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Fuzzy Collineations of Fuzzy Projective Planes

Elif Altıntaş[1*] and Ayşe Bayar[2]

[1]*Department of Software Engineering, Faculty of Engineering, Haliç University, Istanbul, Turkey*
[2]*Department of Mathematics and Computer, Faculty of Science and Letters, Eskişehir Osmangazi University, Eskişehir, Turkey*
[*]*Corresponding author*

Abstract

In this paper, the fuzzy counterparts of the collineations defined in the classical projective planes are defined in fuzzy projective planes. The properties of fuzzy projective plane left invariant under the fuzzy collineations are characterized depending on the base point, base line and the membership degrees of fuzzy projective plane.

Keywords: *Collineation; Fuzzy Projective Plane; Isomorphism; Projective Plane*
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1. Introduction

The fuzzy concept was first proposed by Zadeh in 1965 [15], and many scientists have contributed to this field since then. The first paper on fuzzy groups was published by Azriel Rosenfeld in 1971 [14]. Fuzzy vector spaces were introduced by Katsaras and Liu in 1977 [8].
The fuzzy correspondings of the maps in the vector space was first given by Abdulhalikov in 1996 [1]. In addition, it has been studied by Abdulhalikov that the fuzzy subspace of fuzzy linear maps is isomorphic to the fuzzy subspace of dual maps.
Projective planes have been fuzzified by Kuijken et al., see [9]. Also a fuzzy group corresponding to the fuzzy projective geometry was created, so that through these fuzzy projective geometries a relationship between fuzzy vector spaces and fuzzy groups was obtained by Kuijken, Maldeghem and Kerre in 1999 [10]. The fuzzy projective plane collineations were described by Kuijken and Maldeghem in 2003 [13]. Then the fuzzy line spreads of the Fano plane are classified and a general existence theorem for line spreads of arbitrary finite projective planes is proved by Akça et. al.[3]. Bayar, Akça and Ekmekçi studied the classification of fuzzy lines of the 3-dimensional projective space obtained from the 4-dimensional fuzzy vector space and they classified the fuzzy planes of the same fuzzy projective space in 2006 [6].
The aim of this study is to define the fuzzy equivalents of collineations defined in classical projective planes in fuzzy projective planes and to prove the properties that are invariant under the fuzzy collineations in fuzzy projective planes. The paper is organized as follows. In Section 2, the concepts of fuzzy isomorphism and fuzzy collineations between two fuzzy projective planes are introduced and then some important results are obtained. In Section 3, we introduce the fuzzy homomorphism of fuzzy projective planes by using the homomorphism in the base projective planes. We research validity of fuzzy counterparts of some classical theorems in the classical projective planes for fuzzy projective planes and examine the properties of collineations under fuzzy collineations. Also, we investigate the invariant properties of the fuzzy projective plane under fuzzy collineations with respect to the base point, the base line and the membership degrees of the fuzzy projective plane.

2. Preliminaries

In this section, some relevant definitions of fuzzy set theory, fuzzy vector space and fuzzy projective space are reminded. First recall that fuzzy sets were introduced by Zadeh in the fundamental paper [15].

Definition 2.1. ([15]) *A fuzzy set λ of a set X is a function $\lambda : X \to [0,1] : x \to \lambda(x)$. The number $\lambda(x)$is called the degree of membership of the point x in λ. The intersection $\lambda \wedge \mu$ of the two fuzzy sets λ and μ on X is given by the fuzzy set $\lambda \wedge \mu : X \to [0,1] : \lambda(x) \wedge \mu(x)$, where $\wedge$ denotes the minimum operator and also $\vee$ denotes the maximum operator.*

Definition 2.2. ([12], $Definition$ 2.2) *Consider a set X and fuzzy sets λ and μ on X. The Cartesian product $\lambda \times \mu$ of the two fuzzy sets is defined as follows:*

$$\begin{array}{rccc} \mu \times \lambda : & X \times X & \to & [0,1] \\ & (x,y) & \to & \mu(x) \wedge \lambda(y) \end{array}$$

Email addresses: **elifaltintaskahriman@halic.edu.tr (Elif Altıntaş), akorkmaz@ogu.edu.tr (Ayşe Bayar)**

Definition 2.3. ([9], $Definition$ 1.1) *Let $\mu : V \to [0,1]$ be a fuzzy set on V which is a vector space over a field K. Then we call μ as a fuzzy vector space on V if and only if $\mu(a.\bar{u}+b.\bar{v}) \geq \mu(\bar{u}) \wedge \mu(\bar{v}), \forall \bar{u},\bar{v} \in V$ and $a,b \in K$.*

Definition 2.4. ([7]) *An (axiomatic) projective plane $\mathscr{P}$ is an incidence structure $(\mathscr{N},\mathscr{D},\circ)$ with $\mathscr{N}$ a set of points, $\mathscr{D}$ a set of lines and $\circ$ an incidence relation, such that the following axioms are satisfied:*
i) *every pair of distinct points are incident with a unique common line;*
ii) *every pair of distinct lines are incident with a unique common point;*
iii) *$\mathscr{P}$ contains a set of four points with the property that no three of them are incident with a common line.*
A closed configuration $\mathscr{S}$ of $\mathscr{P}$ is a subset of $\mathscr{N} \cup \mathscr{D}$ that is closed under taking intersection points of any pair of lines in $\mathscr{S}$ and lines spanned by any pair of distinct points of $\mathscr{S}$. We denote the line in $\mathscr{P}$ spanned by the points p and q by $\langle p,q \rangle$.

Definition 2.5. ([4]) *(Pasch's axiom) An (axiomatic) projective space $\mathscr{S}$ is an incidence structure $(\mathscr{N},\mathscr{D},\circ)$ with $\mathscr{N}$ a set of points, $\mathscr{D}$ a set of lines and $\circ$ an incidence relation, such that the following axioms are satisfied:*
i) *every line is incident with at least two points;*
ii) *every pair of distinct points are incident with a unique common line;*
iii) *given distinct points p,q,r,s,t such that $\langle p,q \rangle = \langle p,r \rangle \neq \langle p,s \rangle = \langle p,t \rangle$, there is a point $x \circ \langle q,s \rangle \cap \langle r,t \rangle$.*

Definition 2.6. ([10], $Definition$ 2.5) *Suppose $\mathscr{P}$ is an $n-$dimensional projective space. A fuzzy set λ on the point set of $\mathscr{P}$ is a fuzzy $n-$dimensional projective space on $\mathscr{P}$ if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$, for all collinear points p,q,r of $\mathscr{P}$. It is denoted as $(\lambda,\mathscr{P})$. The projective space $\mathscr{P}$ is called the underlying (crisp) projective space of $(\lambda,\mathscr{P})$. If $\mathscr{P}$ is a fuzzy point, line, plane, etc., we use underlying point, underlying line, underlying plane, etc., respectively. We will sometimes briefly write λ instead of $(\lambda,\mathscr{P})$.*

In practice, this means that in the point set of a line, all elements have the same degree of membership, but may not be the same. Moreover, more generally speaking, this means that in any subspace U, all points have the same degree of membership, except that they may be in subspace $U^{'}$ of U. All points have the same degree of membership, except for those that may be in a subspace $U^{''}$ of $U^{'}$, etc. [2].

Definition 2.7. ([3], $Definition$ 2.4) *Let $(\lambda,\mathscr{P})$ be a fuzzy projective space and let U be a subspace of $\mathscr{P}$. Then (λ_U,U) is called a fuzzy subspace of $(\lambda,\mathscr{P})$ if $\lambda_U(x) \leq \lambda(x)$ for $x \in U$, and $\lambda_U(x) = 0$ for $x \notin U$.*

As already alluded to above, the following proposition gives the structure of a fuzzy projective line.

Proposition 2.8. ([3], $Proposition$ 2.5) *Let (λ,L) be a fuzzy projective line. Then there are constants $a,b \in]0,1]$, $a \leq b$, and a point z of L such that*
i) *$\lambda(z) = b$,*
ii) *$\lambda(x) = a$, for all $x \neq z$.*

By the previous proposition, every fuzzy projective line admitting points with different membership degrees contains a unique point with maximal membership degree. We will refer to such a point as the base point of the fuzzy line. More generally, the structure of a fuzzy projective space looks as follows.

Proposition 2.9. ([3]) *Let $(\lambda,\mathscr{P})$ be a fuzzy projective space of dimension n. Then there are constants $a_i \in]0,1]$, $i = 0,1,...,n$, with $a_i \leq a_{i+1}$, and a chain of subspaces $(U_i)_{0 \leq i \leq n}$ with $U_i \subseteq U_{i+1}$ and $dimU_i = i$, such that*

$$\begin{array}{llll} \lambda: & \mathscr{P} & \to & [0,1] \\ & x & \to & a_0 \qquad for\ x \in U_0, \\ & x & \to & a_i \qquad for\ x \in U_i \setminus U_{i-1}, \quad i = 1,2,...,n. \end{array}$$

Definition 2.10. ([9]) *Consider the projective plane $\mathscr{P} = (\mathscr{N},\mathscr{D},\circ)$. Suppose $p \in \mathscr{N}$ and $\alpha \in [0,1]$. The fuzzy point (p,α) is the following fuzzy set on the point set $\mathscr{N}$ of $\mathscr{P}$:*

$$\begin{array}{llll} (p,\alpha): & \mathscr{N} & \to & [0,1] \\ & p & \to & \alpha \\ & x & \to & 0 \qquad if\ x \in \mathscr{N} \setminus \{p\}. \end{array}$$

The point p is called the base point of the fuzzy point (p,α).
A fuzzy line (L,β) with base line L is defined in a similar way. Two fuzzy lines (L,α) and (M,β), with $\alpha \wedge \beta > 0$, intersect in the unique fuzzy point $(L \cap M, \alpha \wedge \beta)$. Dually, the fuzzy points (p,λ) and (q,μ), with $\lambda \wedge \mu > 0$, span the unique fuzzy line $(\langle p,q \rangle, \lambda \wedge \mu)$.

Definition 2.11. ([11], $Definition$ 3.1) *Suppose $\mathscr{P}$ is a projective plane $(\mathscr{N},\mathscr{D},\circ)$. The fuzzy set λ on $\mathscr{N} \cup \mathscr{D}$ is a fuzzy projective plane on $\mathscr{P}$ if*
i) *$\lambda(L) \geq \lambda(p) \wedge \lambda(q), \forall p,q : \langle p,q \rangle = L$ and*
ii) *$\lambda(p) \geq \lambda(L) \wedge \lambda(M), \forall L,M : L \cap M = p$.*

Definition 2.12. ([1]) *Let E and L be vector spaces over the same field F, and let $\mu : E \to [0,1]$, $\lambda : L \to [0,1]$ be fuzzy subspaces. If $\lambda(\varphi(x)) \geq \mu(x)$ for all $x \in E$, we say that a linear map $\varphi : E \to L$ is fuzzy linear from the fuzzy subspace μ to fuzzy subspace λ. The space of fuzzy linear maps from μ to λ is denoted by $FHom(\mu,\lambda)$.*

Definition 2.13. ([1], $Definition$ 2.7) *Let (E_1,μ_1) and (E_2,μ_2) are two fuzzy vector spaces. If there exists an isomorphism $\varphi : E_1 \to E_2$ with the property $\mu_1(x) = \mu_2(\varphi(x))$ for all $x \in E_1$, $\mu_1 : E_1 \to [0,1]$ and $\mu_2 : E_2 \to [0,1]$ are isomorphic.*

3. Collineations of Fuzzy Projective Planes

In the present section, we will investigate collineations of fuzzy projective planes. Compared to isomorphisms, collineations of projective plane have the advantages and exemplified. In projective planes, a collineation is a point-to-point and line-to-line transformation that preserves the relation of incidence. Thus it transforms ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on. Clearly, it is a self-dual concept, the inverse of a collineation, and the product of two collineations is again a collineation [5]. Our aim is now to define the fuzzy counterparts of homomorphism and isomorphism defined in vector spaces in fuzzy projective planes and to apply theorems about the properties of collineations in projective plane to fuzzy projective plane. Furthermore, we will show that each collineation can be uniquely extended to a fuzzy projective collineation.
The definitions of homomorphism, isomorphism and collineation in projective planes can be adopted to fuzzy projective planes as follows:

Definition 3.1. *Let $[\mathscr{P},\lambda]$ and $[\mathscr{P}',\mu]$ be two fuzzy projective planes with the base planes $\mathscr{P}=(\mathscr{N},\mathscr{D},\circ)$ and $\mathscr{P}'=\left(\mathscr{N}',\mathscr{D}',\circ'\right)$, respectively. Suppose that f is a homomorphism of a projective plane $\mathscr{P}$ into a projective plane $\mathscr{P}'$. $\bar{f}$ is called as a fuzzy homomorphism from $[\mathscr{P},\lambda]$ into $[\mathscr{P}',\mu]$ if $\bar{f}(p,\alpha)=(f(p),\beta)$ for all $(p,\alpha)\in[\mathscr{P},\lambda]$ where $\lambda(p)=\alpha$, $\mu(f(p))=\beta$ and $\alpha\le\beta$. If f is an isomorphism of $\mathscr{P}$ into $\mathscr{P}'$ and $\alpha=\beta$, then $\bar{f}$ is called as a fuzzy isomorphism between the fuzzy projective planes $[\mathscr{P},\lambda]$ and $[\mathscr{P}',\mu]$. Also if $\mathscr{P}=\mathscr{P}'$, $\bar{f}$ is called a fuzzy collineation.*

Theorem 3.2. *Let $\bar{f}:[\mathscr{P},\lambda]\to[\mathscr{P}',\mu]$ be fuzzy isomorphism, the following holds:*
(i) *For any pair of fuzzy points (p_1,α_1) and (p_2,α_2), $p_1\ne p_2$ in $[\mathscr{P},\lambda]$,*

$$\bar{f}(\langle(p_1,\alpha_1),(p_2,\alpha_2)\rangle)=\left\langle\bar{f}(p_1,\alpha_1),\bar{f}(p_2,\alpha_2)\right\rangle.$$

(ii) *For any pair of fuzzy lines (L,β_1) and (M,β_2), $L\ne M$ in $[\mathscr{P},\lambda]$,*

$$\bar{f}((L,\beta_1)\cap(M,\beta_2))=\bar{f}(L,\beta_1)\cap\bar{f}(M,\beta_2).$$

(iii) *For any fuzzy point (p,α) and fuzzy line (L,β) in $[\mathscr{P},\lambda]$, if p is not on L, then the fuzzy point $\bar{f}((p,\alpha))$ is not on $\bar{f}((L,\beta))$ in $[\mathscr{P}',\mu]$.*

Proof. **(i)** Let $\bar{f}$ be a fuzzy isomorphism between $[\mathscr{P},\lambda]$ and $[\mathscr{P}',\mu]$. The fuzzy line spanned by the fuzzy points (p_1,α_1) and (p_2,α_2) with the distinct base points $p_1,\ p_2$ is $\langle(p_1,\alpha_1),(p_2,\alpha_2)\rangle=(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2)$. Since f is an isomorphism between the base projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(p_1)\ne f(p_2)$. So $\bar{f}(p_1,\alpha_1)\ne\bar{f}(p_2,\alpha_2)$. Using the definitions of $\bar{f}$ and f,

$$\bar{f}(\langle(p_1,\alpha_1),(p_2,\alpha_2)\rangle)=(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2)=(\langle f(p_1),f(p_2)\rangle,\alpha_1\wedge\alpha_2)=\langle(f(p_1),\alpha_1),(f(p_2),\alpha_2)\rangle=\left\langle\bar{f}(p_1,\alpha_1),\bar{f}(p_2,\alpha_2)\right\rangle.$$

(ii) Let $\bar{f}$ be a fuzzy isomorphism between $[\mathscr{P},\lambda]$ and $[\mathscr{P}',\mu]$. The intersection point of the fuzzy lines (L,β_1) and (M,β_2) with the distinct base lines $L,\ M$ is $(L,\beta_1)\cap(M,\beta_2)=(L\cap M,\beta_1\wedge\beta_2)$. Since f is isomorphism between the projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(L)$ is different from $f(M)$. So $\bar{f}(L,\beta_1)$ is different from $\bar{f}(M,\beta_2)$. Using the definition of $\bar{f}$ and f

$$\bar{f}((L,\beta_1)\cap(M,\beta_2))=(f(L\cap M),\beta_1\wedge\beta_2)=(f(L)\cap f(M),\beta_1\wedge\beta_2)=((f(L),\beta_1)\cap(f(M),\beta_2))=\bar{f}(L,\beta_1)\cap\bar{f}(M,\beta_2).$$

(iii) Suppose that the fuzzy point $\bar{f}((p,\alpha))$ is on the fuzzy line $\bar{f}((L,\beta))$ when the base point p is not on the base line L. Then the fuzzy point (p,α) is not on the fuzzy line (L,β). From definitions of f and $\bar{f}$, $\bar{f}((p,\alpha))=(f(p),\alpha)$ and $\bar{f}((L,\beta))=(f(L),\beta)$. Since the fuzzy point $\bar{f}((p,\alpha))$ is on the fuzzy line $\bar{f}((L,\beta))$ and f is an isomorphism, $f(p)\circ f(L)$ and $p\circ L$ are obtained. This contradicts the hypothesis. □

From now on, we considered the fuzzy projective plane $[\mathscr{P},\lambda]$ with the base plane $\mathscr{P}$ and λ in the following form:

$$\begin{array}{lllll}\lambda: & \mathscr{P} & \to & [0,1] & \\ & q & \to & a_0 & \\ & p & \to & a_1, & p\in L\setminus\{q\}\\ & p & \to & a_2, & p\in\mathscr{P}\setminus\{L\}\end{array}$$

where L is a projective line of $\mathscr{P}$ contains q and $a_0\ge a_1\ge a_2$, $a_i\in[0,1]$, $i=0,1,2$.

The fuzzy point (q,a_0) and the fuzzy line (L,a_1) are called the base point, the base line of the fuzzy projective plane $[\mathscr{P},\lambda]$, respectively. The invariant properties under any fuzzy collineation in $[\mathscr{P},\lambda]$ depending on the base line, the base point and the membership degrees of $[\mathscr{P},\lambda]$ are investigated in detail with the following theorems.

Theorem 3.3. *Suppose that $\bar{f}$ is a fuzzy collineation of $[\mathscr{P},\lambda]$ defined by the collineation f of the base plane $\mathscr{P}$. Then,*
(i) *If $a_0\ne a_1\ne a_2$, then the fuzzy collineation $\bar{f}$ leaves invariant the base point and the base line of $[\mathscr{P},\lambda]$.*
(ii) *If $a_0\ne a_1=a_2$, then the base point is invariant and the base line turns into a fuzzy line passing through the base point under the fuzzy collineation $\bar{f}$.*

Proof. **(i)** Let $a_0\ne a_1\ne a_2$.
The image of the base point (q,a_0) is $\bar{f}(q,a_0)=(f(q),a_0)$. Since there is no other point which has membership degree a_0 in $[\mathscr{P},\lambda]$, $(f(q),a_0)$ must be the base point. So $f(q)=q$, $\bar{f}(q,a_0)=(q,a_0)$.

Since $(L,a_1)=\langle (q,a_0),(p,a_1)\rangle \ni p\circ L$, $p\neq q$ and from Theorem 3.2 i), the base line is

$$\begin{aligned}\bar{f}((L,a_1)) &= \langle \bar{f}(q,a_0),\bar{f}(p,a_1)\rangle = \langle (f(q),a_0),(f(p),a_1)\rangle, \quad (f(q)=q)\\ &= \langle (q,a_0),(f(p),a_1)\rangle = (\langle q,f(p)\rangle, a_0\wedge a_1)\\ &= (\langle q,f(p)\rangle, a_1).\end{aligned}$$

Since there is no other line with the membership degree a_1, $\bar{f}((L,a_1))=(\langle q,f(p_i)\rangle,a_1)=(L,a_1)$ is obtained. So the base point and the base line are invariant under the fuzzy collineation $\bar{f}$.
The converse of this proposition is not true. While the base point and the base line are invariant, the membership degrees can be different or equal.

(ii) Let $a_0\neq a_1=a_2$.
The image of the base point (q,a_0) is $\bar{f}(q,a_0)=(f(q),a_0)$. Since there is no other line with the membership degree a_0 in $[\mathscr{P},\lambda]$, $(f(q),a_0)$ must be base point. So $f(q)=q$ and $\bar{f}(q,a_0)=(f(q),a_0)=(q,a_0)$. Since $\bar{f}$ is a fuzzy isomorphism, $\bar{f}(q,a_0)\circ\bar{f}(L,a_1)$. Hence, the base point (q,a_0) is on $(f(L),a_1)$. $\bar{f}(L,a_1) = \langle \bar{f}(q,a_0),\bar{f}(p,a_1)\rangle = \langle (f(q),a_0),(f(p),a_1)\rangle = (\langle q,f(p)\rangle,a_0\wedge a_1) = (\langle q,f(p)\rangle,a_1)$. So the base line L,a_1 turns into a fuzzy line through the base point q,a_0 under the fuzzy collineation $\bar{f}$. □

The following theorem states the properties of $\bar{f}$ fuzzy collineation while the base point is invariant.

Theorem 3.4. *Suppose that $\bar{f}$ be a fuzzy collineation of $[\mathscr{P},\lambda]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point (q,a_0) be invariant under the fuzzy collineation $\bar{f}$.*

***(i)** If the base line (L,a_1) is invariant under $\bar{f}$, $[\mathscr{P},\lambda]$ has at most three membership degrees.*

***(ii)** If the base line (L,a_1) turns into a line other than itself passing through the base point (q,a_0), there are at most two membership degrees in $[\mathscr{P},\lambda]$ such that $a_0\geq a_1=a_2$.*

***(iii)** The base line (L,a_1) does not turn into a fuzzy line that does not pass through the base point (q,a_0) under $\bar{f}$ in $[\mathscr{P},\lambda]$.*

Proof. **(i)** Let the base point (q,a_0) and the base line (L,a_1) be invariant under the fuzzy collineation $\bar{f}$. Then $\bar{f}(q,a_0)=(q,a_0)$. The image point $\bar{f}(p,a_1)$ of the fuzzy point (p,a_1) on the base line (L,a_1) is $(f(p),a_1)$ and is on the base line (L,a_1). So, if $a_0\neq a_1\neq a_2$ is taken, there are at most three membership degrees in $[\mathscr{P},\lambda]$.

(ii) Let the base point (q,a_0) be invariant and the base line turns into a line other than the base line passing through the base point $[\mathscr{P},\lambda]$. Since the base point (q,a_0) on (L,a_1) and $\bar{f}$ is a fuzzy collineation, the image of the base point (q,a_0) is also on the image of the base line $(f(L),a_1)$. Due to the fact that $L\neq f(L)$ and the line $f(L)$ passes through points of degree of membership a_2 not on the base line, the membership degree of $f(L)$ is a_2. So, $a_1=a_2$ is obtained. Consequently, $[\mathscr{P},\lambda]$ has at most two membership degrees.

(iii) Since the base point is on the base line, its image is on the image of the base line. However, the being invariant of the base point gives rise to that the image line has to pass through the base point. □

Theorem 3.5. *Suppose that $\bar{f}$ is a fuzzy collineation of $[\mathscr{P},\lambda]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point is not invariant and turns into a fuzzy point on the base line under the fuzzy collineation $\bar{f}$.*

***(i)** If the base line is invariant under the fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda]$, among the membership degree a_i, $i=0,1,2$, there is a relationship $a_0=a_1\geq a_2$.*

***(ii)** If the base point turns into a fuzzy point on the non-invariant base line other than itself under the fuzzy collineation $\bar{f}$, then there is one membership degree in $[\mathscr{P},\lambda]$.*

***(iii)** If the base point of (q,a_0) turns into any point not on the base line under the collineation f in $\mathscr{P}$, then there is only one membership degree in $[\mathscr{P},\lambda]$.*

Proof. **(i)** Let the base point q of the fuzzy point (q,a_0) be not invariant and turn into another point on the base line L, $\bar{f}(q,a_0)=(f(q),a_1)$. Suppose that the base line L is invariant under the collineation f. It is clear that the fuzzy point (q,a_0) turns into the fuzzy point (p,a_1) with $p\circ L$, $q\neq p$. Since $\bar{f}$ is a fuzzy collineation, $a_0=a_1$. Hence, there are at most two membership degree in $[\mathscr{P},\lambda]$.

(ii) Since the fuzzy point (q,a_0) turns into the fuzzy point (p,a_1), $p\neq q$ on (L,a_1). It is clear that $a_0=a_1$. Next suppose that $f(L)\neq L$. So any fuzzy point different from the base point on the base line L with membership degree a_1 turns into any other fuzzy point with membership degree a_2. Since $\bar{f}$ is a fuzzy isomorphism, then $a_1=a_2$. Hence, $a_0=a_1=a_2$.

(iii) Since the base point turns into a point not on the base line, the image of (q,a_0) under $\bar{f}$ is the fuzzy point (p,a_2) with $p\,\not\circ L$. It is clearly $a_0=a_2$. If we use this equality and the condition $a_0\geq a_1\geq a_2$ among the membership degrees in $[\mathscr{P},\lambda]$, $a_0=a_1=a_2$ is obtained. □

Proposition 3.6. *If the base point q of (q,a_0) turns into any point not on the base line L of (L,a_1) under the collineation f, the base line (L,a_1) of $[\mathscr{P},\lambda]$ is not invariant under the fuzzy collineation $\bar{f}$.*

Proof. Let the base point q turn into any point not on the base line L under the collineation f of $\mathscr{P}$. Since the fuzzy base point (q,a_0) turns into $(f(q),a_2)$ such that $f(q)$ is not on L, then the base line $L=\langle q,p\rangle$ spanning by the points p and q turns into $f(L)=\langle f(q),f(p)\rangle\neq L$ under the collineation f. Hence, (L,a_1) is not invariant. □

Theorem 3.7. *Suppose that $\bar{f}$ is a fuzzy collineation of $[\mathscr{P},\lambda]$ defined by the collineation f of the base plane $\mathscr{P}$.*

***(i)** If two distinct points p_1 and p_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$, the fuzzy line spanned by fuzzy points (p_1,α_1) and (p_2,α_2) is invariant under the fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda]$.*

***(ii)** If two distinct lines L_1 and L_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of L_1 and L_2 is not on the base line L in $\mathscr{P}$, then the intersection point of the fuzzy lines (L_1,α_1) and (L_2,α_2) is invariant under the fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda]$.*

***(iii)** Suppose that two distinct lines L_1 and L_2 different from the base line L in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of these lines is on the base line L in $\mathscr{P}$. If the intersection point of the fuzzy lines (L_1,α_1) and (L_2,α_2) is* *invariant under the fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda]$. Then, there is a relationship $a_0 = a_1 = a_2$ or $a_1 = a_2$ among the membership degrees in $[\mathscr{P},\lambda]$.*

Proof. **(i)** Let the base points p_1 and p_2 in $\mathscr{P}$ of (p_1,α_1) and (p_2,α_2) in $[\mathscr{P},\lambda]$ be invariant under the collineation f of $\mathscr{P}$. Then by the definition of fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda]$, $\bar{f}(p_1,\alpha_1) = (f(p_1),\alpha_1) = (p_1,\alpha_1)$ and $\bar{f}(p_2,\alpha_2) = (f(p_2),\alpha_2) = (p_2,\alpha_2)$.
For any pair $((p_1,\alpha_1),(p_2,\alpha_2))$ of fuzzy points, $p_1 \neq p_2$, the fuzzy line $(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2)$ spanned by them, also belongs to the fuzzy projective plane $[\mathscr{P},\lambda]$. By using the definition of $\bar{f}$ of $[\mathscr{P},\lambda]$ and the remaining invariant of the points p_1 and p_2 under the collineation f in $\mathscr{P}$, the image of the fuzzy line $(\langle(p_1,\alpha_1),(p_2,\alpha_2)\rangle)$ under the fuzzy collineation $\bar{f}$ is $(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2) = (\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2)$. Hence, the fuzzy line $(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2)$ is invariant under the fuzzy collineation $\bar{f}$.

(ii) Let the base lines L_1 and L_2 in $\mathscr{P}$ of (L_1,α_1) and (L_2,α_2) in $[\mathscr{P},\lambda]$ be invariant under the collineation f of $\mathscr{P}$. Since $L_1 \neq L_2 \neq L$, the membership degrees $\alpha_i = a_2$, $i = 1,2$. By the definition of $\bar{f}$ and being invariant of the lines L_1 and L_2 under the collineation f in $\mathscr{P}$, $\bar{f}(L_1,a_2) = (L_1,a_2)$ and $\bar{f}(L_2,a_2) = (L_2,a_2)$. The image of the intersection fuzzy point $(L_1\cap L_2,a_2)$ under $\bar{f}$ is invariant because of $(f(L_1\cap L_2),a_2) = (f(L_1)\cap f(L_2),a_2)$. It is implies that the fuzzy point $(L_1\cap L_2,a_2)$ remains invariant under the fuzzy collineation $\bar{f}$.

(iii) Let the different base lines L_1 and L_2 of (L_1,α_1) and (L_2,α_2) be invariant under the collineation f in $\mathscr{P}$. Since $L_1 \neq L_2 \neq L$, $\alpha_i = a_2$, $i = 1,2$. If the intersection point of (L_1,a_2) and (L_2,a_2) is (p,a_1) on the base line (L,a_1) of $[\mathscr{P},\lambda]$. If the intersection point is (q,a_0), then $a_0 = a_1 = a_2$. If the intersection point is (p,a_1), then $a_1 = a_2$ is obtained. □

Theorem 3.8. *Suppose that $\bar{f}$ be a fuzzy collineation of $[\mathscr{P},\lambda]$ defined by the collineation f of the base plane $\mathscr{P}$. In this case,*

(i) *If M is a pointwise invariant line under the collineation f in the base projective plane $\mathscr{P}$, then the corresponding fuzzy line (M,β) is also pointwise invariant under the fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda]$.*

(ii) *If two distinct lines L_1 and L_2 are pointwise invariant under the collineation f of the base plane $\mathscr{P}$, then the intersection point of the fuzzy lines (L_1,α_1) and (L_2,α_2) is invariant under the fuzzy collineation $\bar{f}$.*

(iii) *If the base line L and L_1, $L_1 \neq L$ are pointwise invariant lines under the collineation f of the base plane $\mathscr{P}$, then the fuzzy collineation $\bar{f}$ defined f is the identity collineation in $[\mathscr{P},\lambda]$.*

Proof. **(i)** Let the base line M of the fuzzy line (M,β) in $[\mathscr{P},\lambda]$ be pointwise invariant under the collineation f of $\mathscr{P}$. From the definition of $\bar{f}$ and being pointwise invariant of M under collineation $f,\bar{f}(p,\alpha) = (f(p),\alpha) = (p,\alpha)$ for every fuzzy point (p,α) on (M,β). Hence the fuzzy line (M,β) is pointwise invariant in $[\mathscr{P},\lambda]$.

(ii) Let the base lines L_1 and L_2 of the fuzzy lines (L_1,α_1) and (L_2,α_2) in $[\mathscr{P},\lambda]$ be pointwise invariant under the collineation f of the base plane $\mathscr{P}$, respectively. From (i), the fuzzy lines (L_1,α_1) and (L_2,α_2) are pointwise invariant under the fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda]$. Since (L_i,α_i) are pointwise invariant, hence the intersection point of (L_1,α_1) and (L_2,α_2) is invariant.

(iii) Let the base line L and L_1, $L_1 \neq L$ be pointwise invariant under the collineation f of $\mathscr{P}$. So $\alpha_1 = a_2$ is obtained. It is well-known that if there are two distinct pointwise lines under a collineation of projective plane $\mathscr{P}$, then the collineation f is the identity collineation. From *i)* (L,a_1) and (L_1,a_2), are pointwise invariant. The image of (p,a_2) such that $p \not\in L$ and $p \not\in L_1$ is $\bar{f}(p,a_2) = (f(p),a_2) = (p,a_2)$ under $\bar{f}$. Hence every fuzzy point in $[\mathscr{P},\lambda]$ should be invariant, and this means that $\bar{f}$ is the identity collineation of $[\mathscr{P},\lambda]$. □

Corollary If f is the identity collineation of $\mathscr{P}$, $\bar{f}$ is the identity collineation of $[\mathscr{P},\lambda]$.

4. Conclusion

In this study, the concepts of fuzzy isomorphism and fuzzy collineations are introduced and then some important results are obtained. It is seen that the base point, the base line and the membership degrees of fuzzy projective plane have a important role on holding the fuzzy versions of some classical properties related to collineations of projective plane by collineations in fuzzy projective planes. Consequently, the obtained results related to the fuzzy isomorphisms and fuzzy collineations characterized according to the base point, the base line and the membership degree in fuzzy projective planes have an important effect on enriching the theory of fuzzy geometries. And also, the concepts of fuzzy isomorphism and fuzzy collineations are ready-made for generalization to other fuzzy geometries.

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