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A NEW SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL INCLUSION PROBLEMS IN SEMI-INNER PRODUCT SPACES

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Abstract. In this work we reflect a new system of generalized nonlinear variational inclusion problems in 2-uniformly smooth Banach spaces. By using resolvent operator technique, we offer an iterative algorithm for figuring out the approximate solution of the said system. The motive of this paper is to review the convergence analysis of a system of generalized nonlinear variational inclusion problems in 2-uniformly smooth Banach spaces. The proposition used in this paper can be considered as an extension of propositions for examining the existence of solution for various classes of variational inclusions considered and studied by many authors in 2-uniformly smooth Banach spaces.

1. Introduction

In recent past, variational inequalities have been elongated in dissimilar directions and sections of studies, using peculiar and ingenious techniques. One of such conception is variational inclusions. Numerous problems that exist in engineering, optimization and control situations can be designed by free boundary problems which conveys to variational inequality and variational inclusion problems. For details, please refer [1-5, 8-14, 18, 20-23, 25, 26].

2. Resolvent Operator and Formulation of Problem

Let X be a real 2-uniformly smooth Banach space equipped with norm $\|.\|$ and a semi-inner product $[.,.]$. Let $C(X)$ be the family of all nonempty compact subsets of X and 2^X be the power set of X.

We need the following definitions and results from the literature.

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Definition 1. *Let X be a vector space over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following:*

(i) $[x+y,z] = [x,z] + [y,z], \ \forall x,y,z \in X$;
(ii) $[\lambda x, y] = \lambda[x,y], \ \forall \lambda \in F$ *and* $x,y \in X$;
(iii) $[x,x] > 0$, *for* $x \neq 0$;
(iv) $|[x,y]|^2 \leq [x,x][y,y]$.

The pair $(X,[.,.])$ is called a semi-inner product space.

We observe that $||x|| = [x,x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [7] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x,y] = 0$ if and only if y is orthogonal to x, that is if and only if $||y|| \leq ||y+\lambda x||, \ \forall$ scalars λ.
(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x,y], \ \forall x \in X$.
(iii) The semi-inner product is continuous, that is for each $x,y \in X$, we have $\text{Re}[y, x+\lambda y] \to \text{Re}[y,x]$ as $\lambda \to 0$.

The sequence space $l^p, \ p > 1$ and the function space $L^p, \ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 1. [19] *The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[x,y] = \frac{1}{||y||_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \ x,y \in l^p.$$

Example 2. [7, 19] *The real Banach space $L^p(X,\mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[f,g] = \frac{1}{||g||_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x)) d\mu, \ f,g \in L^p.$$

Definition 2. [19, 24] *Let X be a real Banach space. Then:*

(i) *The modulus of smoothness of X is defined as*

$$\rho_X(t) = \sup \left\{ \frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = t, \ t > 0 \right\}.$$

(ii) *X is said to be uniformly smooth if* $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$.

(iii) *X is said to be p-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\ t^p,\ p > 1$. Clearly, X is 2-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\ t^2$.*

Lemma 1. [19, 24] *Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is 2-uniformly smooth.*

(ii) *There is a constant $c > 0$ such that for every $x, y \in X$, the following inequality holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, f_x\rangle + c||y||^2,$$

where $f_x \in J(x)$ and $J(x) = \{x^\star \in X^\star : \langle x, x^\star\rangle = ||x||^2$ and $||x^\star|| = ||x||\}$ is the normalized duality mapping.

Remark 1. [19] *Every normed linear space is a semi-inner product space (see[15]). In fact by Hahn Banach theorem, for each $x \in X$, there exists atleast one functional $f_x \in X^\star$ such that $\langle x, f_x\rangle = ||x||^2$. Given any such mapping f from X into $X^\star$, we can verify that $[y, x] = \langle y, f_x\rangle$ defines a semi-inner product. Hence we can write (ii) of above Lemma as*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + c||y||^2,\ \forall x, y \in X.$$

The constant c is chosen with best possible minimum value. We call c, as the constant of smoothness of X.

Example 3. *The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + (p - 1)||y||^2.$$

Here the constant of smoothness is $p - 1$.

Definition 3. [16, 19] *Let X be a real 2-uniformly smooth Banach space. A mapping $S : X \to X$ is said to be:*

(i) *monotone, if $[Sx - Sy, x - y] \geq 0,\ \forall x, y \in X$,*

(ii) *strictly monotone, if $[Sx - Sy, x - y] > 0,\ \forall x, y \in X$, and equality holds if and only if $x = y$,*

(iii) *r-strongly monotone if there exists a positive constant $r > 0$ such that*

$$[Sx - Sy, x - y] \geq r||x - y||^2,\ \forall x, y \in X,$$

(iv) *δ-Lipschitz continuous, if there exists a constant $\delta > 0$ such that*

$$\|S(x) - S(y)\| \leq \delta\|x - y\|,\ \forall x, y \in X,$$

(v) *η-monotone, if $[Sx - Sy, \eta(x, y)] \geq 0,\ \forall x, y \in X$,*

(vi) *strictly η-monotone, if $[Sx - Sy, \eta(x, y)] > 0,\ \forall x, y \in X$, and equality holds if and only if $x = y$,*

(vii) *r-strongly η-monotone if there exists a positive constant $r > 0$ such that*

$$[Sx - Sy, \eta(x, y)] \geq r||x - y||^2,\ \forall x, y \in X,$$

(viii) *ξ-cocoercive if there exists a constant $\xi > 0$ such that*

$$[Sx - Sy, x - y] \geq \xi ||Sx - Sy||^2, \ \forall x, y \in X,$$

(ix) *relaxed (ξ, δ)-cocoercive if there exist two constants $\xi, \delta > 0$ such that*

$$[Sx - Sy, x - y] \geq -\xi ||Sx - Sy||^2 + \delta \|x - y\|^2, \ \forall x, y \in X.$$

For $\xi = 0$ S is δ-strongly monotone.

This class of mappings is more general than the class of strongly monotone mappings.

Definition 4. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^X$ be multi-valued mapping. Then*

(i) *η is said to be accretive, if*

$$\Big[\eta(x, y), x - y\Big] \geq 0, \ \forall x, y \in X.$$

(ii) *η is said to be strictly accretive, if*

$$\Big[\eta(x, y), x - y\Big] > 0, \ \forall x, y \in X.$$

and equality holds only when $x = y$.

(iii) *η is said to be r-strongly-accretive if there exists a constant $r > 0$ such that*

$$\Big[\eta(x, y), x - y\Big] \geq r\|x - y\|^2, \ \forall x, y \in X.$$

(iv) *η is said to be m-Lipschitz continuous, if there exists a constant $m > 0$ such that*

$$\|\eta(x, y)\| \leq m\|x - y\|, \ \forall x, y \in X,$$

(v) *M is said to be η-accretive in the first argument if*

$$\Big[u - v, \eta(x, y)\Big] \geq 0, \ \forall x, y \in X, \ \forall u \in M(x, t), v \in M(y, t), \text{ for each fixed } t \in X,$$

(vi) *μ-strongly η-accretive if there exists a positive constant $\mu > 0$ such that*

$$[u - v, \eta(x, y)] \geq \mu\|x - y\|^2, \ \forall x, y \in X, u \in M(x, t), v \in M(y, t).$$

Definition 5. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be single-valued mappings, $M : X \times X \to 2^X$ be a multi-valued mapping, then M is said to be $m - \eta-$accretive mapping if for each fixed $t \in X, M(., t)$ is η-accretive in the first argument and $(I + \rho M(., t))X = X$, $\forall \rho > 0$.*

Theorem 1. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m - \eta-$accretive mapping. If the following inequality : $\Big[u - v, \eta(x, y)\Big] \geq 0$, holds $\forall (y, v) \in$ Graph $(M(., t))$, then $(x, u) \in$ Graph$(M(., t))$, where Graph $(M(., t)) := \{(x, u) \in X \times X : u \in M(x, t)\}$.*

Theorem 2. *Let $\eta : X \times X \to X$ be q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then the mapping $(I+\rho M(.,t))^{-1}$ is single-valued, $\forall\ \rho > 0$.*

Definition 6. *Let $\eta : X \times X \to X$ be single-valued mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then for each fixed $t \in X$, the resolvent operator $R_{\rho,\eta}^{M(.,t)} : X \to X$ is defined by*

$$R_{\rho,\eta}^{M(.,t)}(x) = (I+\rho M(.,t))^{-1}(x),\ \forall x \in X.$$

Theorem 3. *Let $\eta : X \times X \to X$ be p-Lipschitz continuous and q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then for each fixed $t \in X$ the resolvent operator of M, $R_{\rho,\eta}^{M(.,t)}(x) = (I+\rho M(.,t))^{-1}(x)$ is $\frac{p}{q}-$Lipschitz continuous, that is,*

$$\left\| R_{\rho,\eta}^{M(.,t)}(x) - R_{\rho,\eta}^{M(.,t)}(y) \right\| \le L\|x-y\|,\ \forall x,y,t \in X.$$

where $L = \frac{p}{q}$.

Definition 7. *The Hausdorff metric $D(\cdot,\cdot)$ on $CB(X)$, is defined by*

$$D(A,B) = \max\left\{ \sup_{u\in A}\inf_{v\in B} d(u,v),\ \sup_{v\in B}\inf_{u\in A} d(u,v) \right\},\ A,B \in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 8. [6] *A set-valued mapping $T : X \to CB(X)$ is said to be γ-D-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that*

$$D(T(x),T(y)) \le \gamma\|x-y\|,\ \forall x,y \in X.$$

Theorem 4. [17] *Let $T : X \to CB(X)$ be a set-valued mapping on X and (X,d) be a complete metric space. Then:*

(i) *For any given $\nu > 0$ and for any given $u,v \in X$ and $x \in T(u)$, there exists $y \in T(v)$ such that*

$$d(x,y) \ \le\ (1+\nu)D(T(u),T(v));$$

(ii) *If $T : X \to C(X)$, then (i) holds for $\nu = 0$, (where $C(X)$ denotes the family of all nonempty compact subsets of X).*

Lemma 2. *Let $\{b^n\}$ be a sequence of nonnegative real numbers such that*

$$b^{n+1} \le (1-a^n)b^n + c^n + h^n,\ \forall n \ge n_0,$$

where n_0 is a nonnegative integer, $\{a^n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} a^n = \infty$, $c^n = o(a^n)$ and $\sum_{n=0}^{\infty} h^n < \infty$. Then $\lim_{n\to\infty} b^n = 0$.

Definition 9. *A mapping $S : X\times X\times X \to X$ is said to be relaxed $(\xi,\delta)-$cocoercive if there exist constants $\xi,\delta > 0$ such that*

$$\Big[S(x,y,z) - S(x_1,y_1,z_1), x - x_1\Big] \geq -\xi\Big\|S(x,y,z) - S(x_1,y_1,z_1)\Big\|^2 + \delta\Big\|x - x_1\Big\|^2,$$

$$\forall x, x_1, y, y_1, z, z_1 \in X. \tag{1}$$

Definition 10. *A mapping $S : X\times X\times X \to X$ is said to be β-Lipschitz continuous in the first variable if there exist constant $\beta > 0$ such that*

$$\Big\|S(x,y,z) - S(x_1,y_1,z_1)\Big\| \leq \beta\Big\|x - x_1\Big\|,\ \forall x, x_1, y, y_1, z, z_1 \in X. \tag{2}$$

Now, we formulate our main problem.
For each $i = 1, 2, 3$, let $N_i : X\times X\times X \to X,\ f_i : X \to X, \eta_i : X\times X \to X$ be single-valued mappings. Let $A_i, B_i, F_i : X \to C(X)$ be set-valued mappings. Suppose that $M_i : X \times X \to 2^X$ is $m_i - \eta_i-$ accretive mapping. Then we consider the following system of generalized nonlinear variational inclusion problems (in short, SGNVIP): Find $(x_1, x_2, x_3) \in X \times X \times X, u_i \in A_i(x_i), v_i \in B_i(x_i), w_i \in F_i(x_i)$ such that

$$\left.\begin{aligned}
&0 \in f_1(x_1) - f_1(x_2) + \rho_1\{N_1(u_2, u_3, u_1) + M_1(f_1(x_1), x_1)\}\\
&0 \in f_2(x_2) - f_2(x_3) + \rho_2\{N_2(v_3, v_1, v_2) + M_2(f_2(x_2), x_2)\}\\
&0 \in f_3(x_3) - f_3(x_1) + \rho_3\{N_3(w_1, w_2, w_3) + M_3(f_3(x_3), x_3)\},\ \forall \rho_i > 0.
\end{aligned}\right\} \tag{3}$$

Special Cases:

I. If in problem (3), $f_1(x_1) = G(x), f_1(x_2) = H(x)$, such that $G, H : X \to X, f_2 = f_3 \equiv 0, N_1 = N_2 = N_3 \equiv 0, \rho_1 = \rho_2 = \rho_3 = 1$, then problem (3) reduces to the following problem: Find $x \in X$ such that

$$0 \in G(x) - H(x) + M(G(x), x). \tag{4}$$

This type of problem has been considered and studied by Sahu *et al.*[19].

3. Iterative Algorithm

First, we give the following technical lemma:

Lemma 3. *Let X be a real 2-uniformly smooth Banach space. Let for each $i \in \{1, 2, 3\}$ N_i, f_i, η_i be single-valued mappings. Let $A_i, B_i, F_i : X \to C(X)$ be set-valued mappings, $M_i : X \times X \to 2^X$ be $m_i - \eta_i-$accretive mappings. Then (x_i, u_i, v_i, w_i) where $x_i \in X, u_i \in A_i(x_i), v_i \in B_i(x_i), w_i \in F_i(x_i)$ is a solution of*

(3) if and only if (x_i, u_i, v_i, w_i) satisfies

$$\left.\begin{aligned} f_1(x_1) &= R^{M_1(.,x_1)}_{\rho_1,\eta_1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ f_2(x_2) &= R^{M_2(.,x_2)}_{\rho_2,\eta_2}\Big\{f_2(x_3) - \rho_2 N_2(v_3,v_1,v_2)\Big\} \\ f_3(x_3) &= R^{M_3(.,x_3)}_{\rho_3,\eta_3}\Big\{f_3(x_1) - \rho_3 N_3(w_1,w_2,w_3)\Big\} \end{aligned}\right\} \tag{5}$$

where $R^{M_i(.,x_i)}_{\rho_i,\eta_i} = \Big(I + \rho_i M_i(.,x_i)\Big)^{-1}$ *are the resolvent operators.*

Proof. Let (x_i, u_i, v_i, w_i) is a solution of (3), then we have

$$\begin{aligned} & f_1(x_1) = R^{M_1(.,x_1)}_{\rho_1,\eta_1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ \iff \quad & f_1(x_1) = \Big(I + \rho_1 M_1(.,x_1)\Big)^{-1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ \iff \quad & f_1(x_1) + \rho_1 M_1(f_1(x_1),x_1) = \Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ \iff \quad & 0 \in f_1(x_1) - f_1(x_2) + \rho_1\{N_1(u_2,u_3,u_1) + M_1(f_1(x_1),x_1)\}. \end{aligned}$$

Proceeding likewise by using (5), we have

$$\begin{aligned} & f_2(x_2) = R^{M_2(.,x_2)}_{\rho_2,\eta_2}\Big\{f_2(x_3) - \rho_2 N_2(v_3,v_1,v_2)\Big\} \\ & \iff 0 \in f_2(x_2) - f_2(x_3) + \rho_2\{N_2(v_3,v_1,v_2) + M_2(f_2(x_2),x_2)\} \end{aligned}$$

and

$$\begin{aligned} & f_3(x_3) = R^{M_3(.,x_3)}_{\rho_3,\eta_3}\Big\{f_3(x_1) - \rho_3 N_3(w_1,w_2,w_3)\Big\} \\ & \iff 0 \in f_3(x_3) - f_3(x_1) + \rho_3\{N_3(w_1,w_2,w_3) + M_3(f_3(x_3),x_3)\}. \end{aligned}$$

□

Lemma 3 allows us to suggest the following iterative algorithm for finding the approximate solution of (3).

Iterative Algorithm 1. *For each $i = \{1,2,3\}$ given $\{x_i^0, u_i^0, v_i^0, w_i^0\}$ where $x_i^0 \in X_i, u_i^0 \in A_i(x_i^0), v_i^0 \in B_i(x_i^0), w_i^0 \in F_i(x_i^0)$ compute the sequences $\{x_i^n, u_i^n, v_i^n, w_i^n\}$ defined by the iterative schemes*

$$f_3(x_3^n) = R^{M_3(.,x_3^n)}_{\rho_3,\eta_3}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n,w_2^n,w_3^n)\Big\}$$

$$f_2(x_2^n) = R^{M_2(.,x_2^n)}_{\rho_2,\eta_2}\Big\{f_2(x_3^n) - \rho_2 N_2(v_3^n,v_1^n,v_2^n)\Big\}$$

$$x_1^{n+1} = (1-\alpha^n)x_1^n + \alpha^n\Big(x_1^n - f_1(x_1^n) + R^{M_1(.,x_1^n)}_{\rho_1,\eta_1}\Big\{f_1(x_2^n) - \rho_1 N_1(u_2^n,u_3^n,u_1^n)\Big\}\Big)$$

where α^n is a sequence of real numbers such that $\sum\limits_{n=0}^{\infty} \alpha^n = \infty, \ \forall n \geq 0.$

4. Existence of Solution and Convergence Analysis

Theorem 5. *For each $i \in \{1,2,3\}$, let X be a real 2-uniformly smooth Banach space with k as constant of smoothness. Let $N_i : X \times X \times X \to X$ be a relaxed (ξ_i, δ_i)-cocoercive and ν_i-Lipschitz continuous in the first argument. Let f_i be a relaxed (r_i, s_i)-cocoercive and β_i-Lipschitz continuous in the first argument. Let $A_i, B_i, F_i : X_i \to C(X_i)$ be set-valued mappings such that A_i is $L_{A_i} - D-$Lipschitz continuous, B_i is $L_{B_i} - D-$Lipschitz continuous and F_i is $L_{F_i} - D-$Lipschitz continuous. In addition, if there are constants $t_i > 0$ such that*

$$\left\| R^{M_i(.,x_i^n)}_{\rho_i,\eta_i}(z_i) - R^{M_i(.,x_i)}_{\rho_i,\eta_i}(z_i) \right\|_i \leq t_i \|x_i^n - x_i\|_i, \ \forall z_i \in X_i \tag{6}$$

and

$$1 - (t_2 + \Phi_5) > 0, \ 1 - (t_3 + \Phi_6) > 0$$

such that

$$0 < \left(\Phi_4 + \Phi_4 \frac{L_1 L_2 L_3 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} \right.$$

$$\left. + \frac{L_1 L_2 L_3 \Phi_1 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} + t_1 \right) < 1, \tag{7}$$

where

$$\Phi_1 = \sqrt{1 + 2\rho_1(\xi_1 \nu_1^2 L_{A_2}^2 - \delta_1) + k\rho_1^2 \nu_1^2 L_{A_2}^2}; \ \Phi_2 = \sqrt{1 + 2\rho_2(\xi_2 \nu_2^2 L_{B_3}^2 - \delta_2) + k\rho_2^2 \nu_2^2 L_{B_3}^2}.$$

$$\Phi_3 = \sqrt{1 + 2\rho_3(\xi_3 \nu_3^2 L_{F_1}^2 - \delta_3) + k\rho_3^2 \nu_3^2 L_{F_1}^2}; \ \Phi_4 = \sqrt{1 + 2(r_1 \beta_1^2 - s_1) + k\beta_1^2}.$$

$$\Phi_5 = \sqrt{1 + 2(r_2 \beta_2^2 - s_2) + k\beta_2^2}; \ \Phi_6 = \sqrt{1 + 2(r_3 \beta_3^2 - s_3) + k\beta_3^2}.$$

Then the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{w_i^n\}$ generated by above iterative algorithm 1 converges strongly to (x_i, u_i, v_i, w_i) where (x_i, u_i, v_i, w_i) is a solution of above problem (3).

Proof. From Lemma 3, Iterative Algorithm 1, (6) and by using Theorem 3, it follows that

$$\left\| x_1^{n+1} - x_1 \right\|$$

$$
\begin{aligned}
= \quad & \Big\| (1-\alpha^n)x_1^n + \alpha^n \Big(x_1^n - f_1(x_1^n) + R_{\rho_1,\eta_1}^{M_1(.,x_1^n)} \Big\{ f_1(x_2^n) - \rho_1 N_1(u_2^n, u_3^n, u_1^n) \Big\} \Big) \\
& - \Big[(1-\alpha^n)x_1 + \alpha^n \Big(x_1 - f_1(x_1) + R_{\rho_1,\eta_1}^{M_1(.,x_1)} \Big\{ f_1(x_2) - \rho_1 N_1(u_2, u_3, u_1) \Big\} \Big) \Big] \Big\| \\
\leq \quad & (1-\alpha^n) \Big\| x_1^n - x_1 \Big\| + \alpha^n \Big\| (x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1)) \Big\| \\
& + \alpha^n \Big\| R_{\rho_1,\eta_1}^{M_1(.,x_1^n)} \Big\{ f_1(x_2^n) - \rho_1 N_1(u_2^n, u_3^n, u_1^n) \Big\} \\
& - R_{\rho_1,\eta_1}^{M_1(.,x_1^n)} \Big\{ f_1(x_2) - \rho_1 N_1(u_2, u_3, u_1) \Big\} \\
& + R_{\rho_1,\eta_1}^{M_1(.,x_1^n)} \Big\{ f_1(x_2) - \rho_1 N_1(u_2, u_3, u_1) \Big\} \\
& - R_{\rho_1,\eta_1}^{M_1(.,x_1)} \Big\{ f_1(x_2) - \rho_1 N_1(u_2, u_3, u_1) \Big\} \Big\| \\
\leq \quad & (1-\alpha^n) \Big\| x_1^n - x_1 \Big\| + \alpha^n \Big\| (x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1)) \Big\| \\
& + \alpha^n L_1 \Big\| f_1(x_2^n) - f_1(x_2) - \rho_1 \Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1) \Big) \Big\| \\
& + \alpha^n t_1 \| x_1^n - x_1 \| \\
\leq \quad & (1-\alpha^n) \Big\| x_1^n - x_1 \Big\| + \alpha^n \Big\| (x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1)) \Big\| \\
& + \alpha^n L_1 \Big\| (x_2^n - x_2) - (f_1(x_2^n) - f_1(x_2) \Big\| \\
& + \alpha^n L_1 \Big\| (x_2^n - x_2) - \rho_1 \Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1) \Big) \Big\| \\
& + \alpha^n t_1 \| x_1^n - x_1 \|. \qquad (8)
\end{aligned}
$$

Since N_1 is relaxed (ξ_1, δ_1)-cocoercive and ν_1-Lipschitz continuous in the first argument, therefore by using Remark 1 , it follows that

$$
\Big\| (x_2^n - x_2) - \rho_1 \Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1) \Big) \Big\|^2
$$

$$
\begin{aligned}
= &\ \Big\|x_2^n - x_2\Big\|^2 - 2\rho_1\Big[N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1), x_2^n - x_2\Big] \\
&+k\rho_1^2\Big\|N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big\|^2 \\
\leq &\ \Big\|x_2^n - x_2\Big\|^2 - 2\rho_1\Big\{-\xi_1\Big\|N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big\|^2 + \delta_1\Big\|x_2^n - x_2\Big\|^2\Big\} \\
&+k\rho_1^2\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
\leq &\ \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
&-2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
\leq &\ \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2\Big(D(A_2(x_2^n), A_2(x_2))\Big)^2 \\
&-2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2\Big(D(A_2(x_2^n), A_2(x_2))\Big)^2 \\
\leq &\ \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2L_{A_2}^2\Big\|x_2^n - x_2\Big\|^2 \\
&-2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2L_{A_2}^2\Big\|x_2^n - x_2\Big\|^2 \\
\leq &\ \Big(1 + 2\rho_1(\xi_1\nu_1^2L_{A_2}^2 - \delta_1) + k\rho_1^2\nu_1^2L_{A_2}^2\Big)\Big\|x_2^n - x_2\Big\|^2
\end{aligned}
$$

$$
\Longrightarrow \Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big)\Big\| \leq \Phi_1\Big\|x_2^n - x_2\Big\| \tag{9}
$$

where

$$
\Phi_1 = \sqrt{1 + 2\rho_1(\xi_1\nu_1^2L_{A_2}^2 - \delta_1) + k\rho_1^2\nu_1^2L_{A_2}^2}.
$$

Also

$$
\begin{aligned}
\Big\|x_2^n - x_2\Big\| &= \Big\|(x_2^n - x_2) - (f_2(x_2^n) - f_2(x_2)) + (f_2(x_2^n) - f_2(x_2))\Big\| \\
&\leq \Big\|(x_2^n - x_2) - (f_2(x_2^n) - f_2(x_2))\Big\| \\
&\quad +\Big\|f_2(x_2^n) - f_2(x_2)\Big\|. \qquad (10)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\Big\|f_2(x_2^n) - f_2(x_2)\Big\| \\
= \quad & \Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n) - \rho_2 N_2(v_3^n, v_1^n, v_2^n)\Big\} - R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3) - \rho_2 N_2(v_3, v_1, v_2)\Big\}\Big\| \\
= \quad & \Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n) - \rho_2 N_2(v_3^n, v_1^n, v_2^n)\Big\} \\
& -R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3) - \rho_2 N_2(v_3, v_1, v_2)\Big\}\Big\| \\
\leq \quad & \Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n) - \rho_2 N_2(v_3^n, v_1^n, v_2^n)\Big\} \\
& -R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3) - \rho_2 N_2(v_3, v_1, v_2)\Big\}\Big\| \\
& +\Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3) - \rho_2 N_2(v_3, v_1, v_2)\Big\} \\
& -R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3) - \rho_2 N_2(v_3, v_1, v_2)\Big\}\Big\| \\
\leq \quad & L_2\Big\|f_2(x_3^n) - f_2(x_3) - \rho_2\Big(N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big)\Big\| \\
& +t_2\Big\|x_2^n - x_2\Big\| \\
\leq \quad & L_2\Big\|(x_3^n - x_3) - \Big(f_2(x_3^n) - f_2(x_3)\Big)\Big\| \\
& +L_2\Big\|(x_3^n - x_3) - \rho_2\Big(N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big)\Big\| \\
& +t_2\Big\|x_2^n - x_2\Big\|.
\end{aligned}
\tag{11}
$$

Since N_2 is relaxed (ξ_2, δ_2)-cocoercive and ν_2-Lipschitz continuous in the first argument, therefore by using Remark 1, we have

$$\Big\|(x_3^n - x_3) - \rho_2\Big(N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big)\Big\|^2$$

$$\begin{aligned}
&= \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big[N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2), x_3^n - x_3\Big] \\
&\quad + k\rho_2^2\Big\|N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big\{-\xi_2\Big\|N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big\|^2 + \delta_2\Big\|x_3^n - x_3\Big\|^2\Big\} \\
&\quad + k\rho_2^2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big\{-\xi_2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 + \delta_2\Big\|x_3^n - x_3\Big\|^2\Big\} \\
&\quad + k\rho_2^2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 + 2\rho_2\xi_2\nu_2^2\Big(D(B_3(x_3^n), B_3(x_3))\Big)^2 \\
&\quad - 2\rho_2\delta_2\Big\|x_3^n - x_3\Big\|^2 + k\rho_2^2\nu_2^2\Big(D(B_3(x_3^n), B_3(x_3))\Big)^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 + 2\rho_2\xi_2\nu_2^2 L_{B_3}^2\Big\|x_3^n - x_3\Big\|^2 \\
&\quad - 2\rho_2\delta_2\Big\|x_3^n - x_3\Big\|^2 + k\rho_2^2\nu_2^2 L_{B_3}^2\Big\|x_3^n - x_3\Big\|^2 \\
&\leq \Big(1 + 2\rho_2(\xi_2\nu_2^2 L_{B_3}^2 - \delta_2) + k\rho_2^2\nu_2^2 L_{B_3}^2\Big)\Big\|x_3^n - x_3\Big\|^2
\end{aligned}$$

$$\Longrightarrow \Big\|(x_3^n - x_3) - \rho_2\Big(N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big)\Big\| \leq \Phi_2\Big\|x_3^n - x_3\Big\| \tag{12}$$

where

$$\Phi_2 = \sqrt{1 + 2\rho_2(\xi_2\nu_2^2 L_{B_3}^2 - \delta_2) + k\rho_2^2\nu_2^2 L_{B_3}^2}.$$

Since f_2 is relaxed (r_2, s_2)-cocoercive and β_2-Lipschitz continuous, therefore by using Remark 1, we have

$$\Big\|x_3^n - x_3 - (f_2(x_3^n) - f_2(x_3))\Big\|^2$$

$$\begin{aligned}
&= \left\|x_3^n - x_3\right\|^2 - 2\Big[f_2(x_3^n) - f_2(x_3), x_3^n - x_3\Big] + k\left\|f_2(x_3^n) - f_2(x_3)\right\|^2 \\
&\leq \left\|x_3^n - x_3\right\|^2 - 2\Big\{ - r_2\left\|f_2(x_3^n) - f_2(x_3)\right\|^2 + s_2\left\|x_3^n - x_3\right\|^2\Big\} + k\beta_2^2\left\|x_3^n - x_3\right\|^2 \\
&\leq \left\|x_3^n - x_3\right\|^2 + 2r_2\beta_2^2\left\|x_3^n - x_3\right\|^2 - 2s_2\left\|x_3^n - x_3\right\|^2 + k\beta_2^2\left\|x_3^n - x_3\right\|^2 \\
&\leq \Big(1 + 2(r_2\beta_2^2 - s_2) + k\beta_2^2\Big)\left\|x_3^n - x_3\right\|^2
\end{aligned}$$

$$\Longrightarrow \left\|x_3^n - x_3 - (f_2(x_3^n) - f_2(x_3))\right\| \leq \Phi_5\left\|x_3^n - x_3\right\| \tag{13}$$

where

$$\Phi_5 = \sqrt{1 + 2(r_2\beta_2^2 - s_2) + k\beta_2^2}.$$

Similarly

$$\left\|x_2^n - x_2 - (f_2(x_2^n) - f_2(x_2))\right\| \leq \Phi_5\left\|x_2^n - x_2\right\|. \tag{14}$$

Substituting (12), (13) in (11), we have

$$\left\|f_2(x_2^n) - f_2(x_2)\right\| \leq L_2(\Phi_2 + \Phi_5)\left\|x_3^n - x_3\right\| + t_2\left\|x_2^n - x_2\right\|. \tag{15}$$

Combining (10), (14) and (15), we have

$$\begin{aligned}
\left\|x_2^n - x_2\right\| &\leq \Phi_5\left\|x_2^n - x_2\right\| + L_2(\Phi_2 + \Phi_5)\left\|x_3^n - x_3\right\| + t_2\left\|x_2^n - x_2\right\| \\
&\leq (\Phi_5 + t_2)\left\|x_2^n - x_2\right\| + L_2(\Phi_2 + \Phi_5)\left\|x_3^n - x_3\right\|. \qquad (16)
\end{aligned}$$

Again, we have

$$\begin{aligned}
\left\|x_3^n - x_3\right\| &= \left\|(x_3^n - x_3) - (f_3(x_3^n) - f_3(x_3)) + (f_3(x_3^n) - f_3(x_3))\right\| \\
&\leq \left\|(x_3^n - x_3) - (f_3(x_3^n) - f_3(x_3))\right\| + \left\|(f_3(x_3^n) - f_3(x_3))\right\|. \qquad (17)
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\|f_3(x_3^n) - f_3(x_3)\right\| \\
&= \left\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n, w_2^n, w_3^n)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\right\| \\
&\leq \left\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n, w_2^n, w_3^n)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\right\| \\
&\quad + \left\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\right\| \\
&\leq L_3\left\|f_3(x_1^n) - f_3(x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\right\| + t_3\left\|x_3^n - x_3\right\| \\
&\leq L_3\left\|x_1^n - x_1 - \Big(f_3(x_1^n) - f_3(x_1)\Big)\right\|
\end{aligned}$$

$$+L_3\Big\|x_1^n - x_1 - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\| + t_3\Big\|x_3^n - x_3\Big\|. \qquad (18)$$

Since N_3 is relaxed (ξ_3, δ_3)-cocoercive and ν_3-Lipschitz continuous in the first argument, therefore by using Remark 1, we have

$$\begin{aligned}
&\Big\|(x_1^n - x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\|^2 \\
= \;& \Big\|x_1^n - x_1\Big\|^2 - 2\rho_3\Big[N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3), x_1^n - x_1\Big] \\
& + k\rho_3^2\Big\|N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 - 2\rho_3\Big\{-\xi_3\Big\|N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big\|^2 + \delta_3\Big\|x_1^n - x_1\Big\|^2\Big\} \\
& + k\rho_3^2\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2\Big(D(F_1(x_1^n), F_1(x_1))\Big)^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2\Big(D(F_1(x_1^n), F_1(x_1))\Big)^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2L_{F_1}^2\Big\|x_1^n - x_1\Big\|^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2L_{F_1}^2\Big\|x_1^n - x_1\Big\|^2 \\
\leq \;& \Big(1 + 2\rho_3(\xi_3\nu_3^2L_{F_1}^2 - \delta_3) + k\rho_3^2\nu_3^2L_{F_1}^2\Big)\Big\|x_1^n - x_1\Big\|^2
\end{aligned}$$

$$\Longrightarrow \Big\|(x_1^n - x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\| \leq \Phi_3\Big\|x_1^n - x_1\Big\| \qquad (19)$$

where

$$\Phi_3 = \sqrt{1 + 2\rho_3(\xi_3\nu_3^2L_{F_1}^2 - \delta_3) + k\rho_3^2\nu_3^2L_{F_1}^2}.$$

Since f_3 is relaxed (r_3, s_3)-cocoercive and β_3-Lipschitz continuous, therefore by using Remark 1, it follows that

$$\Big\|x_1^n - x_1 - (f_3(x_1^n) - f_3(x_1))\Big\|^2$$

$$
\begin{aligned}
&= \left\|x_1^n - x_1\right\|^2 - 2\Big[f_3(x_1^n) - f_3(x_1), x_1^n - x_1\Big] \\
&\quad + k\left\|f_3(x_1^n) - f_3(x_1)\right\|^2 \\
&\leq \left\|x_1^n - x_1\right\|^2 - 2\Big\{-r_3\left\|f_3(x_1^n) - f_3(x_1)\right\|^2 + s_3\left\|x_1^n - x_1\right\|^2\Big\} \\
&\quad + k\beta_3^2\left\|x_1^n - x_1\right\|^2 \\
&\leq \left\|x_1^n - x_1\right\|^2 + 2r_3\beta_3^2\left\|x_1^n - x_1\right\|^2 \\
&\quad - 2s_3\left\|x_1^n - x_1\right\|^2 + k\beta_3^2\left\|x_1^n - x_1\right\|^2 \\
&\leq \Big(1 + 2(r_3\beta_3^2 - s_3) + k\beta_3^2\Big)\left\|x_1^n - x_1\right\|^2 \\
&\quad \Longrightarrow \left\|x_1^n - x_1 - (f_3(x_1^n) - f_3(x_1))\right\| \leq \Phi_6\left\|x_1^n - x_1\right\|, \qquad (20)
\end{aligned}
$$

where

$$\Phi_6 = \sqrt{1 + 2(r_3\beta_3^2 - s_3) + k\beta_3^2}.$$

Similarly

$$\left\|x_3^n - x_3 - (f_3(x_3^n) - f_3(x_3))\right\| \leq \Phi_6\left\|x_3^n - x_3\right\|. \qquad (21)$$

Substituting (19), (20) in (18), we have

$$\left\|f_3(x_3^n) - f_3(x_3)\right\| \leq L_3(\Phi_3 + \Phi_6)\left\|x_1^n - x_1\right\| + t_3\left\|x_3^n - x_3\right\|. \qquad (22)$$

Combining (17), (21) and (22), we have

$$
\begin{aligned}
&\left\|x_3^n - x_3\right\| \\
&\leq \Phi_6\left\|x_3^n - x_3\right\| + L_3\Big(\Phi_3 + \Phi_6\Big)\left\|x_1^n - x_1\right\| + t_3\left\|x_3^n - x_3\right\| \\
&\leq \Big(\Phi_6 + t_3\Big)\left\|x_3^n - x_3\right\| + L_3\Big(\Phi_3 + \Phi_6\Big)\left\|x_1^n - x_1\right\| \\
&\Longrightarrow \Big(1 - (t_3 + \Phi_6)\Big)\left\|x_3^n - x_3\right\| \leq L_3\Big(\Phi_3 + \Phi_6\Big)\left\|x_1^n - x_1\right\| \\
&\quad \Longrightarrow \left\|x_3^n - x_3\right\| \leq \frac{L_3\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_3 + \Phi_6)\Big)}\left\|x_1^n - x_1\right\|. \qquad (23)
\end{aligned}
$$

Substituting (23) in (16), we have

$$\Big\|x_2^n - x_2\Big\|$$
$$\leq (t_2+\Phi_5)\Big\|x_2^n - x_2\Big\| + \frac{L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_3+\Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|$$
$$\Longrightarrow \Big(1-(t_2+\Phi_5)\Big)\Big\|x_2^n - x_2\Big\| \leq \frac{L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_3+\Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|$$
$$\Longrightarrow \Big\|x_2^n - x_2\Big\| \leq \frac{L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|. \tag{24}$$

Substituting (24) in (9),

$$\Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2,u_3,u_1)\Big)\Big\|$$
$$\leq \frac{L_2L_3\Phi_1\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|. \tag{25}$$

Since f_1 is relaxed (r_1, s_1)-cocoercive and β_1-Lipschitz continuous, therefore following the same procedure as in (13), (20), we have

$$\Big\|x_1^n - x_1 - (f_1(x_1^n) - f_1(x_1))\Big\| \leq \Phi_4\Big\|x_1^n - x_1\Big\|, \tag{26}$$

and similarly, we have

$$\Big\|x_2^n - x_2 - (f_1(x_2^n) - f_1(x_2))\Big\| \leq \Phi_4\Big\|x_2^n - x_2\Big\|, \tag{27}$$

where

$$\Phi_4 = \sqrt{1+2(r_1\beta_1^2 - s_1) + k\beta_1^2}.$$

Combining (24) and (27)

$$\Big\|x_2^n - x_2 - (f_1(x_2^n) - f_1(x_2))\Big\|$$
$$\leq \Phi_4\frac{L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|. \tag{28}$$

Substituting (25), (26), (28) in (8), it follows that

$$\Big\|x_1^{n+1} - x_1^n\Big\|$$
$$\leq \Bigg\{(1-\alpha^n) + \alpha^n\Phi_4 + \alpha^n\Phi_4\frac{L_1L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}$$

$$+\alpha^n \frac{L_1 L_2 L_3 \Phi_1 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)} + \alpha^n t_1 \Bigg\} \Big\| x_1^n - x_1 \Big\|$$

$$\leq \Bigg\{ 1 - \alpha^n \Bigg(1 - \Phi_4 - \Phi_4 \frac{L_1 L_2 L_3 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}$$

$$- \frac{L_1 L_2 L_3 \Phi_1 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)} - t_1 \Bigg) \Bigg\} \Big\| x_1^n - x_1 \Big\|$$

$$\leq (1 - \alpha^n (1 - \hbar)) \Big\| x_1^n - x_1 \Big\|. \tag{29}$$

where $\hbar < 1$ by assumption (7). Therefore by using Lemma 2, $\{x_i^n\}$ converges strongly to a solution of (3). This completes the proof. □

5. Conclusion

A new system of generalized nonlinear variational inclusion problems has been introduced in semi-inner product spaces. Using resolvent operator technique, an iterative algorithm has been constructed to solve the proposed system and the convergence analysis of the iterative algorithm has been investigated. The obtained results generalizes many known classes of variational inequalities and variational inclusions in the literature. The results presented can be used for approximation solvability of some different classes of problems in the literature.

Declaration of Competing Interests The author declare that there is no conflict of interest regarding the publication of this article.

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