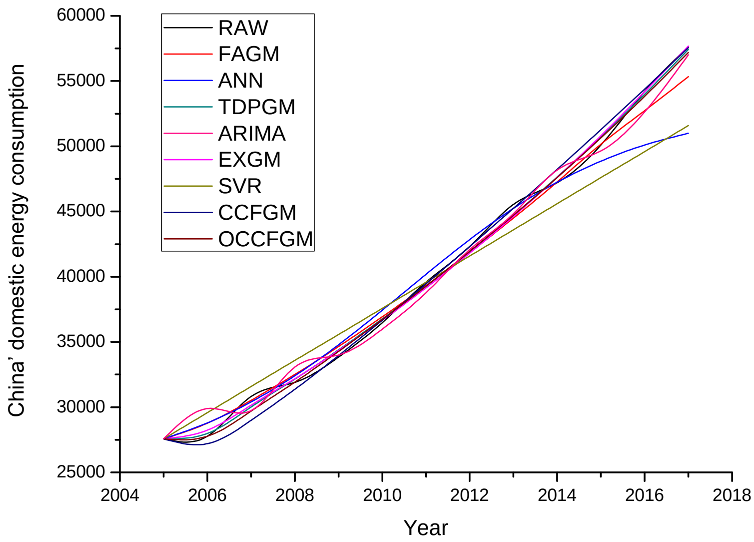

Figure 4. Simulative curves with seven models for domestic energy consumption of China.

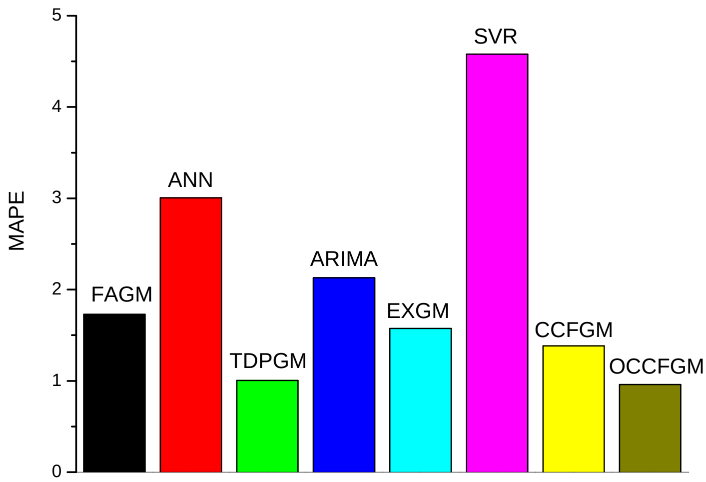

Figure 5. MAPEs of domestic energy consumption of China by FAGM, ANN, TDPGM, ARIMA, EXGM, SVR, CCFGM and OCCFGM models.

Forecasting values of China's domestic energy consumption using OCCFGM(1,1) up to 2025 are listed in Table 2 and Figure 6. The forecasting results show that China's energy consumption growth over our observation period.

Table 2. Forecasting values of China's domestic energy consumption using OCCFGM.

Years	Forecasting values (10^4 tons of standard coal)
2018	60737.6424
2019	64467.6610
2020	68400.8194
2021	72550.1069
2022	76929.0258
2023	81551.6697
2024	86432.7943
2025	91587.8835

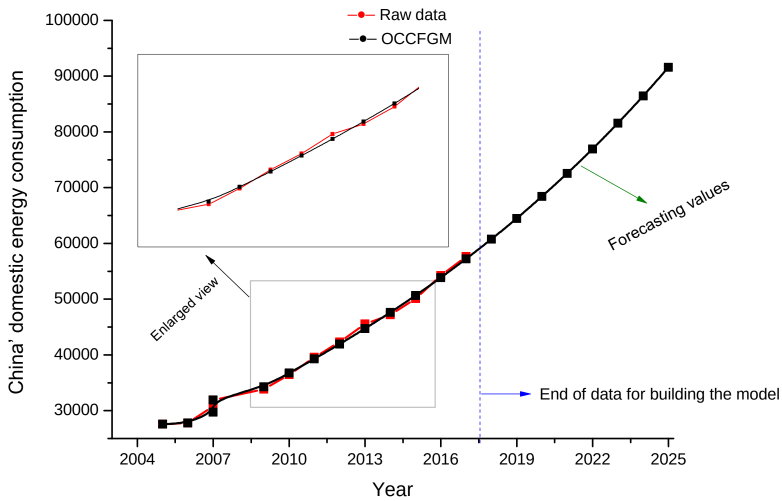

Figure 6. Forecasting results of OCCFGM(1,1) for domestic energy consumption of China.

5.2. Case 2

The CO_2 emission of Russia is studied in this subsection. The raw data set and some columns from paper [36] are used to test for efficacy and applicability of the proposed fractional grey model. Although the raw data of Russia is non-stationarity, the forecasting results show that the novel OCCFGM(1,1) model yielded the lower MAPE compared with other new models.

The optimum α and r values are obtained as $\alpha = 0.84$ and $r = 0.97$ with 1.499 MAPE by using the Brute Force algorithm. MAPEs of all the points within the $\alpha - r$ region are calculated in a short time (nearly 10 seconds) by writing a simple FORTRAN code. The MAPE values of the points in the region and the track of searching for the optimal parameters are graphed in Figure 7.

Furthermore, the OCCFGM(1,1) model compared with the six effective models, including the GM, ARIMA, NGMO, TDPGM, FAGM, FANGM and CFNGM. As can be seen from Table 3, OCCFGM(1,1) yielded the lower MAPE compared with other new models. The simulation of these models for the raw data of Russia's CO_2 emissions is given in Figure 8. The MAPE's of other competing models and the novel model are $1.83\%, 1.61\%, 1.81\%, 1.59\%, 1.66\%, 1.64\%, 1.71\%$ and 1.50%, respectively (see Figure 9). From a short-term forecasting viewpoint, OCCFGM(1,1) produced the lowest Mape compared with the other models, which means that the proposed novel model, OCCFGM(1,1), has highly accurate forecasting power. Forecasting values of Russia's CO_2 emission using OCCFGM(1,1) up to 2025 are listed in Table 4 and Figure 10.

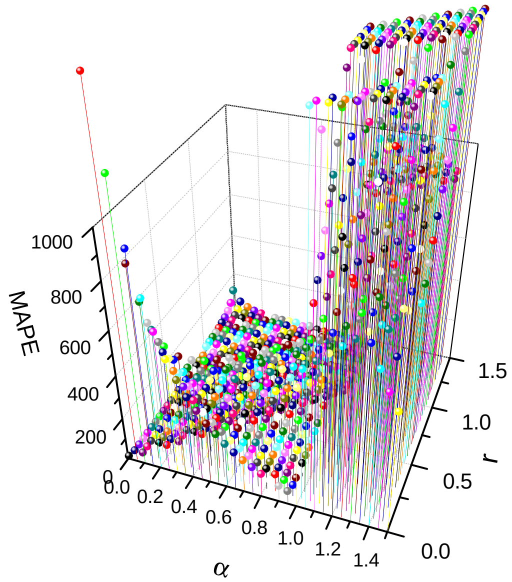

Figure 7. The region of $\alpha - r$ and MAPE values of the all (α, r) points for Case 2.

Table 3. Numerical results for CO_2 emissions of Russia obtained by GM, ARIMA, NGMO, TDPGM, FAGM, FANGM, CFNGM and OCCFGM (million tones).

Year	RAW	GM(1,1)	ARIMA	NGMO	TDPGM	FAGM	FANGM	CFNGM	OCCFGM
2000	1453.30	1453.30	1453.30	1453.30	1453.30	1453.30	1453.30	1453.30	1453.30
2001	1466.50	1485.90	1472,30	1466.69	1458.21	1444.28	1455.65	1477.86	1476.91
2002	1466.30	1488.44	1482,64	1475.27	1472.03	1461.78	1466.31	1459.90	1466.67
2003	1495.20	1491.09	1484,91	1483.21	1483.73	1480.59	1478.18	1477.79	1476.13
2004	1490.30	1492.75	1502,46	1490.56	1493.48	1496.56	1489.58	1490.74	1490.15
2005	1466.60	1496.42	1500,20	1497.37	1501.43	1509.01	1499.94	1500.51	1503.28
2006	1535.00	1499.09	1491,06	1503.67	1507.73	1518.19	1508.99	1508.07	1513.88
2007	1528.10	1501.76	1529,85	1509.50	1512.55	1524.56	1516.63	1514.00	1521.74
2008	1554.30	1504.44	1522,50	1514.90	1516.04	1528.61	1522.83	1518.66	1527.09
2009	1445.30	1507.12	1535,24	1519.90	1518.37	1530.79	1527.57	1522.33	1530.29
2010	1492.20	1509.81	1478,16	1524.53	1519.71	1531.49	1530.90	1525.18	1531.67
2011	1555.90	1512.50	1512,28	1528.81	1520.23	1531.02	1532.82	1527.36	1531.51
2012	1569.10	1515.20	1544,16	1532.77	1520.09	1529.65	1533.39	1528.97	1530.05
2013	1527.70	1517.90	1545,99	1536.44	1519.47	1527.60	1532.65	1530.09	1527.48
2014	1530.80	1520.61	1521,96	1539.84	1518.54	1525.03	1530.63	1530.80	1523.96
2015	1489.50	1523.32	1524,94	1542.99	1517.48	1522.09	1527.39	1531.14	1519.62
2016	1501.50	1526.04	1503,95	1545.90	1516.48	1518.88	1522.98	1531.16	1514.57
2017	1488.40	1528.76	1514,62	1548.60	1515.71	1515.50	1517.42	1530.89	1508.89
2018	1550.80	1531.48	1510,06	1551.09	1515.36	1512.00	1510.78	1530.37	1502.67
MAPE		1.83	1.61	1.81	1.59	1.66	1.64	1.71	1.50

5.3. Case 3

Recently, the fractional order accumulated grey prediction model FAGM(1,1) is applied to predict per capita water consumption of 31 regions in China from 2019 to 2024 [23]. In this subsection, we use the novel OCCFGM(1,1) model to predict future drinking water demand for Turkey. In addition, this subsection presents a systematic predicting

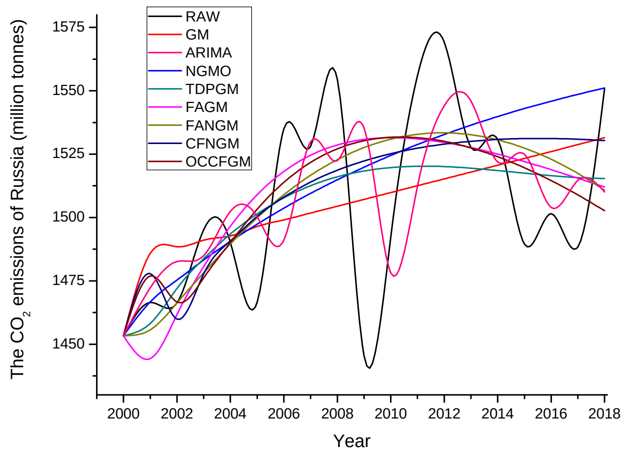

Figure 8. Simulative curves with seven models for CO_2 emissions of Russia.

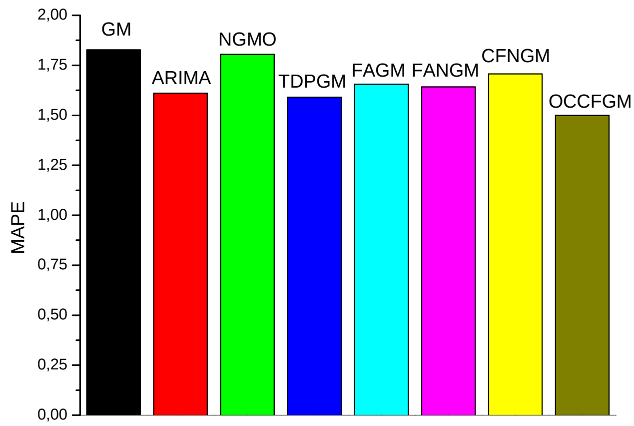

Figure 9. MAPEs of CO_2 emissions of Russia by GM, ARIMA, NGMO, TDPGM, FAGM, FANGM, CFNGM and OCCFGM models.

Table 4. Forecasting values of CO_2 emissions of Russia using OCCFGM.

Years	Forecasting values (million tonnes)
2019	1495.9629
2020	1488.8365
2021	1481.3359
2022	1473.5040
2023	1465.3785
2024	1456.9923
2025	1448.3752

methodology which includes data collection, parameter estimation, result analysis and future drinking water demand for Turkey. Turkish Statistical Institute (TUIK) reports complete, correct and official data. The data includes the amount of total drinking water drawn from resources between 2008 and 2018 (thousand m^3/year) in Turkey.

The optimum α and r values are obtained as $\alpha = 0.58$ and $r = 0.96$ with 0.6547 MAPE by using the Brute Force algorithm. MAPEs of all the points within the $\alpha - r$ region are calculated in a short time (nearly 10 second) by writing a simple FORTRAN code. The MAPE values of the points in the region and the track of searching for the

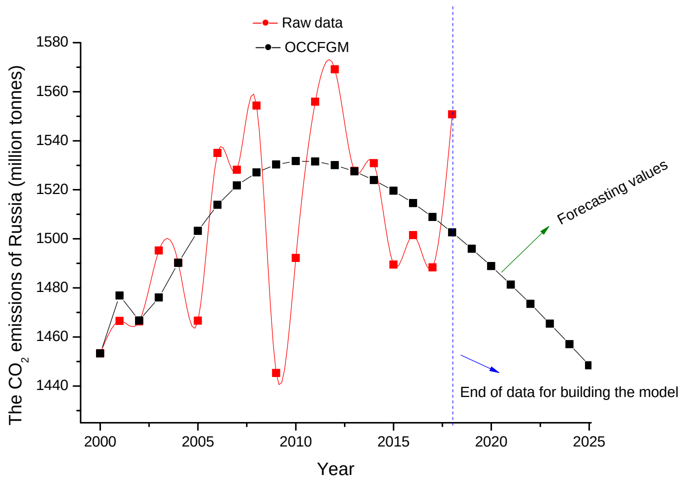

Figure 10. Forecasting results of OCCFGM(1,1) for CO_2 emissions of Russia.

optimal parameters are graphed in Figure 11. In addition, the optimum α value for the CFGM(1,1) model is obtained as $\alpha = 1$ by using the Brute Force algorithm. Hence, the CFGM(1,1) model turn into standard GM(1,1) model for this raw data. For this reason, the forecasting values of these two models are same.

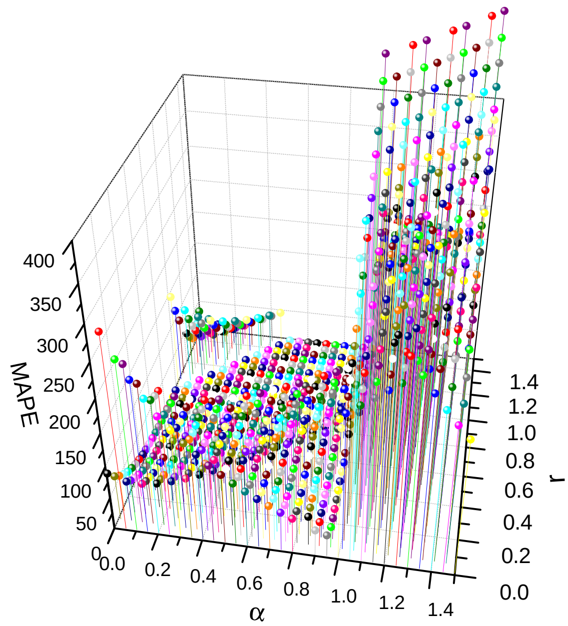

Figure 11. The region of $\alpha - r$ and MAPE values of the all (α, r) points for Case 3.

Furthermore, the OCCFGM(1,1) model compared with the five effective models, including the EXGM, ARIMA, GM, ONGM, TDPGM and CFGM. As can be seen from Table 5, OCCFGM(1,1) yielded the lower MAPE compared with other models. The simulation of these models for the raw data of drinking water consumption of Turkey is given in Figure 12. The MAPEs of other competing models and the novel model are 0.66%, 0.97%, 1.35%, 5.14%, 4.80%, 1.35% and 0.65%, respectively (see Figure 13). From a short-term forecasting viewpoint, OCCFGM(1,1) produced the lowest MAPE compared with the other models, which means that the proposed novel model, OCCFGM(1,1), has highly accurate forecasting power.

Table 5. Numerical results for total drinking water drawn from resources in Turkey obtained by EXGM, ARIMA, GM, ONGM, TDPGM, CFGM and OCCFGM (thousand m^3/year).

Year	Actual Value	EXGM	ARIMA	GM(1,1)	ONGM(1,1)	TDPGM	CFGM	OCCFGM
2008	4546574.00	4546574.00	4546574.00	4546574.00	4546574.00	4546574.00	4546574.00	4546574.00
2010	4784734.00	4778345.14	4823049,39	4670431.88	4235894.33	5220119.92	4670431.88	4768929.14
2012	4936342.00	4942384.36	5018419,60	5007438.87	5181165.10	5254997.91	5007438.87	4936652.24
2014	5237407.00	5289990.29	5179886,59	5368763.47	5574286.36	5593705.01	5368763.47	5307095.84
2016	5838561.27	5721038.75	5745807,76	5756160.36	5737778.48	6025127.79	5756160.36	5765923.61
2018	6193157.54	6234712.30	6232945,90	6171510.88	5805771.93	6393223.44	6171510.88	6256134.72
MAPE		0.66	0.97	1.35	5.14	4.80	1.35	0.65

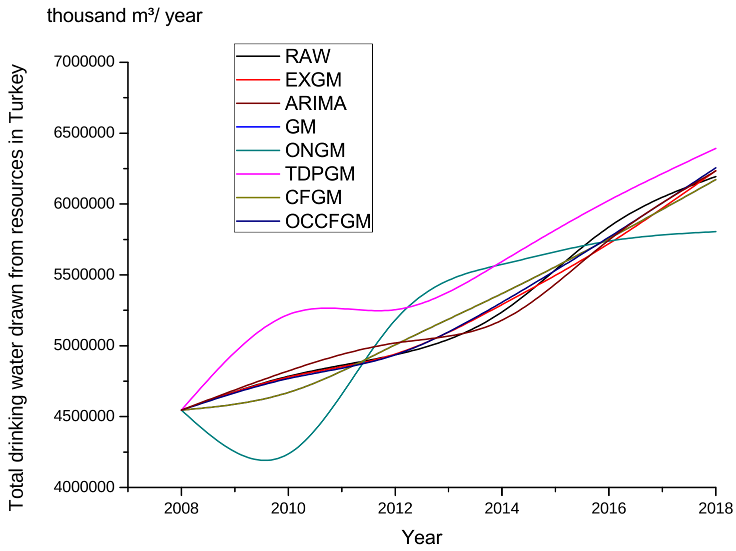

Figure 12. Simulative curves with six models for Case 3.

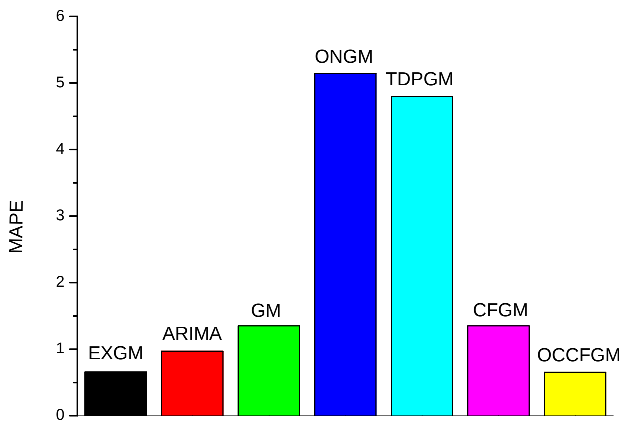

Figure 13. MAPEs of total drinking water drawn from resources in Turkey by EXGM, ARIMA, GM, ONGM, TDPGM, CFGM and OCCFGM models.

Scientific simulation and accurate forecasting of the future drinking water consumption are important to ensure the supply-demand balance of water. Furthermore, Turkey's drinking water consumption during 2020 to 2028 has been forecasted by the OCCFGM(1,1) in this subsection (see Table 6). The forecasting results provide the growth trend of the future drinking water consumption of Turkey, and also offer a guideline for policymaking and project planning (see Figure 6). It can be observed that OCCFGM(1,1) is quite practical to use, with satisfactory accuracy in drinking water consumption forecasting with the smallest MAPE when compared with the comparison models.

Table 6. Forecasting values of total drinking water drawn from sources in Turkey using OCCFGM.

Years	Forecasting values (thousand m^3/ year)
2020	6753647.40
2022	7249829.00
2024	7742626.36
2026	8232539.20
2028	8720907.75

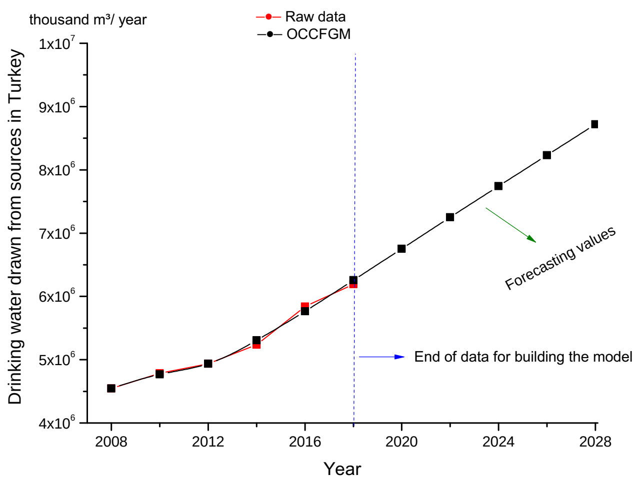

Figure 14. Forecasting results of OCCFGM(1,1) for drinking water demand of Turkey.

6. Conclusions

Since the fractional calculations are most important to the grey prediction model, there are many scholars proposing new methods on the fractional grey models. Hence, a novel optimization for the CCFGM(1,1) model based on the conformable fractional difference and conformable fractional accumulation has been developed in this study. The results of the numerical examples indicated that the proposed grey prediction model aims to achieve very effective performance. The structural parameters (a, b and c) of the model can be dynamically adjusted according to the actual system. The optimal value of fractional order, α and r are calculated by using the brute-force approach. The proposed OCCFGM(1,1) model is suitable for predicting the data sequence with the characteristics of non-homogeneous exponential law. The OCCFGM(1,1) model has achieved higher accuracy than the comparison models such as EXGM, GM, ARIMA, ONGM, TDPGM, CFGM, NGMO, FAGM, FANGM, CFNGM, ANN, SVR and CCFGM. However, they can all be employed for estimations.

Using three case studies in Section 5, we have shown that the MAPE of the OCCFGM(1,1) model is very low. The proposed OCCFGM(1,1) model maybe plays an important role in enriching the theoretical system of grey forecasting theory and it can be used for other real cases of small sample forecasting in the future. In addition, the combination of other fractional forecasting models, especially for the time series with highly effective, is also an interesting direction for next studies.

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