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Curve Couples of Bézier Curves in Euclidean $2-$Space

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Abstract

The goal of this paper is to characterize the evolute, involute and parallel curves of a Bezier curve which is applicable to computer graphics and related subjects. Especially, these curve couples are investigated at the endpoints. Moreover, the curvatures of these curve couples are given.

1. Introduction

Geometry of curves is very essential because it has many important applications in many different areas. Therefore, various curves and surfaces have been studied by many authors for many years. Recently, due to its different structure, Bézier curves have attracted the attention of many researchers. Bézier curves are introduced firstly by Pierre Bézier in 1968. Bézier curves are the most important mathematical representations of curves which are applied to computer graphics and related areas.
C. Huygens, who is also known for his studies in optics, investigated the concepts of evolutes and involutes [1]. In classical differential geometry, the evolute of a regular curve in the Euclidean plane is given by not only the locus of all its centres of the curvature, but also the envelope of normal lines of the regular curve, namely, the locus of singular loci of parallel curves. On the other side, the involute of a regular curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. Two curves are said to be parallel of one another if any curve normal to one is normal to the other. Kılıçoğlu and Şenyurt studied the involute of the cubic Bézier curve in Euclidean $3-$space [2]. In [3], the evolute-involute curve couples of Bézier curves in Euclidean $3-$space are investigated. In this study, curve couples of Bézier curves are examined in the Euclidean $2-$ space in which the Bézier curve couples need not to be unit speed and suitable for giving examples.
The rest part of the paper is given as follows: Section 2 gives some basic notations and definitions for needed throughout the study. Section 3 gives the Serret-Frenet frame of a planar Bézier curve. Section 4 characterizes evolute curve of a planar Bézier curve and investigate at end points. Moreover, the Frenet apparatus of this curve couple is given. Section 5 characterizes involute curve of a planar Bézier curve and investigate at end points. In addition, the Frenet apparatus of this curve couple is handled. Section 6 constructs the parallel curve of a planar Bézier curve. Especially, the Frenet apparatus of this curve couple is given. In the final section, we conclude our work and talk about our future works.

2. Preliminaries

A classical Bézier curve of degree m with control points p_j is defined as

$$B(t)=\sum_{j=0}^{m} p_j B_j^m(t), t\in[0,1] \tag{2.1}$$

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where

$$B_{i,n}(t)=\begin{cases}\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i, & \text{if } 0\le i\le n\\ 0, & \text{otherwise}\end{cases}$$

are called the Bernstein basis functions of degree m. The polygon formed by joining the control points $p_0,p_1,...,p_m$ in the specified order is called the Bézier control polygon.
If a curve is differentiable at its each point in an open interval, in this case a set of orthogonal unit vectors can be obtained. And these unit vectors are called Frenet frame. The rates of these frame vectors along the curve define curvatures of the curves. The set of these vectors and curvatures of a curve, is called Frenet apparatus of the curve.

Definition 2.1. *The first derivative $B'(t)$ of a degree-n Bézier curve $B(t)$ is clearly a degree $m-1$ curve. Such a curve can be written in Bézier form as*

$$B'(t)=m\sum_{i=0}^{m-1}\triangle p_i B_i^{m-1}(t)$$

where $\triangle p_i=p_{i+1}-p_i$, $i=0,1,...,m-1$ are the control points of $B^{'}(t)$[4].

Definition 2.2. *$J:E^2\to E^2$ is a linear transformation which is defined by the following equation*

$$J(P_1,P_2)=(-P_2,P_1)[5].$$

Definition 2.3. *Let $\beta:I\to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T(t),N(t)\}$ and curvature $\kappa(t)$ of $\beta(t)$ for $\forall t\in I$ are defined by the following equations [5]:*

$$T(t)=\frac{\beta'(t)}{\|\beta'(t)\|},N(t)=\frac{J\beta'(t)}{\|\beta'(t)\|},\kappa(t)=\frac{<\beta''(t),J\beta'(t)>}{\|\beta'(t)\|^3}. \tag{2.2}$$

Definition 2.4. *For a plane regular curve $\beta(t)$ with $\kappa\neq 0$, the central curve*

$$\beta^*(t)=\beta(t)+\frac{1}{\kappa(t)}N(t) \tag{2.3}$$

where N is the normal of the curve β is called the evolute of β [6].

Definition 2.5. *For a plane regular curve $\beta(t)$ with $\kappa(t)\neq 0$, $t\in[t_1,t_2]$ and $a\in(t_1,t_2)$*

$$\beta^*(t)=\beta(t)-\frac{\beta'(t)}{\|\beta'(t)\|}\int_a^t\|\beta'(w)\|dw \tag{2.4}$$

is called the involute of β [7].

Definition 2.6. *The parallel at an oriented distance c to the left of a regular curve $\beta(t)$ is defined by the following equation*

$$\beta^*(t)=\beta(t)+cN(t) \tag{2.5}$$

[7].

For further information on curve couples see [6]-[8].

From now on, we will say Bézier curve instead of a non-unit speed planar Bézier curve of degree m throughout the paper.

3. The Serret-Frenet frame of a planar Bézier curve

In this section, the Serret-Frenet frame and curvature of a Bézier curve is given.

Theorem 3.1. *A Bézier curve with control points $p_0,p_1,...,p_m$ has the following Serret-Frenet frame $\{T(t),N(t)\}$ and curvature $\kappa(t)$ of Bézier curve with control points $p_0,p_1,...,p_m$ defined by (2.1) for $\forall t\in\mathbb{R}$ are*

$$T(t) = \frac{\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j}{(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{1}{2}}}, \tag{3.1}$$

$$N(t) = \frac{\sum_{j=0}^{m-1}B_j^{m-1}(t)J\triangle p_j}{(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{1}{2}}} \tag{3.2}$$

and

$$\kappa(t)=\frac{m-1}{m}\frac{\sum_{j=0}^{m-2}B_j^{m-2}(t)\sum_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>}{(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}} \tag{3.3}$$

where $\triangle p_j = p_{j+1}-p_j$ *and* $\triangle^2 p_j = p_{j+2}-2p_{j+1}+p_j$ *[9].*

4. Evolute of a planar Bézier curve

In this section, we characterize evolute curve of a planar Bézier curve and give its curvature. Moreover, we investigate this curve at $t=0$ and $t=1$.

Theorem 4.1. *The evolute* $B^*(t)$ *of a Bézier curve with control points* $p_0,p_1,...,p_m$ *defined by (2.1) for* $\forall t\in R$ *is*

$$B^*(t)=\sum_{j=0}^{m}p_jB_j^m(t)+\frac{m}{m-1}\frac{\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>\sum_{j=0}^{m-1}B_j^{m-1}(t)J\triangle p_j}{\sum_{j=0}^{m-2}B_j^{m-2}(t)\sum_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>}. \tag{4.1}$$

Proof. Taking into consideration the equations (2.1), (3.2) and (3.3) in (2.3), it can be proved. □

Remark 4.2. *The evolute* $B^*(t)$ *of a Bézier curve which is defined by (2.1) with control points* $p_0,p_1,...,p_m$ *is*

$$B^*(0)=p_0+\frac{m}{m-1}\frac{\|\triangle p_0\|^2 J\triangle p_0}{<\triangle p_1,J\triangle p_0>}$$

at $t=0$.

Remark 4.3. *The evolute* $B^*(t)$ *of a Bézier curve which is defined by (2.1) with control points* $p_0,p_1,...,p_m$ *is*

$$B^*(1)=p_m+\frac{m}{m-1}\frac{\|\triangle p_{m-1}\|^2 J\triangle p_{m-1}}{<\triangle p_{m-1},J\triangle p_{m-2}>}$$

at $t=1$.

Theorem 4.4. *The curvature of evolute* $B^*(t)$ *of a Bézier curve with control points* $p_0,p_1,...,p_m$ *defined by (2.1) for* $\forall t\in R$ *is*

$$\kappa_*(t)=\varepsilon_\kappa\left|\frac{(m^2.(m-1)\sum_{j=0}^{m-2}B_j^{m-2}(t)\sum_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>)^3}{\begin{array}{c}m^3(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}.(m^4(m-1)(m-2)\sum_{j=0}^{m-3}B_j^{m-3}(t)\sum_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^3 p_j,J\triangle p_i>)\\ (\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)-3m^7(m-1)^2(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}\\ .[(\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle_x p_j)(\sum_{i=0}^{m-2}B_j^{m-2}(t)\triangle_x^2 p_i)+(\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle_y p_j)(\sum_{i=0}^{m-2}B_j^{m-2}(t)\triangle_y^2 p_i)]\\ .(\sum_{j=0}^{m-2}B_j^{m-2}(t)\sum_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>)\end{array}}\right|$$

where ε_κ *is the sign of the curvature of the Bézier curve,* $\triangle_x p_j=(p_{j+1})_x-(p_j)_x$, $\triangle_y p_j=(p_{j+1})_y-(p_j)_y$, $\triangle_x^2 p_j=(p_{j+2})_x-2(p_{j+1})_x+(p_j)_x$ *and* $\triangle_y^2 p_j=(p_{j+2})_y-2(p_{j+1})_y+(p_j)_y$.

Proof. Taking into consideration the equations (2.2) and (4.1) together, it can be proved. □

Example 4.5. *For given control points* $p_0=(0,0), p_1=(\frac{1}{2},0), p_2=(\frac{1}{2},\frac{1}{2})$, *we have the following quadratic planar Bézier curve* $B(t)$

$$B(t)=\sum_{j=0}^{2}p_jB_j^2(t) \tag{4.2}$$

and the evolute of $B(t)$ *is given by the following equation*

$$B^*(t)=(\frac{3}{2}t^2-2t^3,1-3t+\frac{9}{2}t^2-2t^3).$$

The tanget of $B(t)$ *at* $t=0$ *is* $T=(1,0)$ *and the tanget of* $B^*(t)$ *at* $t=0$ *is* $T^*=(0,1)$. *The tanget of* $B(t)$ *at* $t=1$ *is* $T=(0,1)$ *and the tanget of* $B^*(t)$ *at* $t=1$ *is* $T^*=(1,0)$. *Therefore, the tangets at the end points are perpendicular.*

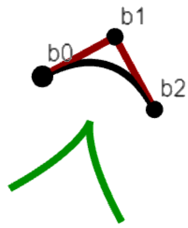

Figure 4.1: Bézier curve and the evolute couple are given by black and green color, respectively.

5. Involute of a planar Bézier curve

In this section, we characterize involute curve of a planar Bézier curve and give its curvature. In addition, we give an example.

Theorem 5.1. *The involute $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$B^*(t) = \sum_{i=0}^{m} p_i B_i^m(t) - \frac{\sum\limits_{i=0}^{m-1} \triangle p_i B_i^{m-1}(t)}{(\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}}} \int_a^t \| m \sum_{i=0}^{m-1} \triangle p_i B_i^{m-1}(w) \| dw. \tag{5.1}$$

Proof. Taking into consideration the equations (2.1) and (3.1) in (2.4), it can be proved. □

Theorem 5.2. *The curvature of involute $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$\kappa_*(t) = \frac{\varepsilon_\kappa}{m \int_a^t (\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(w) B_i^{m-1}(w) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}} dw}$$

where ε_κ is the sign of the curvature of the Bézier curve.

Proof. Taking into consideration the equations (2.2) and (5.1) together, it can be proved. □

Example 5.3. *The equation of involute couple of $B(t)$ which is given by (4.2)*

$$B^*(t) = (t - \frac{t^2}{2} - \frac{(1-t)[\sqrt{2} arcsinh(2t-1) + (4t-2)\sqrt{2t^2-2t+1}]}{4\sqrt{1-2t+2t^2}}, \frac{t^2}{2} - \frac{t[\sqrt{2} arcsinh(2t-1) + (4t-2)\sqrt{2t^2-2t+1}]}{4\sqrt{1-2t+2t^2}})$$

where $a = \frac{1}{2}$.

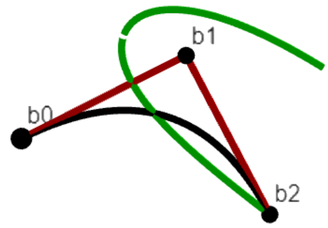

Figure 5.1: Bézier curve and the involute couple are given by black and green color, respectively.

6. Parallel of a planar Bézier curve

In this section, we characterize parallel curve of a planar Bézier curve and give its curvature. Moreover, we investigate this curve at $t = 0$ and $t = 1$.

Theorem 6.1. *The parallel $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$B^*(t) = \sum_{j=0}^{m} p_j B_j^m(t) + \frac{c \sum\limits_{j=0}^{m-1} B_j^{m-1}(t) J \triangle p_j}{(\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}}} \tag{6.1}$$

where c is a constant.

Proof. Taking into consideration the equations (2.1) and (3.2) in (2.5), it can be proved. □

Remark 6.2. *The parallel $B^*(t)$ of a Bézier curve which is defined by (2.1) with control points $p_0, p_1, ..., p_m$ is*

$$B^*(0) = p_0 + \frac{cJ\triangle p_0}{\|\triangle p_0\|}$$

at $t = 0$.

Remark 6.3. *The parallel $B^*(t)$ of a Bézier curve which is defined by (2.1) with control points $p_0, p_1, ..., p_m$ is*

$$B^*(1) = p_m + \frac{cJ\triangle p_{m-1}}{\|\triangle p_{m-1}\|}$$

at $t = 1$.

Theorem 6.4. *The curvature κ_* of parallel curve $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$\kappa_*(t) = \frac{(m-1)\sum_{j=0}^{m-2} B_j^{m-2}(t) \sum_{i=0}^{m-1} B_i^{m-1}(t) < \triangle^2 p_j, J\triangle p_i >}{m(\sum_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{3}{2}} - c(m-1)\sum_{j=0}^{m-2} B_j^{m-2}(t) \sum_{i=0}^{m-1} B_i^{m-1}(t) < \triangle^2 p_j, J\triangle p_i >}.$$

Proof. Taking into consideration the equations (2.2) and (6.1) together, it can be proved. □

Example 6.5. *The equation of parallel couple of $B(t)$ which is given by (4.2)*

$$B^*(t) = (t - \frac{t^2}{2} + \frac{t}{\sqrt{1-2t+2t^2}}, \frac{t^2}{2} - \frac{t}{\sqrt{1-2t+2t^2}})$$

where $c = -1$.

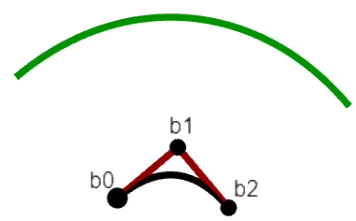

Figure 6.1: Bézier curve and the parallel couple are given by black and green color, respectively.

7. Conclusion

In this paper evolute, involute and parallel curves of a Bézier curve are characterized and investigated at the beginning and the ending points. In addition, these curve couples curvatures are obtained. In our future work, we will study the other curve couples of Bézier curve.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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