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Research Article

Some results on paracontact metric (k, μ)-manifolds with respect to the Schouten-van Kampen connection

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Abstract

In the present paper we study certain symmetry conditions and some types of solitons on paracontact metric (k, μ)-manifolds with respect to the Schouten-van Kampen connection. We prove that a Ricci semisymmetric paracontact metric (k, μ)-manifold with respect to the Schouten-van Kampen connection is an η-Einstein manifold. We investigate paracontact metric (k, μ)-manifolds satisfying $\breve{Q} \cdot \breve{R}_{cur} = 0$ with respect to the Schouten-van Kampen connection. Also, we show that there does not exist an almost Ricci soliton in a $(2n+1)$-dimensional paracontact metric (k, μ)-manifold with respect to the Schouten-van Kampen connection such that $k > -1$ or $k < -1$. In case of the metric is being an almost gradient Ricci soliton with respect to the Schouten-van Kampen connection, then we state that the manifold is either $N(k)$-paracontact metric manifold or an Einstein manifold. Finally, we present some results related to almost Yamabe solitons in a paracontact metric (k, μ)-manifold equipped with the Schouten-van Kampen connection and construct an example which verifies some of our results.

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1. Introduction

Kaneyuki [15] introduced the concept of paracontact metric (for short, pcm) structures in 1985. Recently, pcm manifolds have been studied by many authors, especially after the paper of Zamkovoy [32]. An important class among pcm manifolds is called the (k, μ)-manifold, which satisfies the nullity condition [6] given by

$$R_{cur}(U, W)\xi = \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW), \tag{1.1}$$

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for all U, W vector fields on M, where κ and μ are constants and $h = \frac{1}{2}\mathcal{L}_\xi \Psi$. This class also includes the para-Sasakian manifolds [15,32], the pcm manifolds satisfying $R_{cur}(U, W)\xi = 0$, for all U, W [33].

Symmetry property is one of the essential tools for investigating the geometry of manifolds. Symmetric Riemannian manifolds, that is Riemannian manifolds admitting $\nabla R_{cur} = 0$, where R_{cur} is the curvature tensor and ∇ is the Levi-Civita (for short, LC) connection, were introduced locally by Shirokov. In 1927, Cartan presented a comprehensive theory of symmetric Riemannian manifolds. If the curvature tensor R_{cur} of a manifold satisfies $R_{cur}(U, W) \cdot R_{cur} = 0$, then it is called a semisymmetric manifold. Here, $R_{cur}(U, W)$ is viewed as a derivation of the tensor algebra at each point of the manifold for the tangent vectors U, W. A local classification of semisymmetric manifolds were made by Szabó [27]. In addition, a manifold satisfying $R_{cur}(U, W) \cdot Ric = 0$, where Ric denotes the Ricci tensor of type $(0, 2)$, is called Ricci semisymmetric. Mirzoyan gave a general classification of manifolds of this type in [17]. For certain curvature conditions on pcm (κ, μ)-spaces we refer [16].

A pcm (κ, μ)-manifold admitting a Ricci tensor satisfying $Ric = \lambda_1 g$ (resp., $Ric = \lambda_1 g + \lambda_2 \eta \otimes \eta$) is called *Einstein* (resp., *η-Einstein*) manifold, where λ_1 and λ_2 are constants.

Riemannian manifolds with hyperdistributions and the Schouten van-Kampen (for short, S-vK) connection which is one of the most suitable connection adaptable to the hyperdistributions, were studied by Solov'ev [23–26]. Also see [2,13,21]. Almost pcm manifolds with the S-vK connection and curvature identities of such manifolds were investigated by Olszak [19]

As a generalization of Einstein manifold, an almost Ricci soliton (M, g, λ) was defined as a Riemannian manifold endowed with a complete vector field V satisfying

$$\mathcal{L}_V g + 2Ric + 2\lambda g = 0, \tag{1.2}$$

where $\mathcal{L}$ denotes the Lie derivative, Ric is the Ricci tensor on M and λ is a differentiable function [12]. If λ is negative, zero and positive, then the almost Ricci soliton is called shrinking, steady and expanding, respectively. The concept of the η-Ricci soliton was introduced in [8].

An almost η-Ricci soliton is a Riemannian manifold (M, g, λ, μ) admitting a differentiable vector field V such that the Ricci tensor Ric of M satisfies

$$\mathcal{L}_V g + 2Ric + 2\lambda g + 2\beta \eta \otimes \eta = 0, \tag{1.3}$$

where λ and μ are some differentiable functions. In case of the vector field V is being the gradient of a potential function $-f$, the equation (1.2) reduces to

$$\nabla\nabla f = Ric + \lambda g, \tag{1.4}$$

and an almost Ricci soliton is said to be an almost gradient Ricci soliton.

It was proved in [12,14] that, for 2-dimensional and 3-dimensional cases, a Ricci soliton on a compact manifold is of constant curvature (see also [9] and [10]). For further read we refer [3,4,20,22].

For solving the Yamabe problem, the Yamabe flows were firstly introduced in [12]. Yamabe solitons are self-similar solutions for Yamabe flows and they seem to be as singularity models. More clearly, the Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively. For further read, we refer [1,5,7,11,18,28–31]. As a generalization of Yamabe solitons, an almost Yamabe soliton is a Riemannian manifold (M, g) endowed with a vector field V satisfying [1]

$$\mathcal{L}_V g - 2(r - \delta) g = 0, \tag{1.5}$$

where r is the scalar curvature of M and δ is a differentiable function. An almost Yamabe soliton is called expanding, steady or shrinking, if $\delta < 0$, $\delta = 0$ or $\delta > 0$, respectively. In case of δ is being a constant, then an almost Yamabe soliton induces to a Yamabe soliton.

Moreover, if the Yamabe soliton is of constant scalar curvature Sc, then the Riemannian metric g is said to be a Yamabe metric.

In the present paper, we study certain semisymmetry conditions and some types of solitons in pcm (κ,μ)-manifolds. Following the introduction, Section 2 is devoted to some basic concepts that will be need throughout the paper. In Section 3, some properties of pcm (κ,μ)-manifolds endowed with the S-vK connection are presented. In section 4, we prove that Ricci semisymmetric pcm (κ,μ)-manifold with respect to (for short, wrt) the S-vK connection is an η-Einstein manifold. In section 5, we study pcm (κ,μ)-manifolds satisfying $\breve{Q}\cdot\breve{R}_{cur}=0$ wrt the S-vK connection. In section 6, we investigate almost Ricci soliton and almost η-Ricci soliton types on pcm (κ,μ)-manifolds wrt the S-vK connection. We show that there does not exist an almost Ricci soliton in a pcm (κ,μ)-manifold wrt the S-vK connection with $\kappa>-1$ or $\kappa<-1$. Section 7 is devoted to pcm (κ,μ)-manifolds $(\kappa\neq-1)$ admitting almost gradient Ricci soliton. In Section 8, we obtain some results related to almost Yamabe solitons in a pcm (κ,μ)-manifold and construct an example which verifies some of our results.

2. Preliminaries

Let M be $(2n+1)$-dimensional differentiable manifold endowed with a tensor field Ψ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\eta(\xi)=1,\ \ \Psi^2=I-\eta\otimes\xi,$$

and Ψ induces an almost paracomplex structure on each fibre of $\ker(\eta)$. From the last equation, we get $\Psi\xi=0$, $\eta\circ\Psi=0$ and note that the rank of endomorphism Ψ is $2n$. If an almost paracontact manifold M admits a pseudo-Riemannian metric g such that

$$g(\Psi U,\Psi W)=-g(U,W)+\eta(U)\eta(W), \tag{2.1}$$

for all $U,W\in\Gamma(TM)$, then the manifold is said to be an *almost pcm manifold*. The signature of the pseudo-Riemannian metric g is $(n+1,n)$ and an orthogonal basis $\{U_i,W_j,\xi\}$, namely a Ψ-basis, satisfying $g(U_i,U_j)=\delta_{ij}$, $g(W_i,W_j)=-\delta_{ij}$, $g(U_i,W_j)=0$, $g(\xi,U_i)=g(\xi,W_j)=0$, and $W_i=\Psi U_i$, for any $i,j\in\{1,\ldots,n\}$ can always be constructed for an almost pcm manifold.

The fundamental form of the almost pcm manifold is given by $\theta(U,W)=g(U,\Psi W)$. An almost pcm manifold with $d\eta=\theta$ is called a *pcm manifold*. In a pcm manifold, by help of Lie derivative $\mathcal{L}_\xi$ of the fundamental form, a trace-free symmetric operator h can be defined by $h=\frac{1}{2}\mathcal{L}_\xi\Psi$. This operator [32] anti-commutes with Ψ and satisfies $h\xi=0$, $trh=trh\Psi=0$ and

$$\nabla_U\xi=-\Psi U+\Psi hU, \tag{2.2}$$

$$(\nabla_U\eta)W=g(U,\Psi W)-g(hU,\Psi W), \tag{2.3}$$

where ∇ is the LC connection of the manifold. In addition, $h=0$ if and only if ξ is Killing vector field, which implies that (M,Ψ,ξ,η,g) is said to be a *K-paracontact manifold*. A normal pcm manifold is said to be a *para-Sasakian manifold*. Each para-Sasakian manifold is a K-paracontact manifold and but the converse holds only in 3-dimensional case. We also recall that any para-Sasakian manifold satisfies

$$R_{cur}(U,W)\xi=\eta(U)W-\eta(W)U,$$

where R_{cur} is Riemannian curvature operator given by

$$R_{cur}(U,W)Z=\nabla_U\nabla_WZ-\nabla_W\nabla_UZ-\nabla_{[U,W]}Z.$$

3. Pcm (k,μ)-manifolds with respect to the Schouten-van Kampen connection

A distribution defined by

$$N(\kappa,\mu): p \to N_p(\kappa,\mu) = \left\{ \begin{array}{c} Z \in T_pM \mid R_{cur}(U,W)Z = \kappa(g(W,Z)U - g(U,Z)W) \\ +\mu(g(W,Z)hU - g(U,Z)hW) \end{array} \right\}, \tag{3.1}$$

is called the (κ,μ)-nullity distribution of a pcm manifold (M,Ψ,ξ,η,g) for the pair (κ,μ), where κ and μ are some real constants. In case of the characteristic vector field ξ belonging to the (κ,μ)-nullity distribution, from (3.1) we write

$$R_{cur}(U,W)\xi = \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW),$$

for all $U,W \in \Gamma(TM)$. We refer [6] for basic results of pcm manifolds with the characteristic vector field satisfying the nullity condition (the condition (3.1)), for some real numbers κ and μ.

Lemma 3.1 ([6]). *In a $(2n+1)$-dimensional pcm (κ,μ)-manifold (M,Ψ,ξ,η,g), the followings hold:*

$$h^2 = (\kappa+1)\Psi^2, \tag{3.2}$$

$$(\nabla_U\Psi)W = -g(U,W)\xi + g(hU,W)\xi + \eta(W)U - \eta(W)hU, \quad \text{for } \kappa \neq -1, \tag{3.3}$$

$$\begin{aligned}(\nabla_U h)W = &-\{(1+\kappa)g(U,\Psi W) + g(U,\Psi hW)\}\xi \\ &+\eta(W)\Psi h(hU-U) - \mu\eta(U)\Psi hW,\end{aligned} \quad \text{for } \kappa \neq -1, \tag{3.4}$$

$$\begin{aligned}(\nabla_U\Psi h)W = &\ g(h^2U - hU, W)\xi + \eta(W)(h^2U - hU) \\ &-\mu\eta(U)hW,\end{aligned} \quad \text{for } \kappa > -1, \tag{3.5}$$

$$\begin{aligned}(\nabla_U\Psi h)W = &\ (1+\kappa)g(U,W)\xi - g(hU,W)\xi \\ &+\eta(W)(h^2U - hU) - \mu\eta(U)hW,\end{aligned} \quad \text{for } \kappa < -1, \tag{3.6}$$

$$\begin{aligned}QW = &\ (2(1-n)+n\mu)W + (2(n-1)+\mu)hW \\ &+(2(n-1)+n(2\kappa-\mu))\eta(W)\xi,\end{aligned} \quad \text{for } \kappa \neq -1, \tag{3.7}$$

$$Q\xi = 2n\kappa\xi, \tag{3.8}$$

$$\begin{aligned}(\nabla_U h)W - (\nabla_W h)U = &-(1+\kappa)(2g(U,\Psi W)\xi + \eta(U)\Psi W - \eta(W)\Psi U) \\ &+(1-\mu)(\eta(U)\Psi hW - \eta(W)\Psi hU),\end{aligned} \tag{3.9}$$

$$\begin{aligned}(\nabla_U\Psi h)W - (\nabla_W\Psi h)U = &(1+\kappa)(\eta(W)U - \eta(U)W) \\ &+ (\mu-1)(\eta(W)hU - \eta(U)hW),\end{aligned} \tag{3.10}$$

for any vector fields U, W on M.

Some important subclasses of pcm (κ,μ)-manifolds are given, regarding (1.1), by para-Sasakian manifolds, and pcm manifolds satisfying $R_{cur}(U,W)\xi = 0$. In [33], the authors showed that the pcm manifold $(M^{2n+1},\Psi,\xi,\eta,g)$ with $n > 1$ satisfying the last condition is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature -4. From (3.2), note that $h^2 = 0$ on a pcm (κ,μ)-manifold with $\kappa = -1$.

On the other hand we have two naturally defined distributions in the tangent bundle TM of M as follows:

$$D^H = \ker\eta, \qquad D^V = \operatorname{span}\{\xi\}.$$

Then we have $TM = D^H \oplus D^V$, $D^H \cap D^V = \{0\}$ and $D^H \perp D^V$. This decomposition allows one to define the S-vK connection $\check{\nabla}$ over an almost paracontact metric structure.

The S-vK connection $\breve{\nabla}$ on an almost (para) contact metric manifold with respect to LC-connection ∇ is defined by [23]

$$\breve{\nabla}_U W = \nabla_U W - \eta(W)\nabla_U \xi + (\nabla_U \eta)(W)\xi. \tag{3.11}$$

Thus with the help of the S-vK connection given by (3.11), many properties of some geometric objects connected with the distributions D^H and D^V can be characterized [23–25]. For example g, ξ and η are parallel with respect to $\breve{\nabla}$, that is, $\breve{\nabla}\xi = 0, \breve{\nabla}g = 0, \breve{\nabla}\eta = 0$. Also the torsion $T\breve{o}r$ of $\breve{\nabla}$ is defined by

$$T\breve{o}r(U, W) = \eta(U)\nabla_W \xi - \eta(W)\nabla_U \xi + 2d\eta(U, W)\xi. \tag{3.12}$$

Now we consider a pcm (κ, μ)-manifold wrt the S-vK connection. Firstly, using (2.2) and (2.3) in (3.11), we get

$$\breve{\nabla}_U W = \nabla_U W + \eta(W)\Psi U - \eta(W)\Psi hU + g(U, \Psi W)\xi - g(hU, \Psi W)\xi. \tag{3.13}$$

Let $\breve{R}_{cur}$ be the curvature tensors of the S-vK connection $\breve{\nabla}$ given by $\breve{R}_{cur}(U, W) = [\breve{\nabla}_U, \breve{\nabla}_W] - \breve{\nabla}_{[U,W]}$. Using (3.13) in the definition of $\breve{R}_{cur}(U, W)$, we have

$$\begin{aligned} \breve{R}_{cur}(U, W)Z \;=\; & \breve{\nabla}_U(\nabla_W Z + \eta(Z)\Psi W - \eta(Z)\Psi hW \\ & + g(W, \Psi Z)\xi - g(hW, \Psi Z)\xi) \\ & - \breve{\nabla}_W(\nabla_U Z + \eta(Z)\Psi U - \eta(Z)\Psi hU \\ & + g(U, \Psi Z)\xi - g(hU, \Psi Z)\xi) \\ & - (\nabla_{[U,W]} Z + \eta(Z)\Psi[U, W] - \eta(Z)\Psi h[U, W] \\ & + g([U, W], \Psi Z)\xi - g(h[U, W], \Psi Z)\xi). \end{aligned} \tag{3.14}$$

Using (3.9), (3.10) and (3.3) in (3.14), we have the relation between R_{cur} and $\breve{R}_{cur}$ on M

$$\begin{aligned} \breve{R}_{cur}(U, W)Z \;=\; & R_{cur}(U, W)Z + g(U, \Psi Z)\Psi W - g(W, \Psi Z)\Psi U + g(hW, \Psi Z)\Psi U \\ & - g(hU, \Psi Z)\Psi W + g(W, \Psi Z)\Psi hU - g(U, \Psi Z)\Psi hW \\ & + g(hU, \Psi Z)\Psi hW - g(hW, \Psi Z)\Psi hU \\ & + \kappa\{g(U, Z)\eta(W)\xi - g(W, Z)\eta(U)\xi \\ & + \eta(U)\eta(Z)W - \eta(W)\eta(Z)U\} \\ & + \mu\{g(hU, Z)\eta(W)\xi - g(hW, Z)\eta(U)\xi \\ & + \eta(U)\eta(Z)hW - \eta(W)\eta(Z)hU\} \end{aligned} \tag{3.15}$$

Now from (3.15), we get

$$\begin{aligned} g(\breve{R}_{cur}(U, W)Z, T) \;=\; & g(R_{cur}(U, W)Z, T) + g(U, \Psi Z)g(\Psi W, T) - g(W, \Psi Z)g(\Psi U, T) \\ & + g(hW, \Psi Z)g(\Psi U, T) - g(hU, \Psi Z)g(\Psi W, T) \\ & + g(W, \Psi Z)g(\Psi hU, T) - g(U, \Psi Z)g(\Psi hW, T) \\ & + g(hU, \Psi Z)g(\Psi hW, T) - g(hW, \Psi Z)g(\Psi hU, T) \\ & + \kappa\{g(U, Z)\eta(W)\eta(T) - g(W, Z)\eta(U)\eta(T) \\ & + g(W, T)\eta(U)\eta(Z) - g(U, T)\eta(W)\eta(Z)\} \\ & + \mu\{g(hU, Z)\eta(W)\eta(T) - g(hW, Z)\eta(U)\eta(T) \\ & + g(hW, T)\eta(U)\eta(Z) - g(hU, T)\eta(W)\eta(Z)\}. \end{aligned} \tag{3.16}$$

If we take $U = T = e_i$, $\{i = 1, ..., 2n + 1\}$, in (3.16), where $\{ei\}$ is an orthonormal basis of $\chi(M)$, we get

$$\breve{R}ic(W, Z) = Ric(W, Z) - 2n\kappa\eta(W)\eta(Z) - \mu g(hW, Z), \tag{3.17}$$

where $\breve{Ric}$ and Ric denote the Ricci tensor of the connections $\breve{\nabla}$ and ∇, respectively. As a consequence of (3.17), we get for the Ricci operator $\breve{Q}$

$$\breve{Q}W = QW - 2n\kappa\eta(W)\xi - \mu hW. \tag{3.18}$$

Also if we take $W = Z = e_i$, $\{i = 1, ..., 2n+1\}$, in (3.18), we get

$$\breve{r} = r - 2n\kappa, \tag{3.19}$$

where $\breve{r}$ and r denote the scalar curvatures of the connections $\breve{\nabla}$ and ∇, respectively.

4. Ricci semisymmetric pcm (κ,μ) -manifolds with respect to the Schouten-van Kampen connection

In this section we study Ricci semisymmetric pcm (κ,μ)-manifolds wrt the S-vK connection. Firstly we give the following:

Definition 4.1. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be Ricci semisymmetric if we have

$$R_{cur}(U,W)\cdot Ric = 0,$$

holds on M for all $U, W \in \chi(M)$.

Let M be a Ricci semisymmetric pcm (κ,μ)-manifold with $(\kappa \neq -1)$ wrt the S-vK connection. Then above equation is equivalent to

$$(\breve{R}_{cur}(U,W)\cdot \breve{Ric})(Z,T) = 0,$$

for any $U, W, Z, T \in \chi(M)$. Thus we have

$$\breve{Ric}(\breve{R}_{cur}(U,W)Z,T) + \breve{Ric}(Z,\breve{R}_{cur}(U,W)T) = 0, \tag{4.1}$$

Using (3.15) in (4.1), we get

$$\begin{aligned}
&\kappa\begin{bmatrix} \{g(W,Z)-\eta(W)\eta(Z)\}\breve{Ric}(U,T) \\ -\{g(U,Z)-\eta(U)\eta(Z)\}\breve{Ric}(W,T) \\ +\{g(W,T)-\eta(W)\eta(T)\}\breve{Ric}(U,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{Ric}(W,Z) \end{bmatrix} \\
&+\mu\begin{bmatrix} \{g(W,Z)-\eta(W)\eta(Z)\}\breve{Ric}(hU,T) \\ -\{g(U,Z)-\eta(U)\eta(Z)\}\breve{Ric}(hW,T) \\ +\{g(W,T)-\eta(W)\eta(T)\}\breve{Ric}(hU,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{Ric}(hW,Z) \end{bmatrix} \\
&+g(U,\Psi Z)\breve{Ric}(\Psi W,T) - g(W,\Psi Z)\breve{Ric}(\Psi U,T) \\
&+g(hW,\Psi Z)\breve{Ric}(\Psi U,T) - g(hU,\Psi Z)\breve{Ric}(\Psi W,T) \\
&+g(W,\Psi Z)\breve{Ric}(\Psi hU,T) - g(U,\Psi Z)\breve{Ric}(\Psi hW,T) \\
&+g(hU,\Psi Z)\breve{Ric}(\Psi hW,T) - g(hW,\Psi Z)\breve{Ric}(\Psi hU,T) \\
&+g(U,\Psi T)\breve{Ric}(\Psi W,Z) - g(W,\Psi T)\breve{Ric}(\Psi U,Z) \\
&+g(hW,\Psi T)\breve{Ric}(\Psi U,Z) - g(hU,\Psi T)\breve{Ric}(\Psi W,Z) \\
&+g(W,\Psi T)\breve{Ric}(\Psi hU,Z) - g(U,\Psi T)\breve{Ric}(\Psi hW,Z) \\
&+g(hU,\Psi T)\breve{Ric}(\Psi hW,Z) - g(hW,\Psi T)\breve{Ric}(\Psi hU,Z) = 0.
\end{aligned} \tag{4.2}$$

Putting $U = T = e_i$, $\{i = 1, ..., 2n+1\}$, in (4.2), we obtain

$$\kappa\breve{r}\{g(W,Z)-\eta(W)\eta(Z)\} - 2n\kappa\breve{Ric}(W,Z) - 2n\mu\breve{Ric}(hW,Z) = 0. \tag{4.3}$$

Now putting $W = hW$ in (4.3), we have

$$\kappa\breve{r}g(hW,Z) - 2n\kappa\breve{Ric}(hW,Z) - 2n(\kappa+1)\mu\breve{Ric}(W,Z) = 0. \tag{4.4}$$

Assume that $\kappa \neq -1$ and $\mu \neq 0$. Multiplying with (4.3) by κ and (4.4) by μ, then subtract the results, we obtain

$$2n[\kappa^2 - \mu^2(\kappa+1)]\breve{R}ic(W,Z) = \kappa^2\breve{r}\{g(W,Z) - \eta(W)\eta(Z)\} - \kappa\mu\breve{r}g(hW,Z). \tag{4.5}$$

Using (3.17) in (4.5), we get

$$\begin{aligned} & 2n[\kappa^2 - \mu^2(\kappa+1)]\{Ric(W,Z) - 2n\kappa\eta(W)\eta(Z) - \mu g(hW,Z)\} \\ = \; & \kappa^2\breve{r}\{g(W,Z) - \eta(W)\eta(Z)\} + (\mu - \kappa\mu\breve{r} - 1)g(hW,Z), \end{aligned}$$

i.e.,

$$\begin{aligned} Ric(W,Z) &= \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)}g(W,Z) \\ &\quad +(2n\kappa - \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)})\eta(W)\eta(Z) \\ &\quad +(\mu - \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)})g(hW,Z). \end{aligned} \tag{4.6}$$

Again using (3.7) in (4.6), we have

$$Ric(W,Z) = \frac{A_1 - C_1B_2}{1 - A_2C_1}g(W,Z) + \frac{B_1 - C_1C_2}{1 - A_2C_1}\eta(W)\eta(Z), \tag{4.7}$$

where

$$\begin{aligned} A_1 &= \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)}, & B_1 &= 2n\kappa - \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)}, \\ C_1 &= \mu - \frac{\kappa^2\breve{r}}{2n(\kappa^2 - \mu^2(\kappa+1)}, & A_2 &= \frac{1}{2(n-1)+\mu}, \\ B_2 &= \frac{2(1-n)+n\mu}{2(n-1)+\mu}, & C_2 &= \frac{2(n-1)+n(2\kappa-\mu)}{2(n-1)+\mu}. \end{aligned}$$

Therefore, from (4.7) we have the following:

Theorem 4.2. *Let M be a $(2n+1)$-dimensional pcm (κ,μ)-manifold with $\kappa \neq -1$. If M is a Ricci semisymmetric pcm (κ,μ)-manifold wrt the S-vK connection then the manifold M is an η-Einstein manifold wrt the LC connection provided $\mu \neq 2(1-n)$.*

5. Pcm (κ,μ)-manifolds satisfying $\breve{Q}\cdot\breve{R} = 0$ with respect to the Schouten-van Kampen connection

In this section we study the condition $\breve{Q}\cdot\breve{R}_{cur} = 0$ on pcm (κ,μ)-manifolds wrt the S-vK connection. Firstly we give the following:

$$(\breve{Q}\cdot\breve{R}_{cur})(U,W)Z = \breve{Q}\breve{R}_{cur}(U,W)Z - \breve{R}_{cur}(\breve{Q}U,W)Z - \breve{R}_{cur}(U,\breve{Q}W)Z - \breve{R}_{cur}(U,W)\breve{Q}Z = 0.$$

Then we write

$$\begin{aligned} & g(\breve{Q}\breve{R}_{cur}(U,W)Z,T) - g(\breve{R}_{cur}(\breve{Q}U,W)Z,T) \\ & -g(\breve{R}_{cur}(U,\breve{Q}W)Z,T) - g(\breve{R}_{cur}(U,W)\breve{Q}Z,T) = 0, \end{aligned} \tag{5.1}$$

which infers

$$\begin{aligned} & g(\breve{R}_{cur}(U,W)Z,\breve{Q}T) + g(\breve{R}_{cur}(Z,T)W,\breve{Q}U) \\ & -g(\breve{R}_{cur}(Z,T)U,\breve{Q}W) + g(\breve{R}_{cur}(U,W)T,\breve{Q}Z) = 0. \end{aligned}$$

So we can write

$$\begin{aligned} & \breve{R}ic(\breve{R}_{cur}(U,W)Z,T) + \breve{R}ic(\breve{R}_{cur}(Z,T)W,U) \\ & -\breve{R}ic(\breve{R}_{cur}(Z,T)U,W) + \breve{R}ic(\breve{R}_{cur}(U,W)T,Z) = 0. \end{aligned} \tag{5.2}$$

Now using (3.14) in (5.2), we compute

$$\kappa\begin{bmatrix} \{g(W,T)-\eta(W)\eta(T)\}\breve{R}ic(U,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{R}ic(W,Z) \\ +\{g(W,T)-\eta(W)\eta(T)\}\breve{R}ic(U,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{R}ic(W,Z) \end{bmatrix}$$
$$+\mu\begin{bmatrix} \{g(W,T)-\eta(W)\eta(T)\}\breve{R}ic(hU,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{R}ic(hW,Z) \\ +\{g(W,T)-\eta(W)\eta(T)\}\breve{R}ic(hU,Z) \\ -\{g(U,T)-\eta(U)\eta(T)\}\breve{R}ic(hW,Z) \end{bmatrix}$$
$$\begin{aligned}
&+g(U,\Psi Z)\breve{R}ic(\Psi W,T)-g(W,\Psi Z)\breve{R}ic(\Psi U,T)+g(hW,\Psi Z)\breve{R}ic(\Psi U,T)\\
&-g(hU,\Psi Z)\breve{R}ic(\Psi W,T)+g(W,\Psi Z)\breve{R}ic(\Psi hU,T)-g(U,\Psi Z)\breve{R}ic(\Psi hW,T)\\
&+g(hU,\Psi Z)\breve{R}ic(\Psi hW,T)-g(hW,\Psi Z)\breve{R}ic(\Psi hU,T)+g(U,\Psi T)\breve{R}ic(\Psi W,Z)\\
&-g(W,\Psi T)\breve{R}ic(\Psi U,Z)+g(hW,\Psi T)\breve{R}ic(\Psi U,Z)-g(hU,\Psi T)\breve{R}ic(\Psi W,Z)\\
&+g(W,\Psi T)\breve{R}ic(\Psi hU,Z)-g(U,\Psi T)\breve{R}ic(\Psi hW,Z)+g(hU,\Psi T)\breve{R}ic(\Psi hW,Z)\\
&-g(hW,\Psi T)\breve{R}ic(\Psi hU,Z)+g(\Psi W,Z)\breve{R}ic(U,\Psi T)-g(\Psi W,T)\breve{R}ic(U,\Psi Z)\\
&+g(\Psi W,hT)\breve{R}ic(U,\Psi Z)-g(\Psi W,hZ)\breve{R}ic(U,\Psi T)+g(\Psi W,T)\breve{R}ic(\Psi hZ,U)\\
&-g(Z,\Psi W)\breve{R}ic(\Psi hT,U)+g(hZ,\Psi W)\breve{R}ic(\Psi hT,U)-g(hT,\Psi W)\breve{R}ic(\Psi hZ,U)\\
&-g(\Psi U,Z)\breve{R}ic(W,\Psi T)+g(\Psi U,T)\breve{R}ic(W,\Psi Z)-g(\Psi U,hT)\breve{R}ic(W,\Psi Z)\\
&+g(\Psi U,hZ)\breve{R}ic(W,\Psi T)-g(\Psi U,T)\breve{R}ic(W,\Psi hZ)+g(\Psi U,Z)\breve{R}ic(\Psi hT,W)\\
&-g(hZ,\Psi U)\breve{R}ic(\Psi hT,W)+g(hT,\Psi U)\breve{R}ic(\Psi hZ,W)=0. \qquad (5.3)
\end{aligned}$$

Putting $U=T=e_i$, $\{i=1,...,2n+1\}$, in (5.3), we have

$$\kappa(1-2n)\breve{R}ic(W,Z)+\mu(1-2n)\breve{R}ic(hW,Z)+(\kappa+1)\breve{R}ic(\Psi W,\Psi Z)+\breve{R}ic(W,Z)=0,$$

which entails

$$(2n\kappa+1)\breve{R}ic(W,Z)+\mu(2n-1)\breve{R}ic(hW,Z)-2(\kappa+1)(2n-2+\mu)g(hW,Z)=0. \quad (5.4)$$

Now putting $W=hW$ in (5.4), we have

$$\begin{aligned}&(2n\kappa+1)\breve{R}ic(hW,Z)+\mu(2n-1)(\kappa+1)\breve{R}ic(W,Z)\\ &-2(\kappa+1)(2n-2+\mu)(\kappa+1)\{g(W,Z)-\eta(W)\eta(Z)\}=0.\end{aligned} \quad (5.5)$$

Multiplying (5.4) by $2n\kappa+1$ and (5.5) by $\mu(2n-1)$, we have

$$\begin{aligned}&(2n\kappa+1)^2\breve{R}ic(W,Z)+\mu(2n-1)(2n\kappa+1)\breve{R}ic(hW,Z)\\ &-2(\kappa+1)(2n\kappa+1)(2n-2+\mu)g(hW,Z)=0,\end{aligned} \quad (5.6)$$

and

$$\begin{aligned}&\mu(2n-1)(2n\kappa+1)\breve{R}ic(hW,Z)+\mu(2n-1)^2(\kappa+1)\breve{R}ic(W,Z)\\ &-2(\kappa+1)(2n-2+\mu)\mu(2n-1)(\kappa+1)\{g(W,Z)-\eta(W)\eta(Z)\}=0,\end{aligned} \quad (5.7)$$

respectively. Subtracting (5.6) from (5.7), we get

$$\breve{R}ic(W,Z)=\frac{\lambda_1}{\gamma}g(hW,Z)-\frac{\lambda_2}{\gamma}g(W,Z)+\frac{\lambda_2}{\gamma}\eta(W)\eta(Z), \quad (5.8)$$

where

$$\begin{aligned}\lambda_1 &= 2(2n\kappa+1)(\kappa+1)(2n-2+\mu),\\ \lambda_2 &= 2\mu(\kappa+1)^2(2n-2+\mu),\\ \gamma &= (2n\kappa+1)^2-(2n-1)^2\mu(\kappa+1).\end{aligned}$$

Now using (3.7) in (5.8), we obtain

$$(\frac{\lambda_1}{\gamma(B-\mu)}-1)\breve{R}ic(W,Z)=(\frac{\lambda_1 A}{\gamma(B-\mu)}+\frac{\lambda_2}{\gamma})g(W,Z)+(\frac{\lambda_1(C-2n\kappa)}{\gamma(B-\mu)}-\frac{\lambda_2}{\gamma})\eta(W)\eta(Z),$$

where $A=(2(1-n)+n\mu), B=(2(n-1)+\mu, C=2(n-1)+n(2\kappa-\mu)$. The last equation can be written

$$\breve{R}ic(W,Z)=\rho g(W,Z)+\sigma\eta(W)\eta(Z),$$

where

$$\rho=\frac{\frac{\lambda_1 A}{\gamma(B-\mu)}+\frac{\lambda_2}{\gamma}}{\frac{\lambda_1}{\gamma(B-\mu)}-1},\quad \sigma=\frac{\frac{\lambda_1(C-2n\kappa)}{\gamma(B-\mu)}-\frac{\lambda_2}{\gamma}}{\frac{\lambda_1}{\gamma(B-\mu)}-1}.$$

Thus the manifold M is an η-Einstein manifold wrt the S-vK connection. Hence we have the following:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional pcm (κ,μ)-manifold with $\kappa\neq -1$ satisfying the condition $\breve{Q}\cdot\breve{R}_{cur}=0$ wrt the S-vK connection. Then the manifold M is an η-Einstein manifold wrt the S-vK connection provided $\frac{\lambda_1}{\gamma(B-\mu)}-1\neq 0$.*

6. Almost Ricci solitons and almost η-Ricci solitons on pcm (κ,μ)-manifolds with respect to the Schouten-van Kampen connection

In this section we study almost Ricci solitons and almost η-Ricci soliton in pcm (κ,μ)-manifolds wrt the S-vK connection.

In a pcm (κ,μ)-manifold $(\kappa\neq -1)$ with the S-vK connection, since $\breve{\nabla}g=0$ by using (1.2), we get

$$(\breve{\mathcal{L}}_V g)(U,T)=g(\nabla_U V,T)+g(U,\nabla_T V)=(\mathcal{L}_V g)(U,T), \tag{6.1}$$

where $\breve{\mathcal{L}}$ denotes the Lie derivative on manifold wrt the S-vK connection.

Now we consider an almost Ricci soliton on a pcm (κ,μ)-manifold wrt the S-vK connection. From (1.2), we can write

$$(\breve{\mathcal{L}}_V g+2\breve{R}ic+2\breve{\lambda}g)(U,T)=0. \tag{6.2}$$

Using (6.1) in (6.2), we obtain

$$\left\{\begin{array}{c}(\mathcal{L}_V g)(U,T)+2Ric(U,T)+2\breve{\lambda}g(U,T)\\ -4n\kappa\eta(U)\eta(T)-2\mu g(hU,T)\end{array}\right.=0. \tag{6.3}$$

Thus we have the followings:

Theorem 6.1. *A $(2n+1)$-dimensional pcm (κ,μ)-manifold M bearing an almost Ricci soliton $(V,\breve{\lambda},g)$ wrt the S-vK connection admits an almost η-Ricci soliton $(V,\breve{\lambda},-2n\kappa,g)$ wrt the LC connection provided the manifold is a $N(\kappa)$-pcm manifold.*

Corollary 6.2. *If M is a $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost Ricci soliton $(V,\breve{\lambda},g)$ wrt the S-vK connection, then M admits an almost Ricci soliton $(V,\breve{\lambda},g)$ wrt the LC connection provided the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

Conversely, assume that a pcm (κ,μ)-manifold admits an almost Ricci soliton (V,λ,g) wrt the LC connection. Then, from (1.2) and (3.17), we have

$$\left\{\begin{array}{c}(\mathcal{L}_V g)(U,T)+2\breve{R}ic(U,T)+2\lambda g(U,T)\\ +4n\kappa\eta(U)\eta(T)+2\mu g(hU,T)\end{array}\right.=0.$$

Hence we give the followings:

Theorem 6.3. *Let M be a $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost Ricci soliton (V,λ,g) wrt the LC connection. Then M admits an almost η-Ricci soliton $(V,\lambda,2n\kappa,g)$ wrt the S-vK connection provided the manifold is a $N(\kappa)$-pcm manifold.*

Corollary 6.4. *A $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost Ricci soliton (V,λ,g) wrt the LC connection admits an almost Ricci soliton (V,λ,g) wrt the S-vK connection provided the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

In case of g is being an almost η-Ricci soliton wrt the LC connection, we have the following:

Theorem 6.5. *A $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost η-Ricci soliton (V,λ,β,g) wrt the LC connection admits an almost η-Ricci soliton $(V,\lambda,2n\kappa+\beta,g)$ wrt the S-vK connection provided the manifold is a $N(\kappa)$-pcm manifold.*

Proof. Assume that M is a $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost η-Ricci soliton (V,λ,β,g) wrt the LC connection. From (1.2) and (3.17) we write

$$\begin{gathered}(\mathcal{L}_V g)(U,T)+2\breve{R}ic(U,T)+2\lambda g(U,T)\\ +2(2n\kappa+\beta)\eta(U)\eta(T)+2\mu g(hU,T)=0.\end{gathered}$$

This completes the proof. □

Now we consider the case of the potential vector field being the structure vector field.

Assume that M is a $(2n+1)$-dimensional pcm (κ,μ)-manifold bearing an almost Ricci soliton $(\xi,\breve{\lambda},g)$ wrt the S-vK connection. Using (2.2), (3.17) and (6.1) in (6.2), we write

$$g(U,\Psi hT)+g(QU,T)+\breve{\lambda}g(U,T)-2n\kappa\eta(U)\eta(T)-\mu g(hU,T)=0. \tag{6.4}$$

From (6.4), we get

$$\Psi hU+QU+\breve{\lambda}U-2n\kappa\eta(U)\xi-\mu hU=0. \tag{6.5}$$

By taking covariant derivative of (6.5), we have

$$\begin{gathered}(\nabla_X\Psi h)\,U+\Psi h(\nabla_X U)+(\nabla_X Q)\,U+Q\nabla_X U+X(\breve{\lambda})U+\breve{\lambda}\nabla_X U\\ -2n\kappa\,(g(\nabla_X U,\xi)\xi+g(U,\nabla_X\xi)\xi+\eta(U)\nabla_X\xi)\\ -\mu\,(\nabla_X h)\,U-\mu h\nabla_X U=0,\end{gathered}$$

which implies that

$$\begin{gathered}(\nabla_X\Psi h)\,U+(\nabla_X Q)\,U+X(\breve{\lambda})U\\ -2n\kappa\,(g(U,\nabla_X\xi)\xi+\eta(U)\nabla_X\xi)-\mu\,(\nabla_X h)\,U=0.\end{gathered} \tag{6.6}$$

We have the following cases:

Case 1. Assume that $\kappa>-1$. By using (3.5), (3.3), (3.4) and (2.2) in (6.6), we have

$$\begin{gathered}g(h^2X-hX,U)\xi+\eta(U)\,(h^2X-hX)-\mu\eta(X)hU\\ -2(n-1)(1+\kappa)g(X,\Psi U)\xi-2(n-1)g(X,\Psi hU)\xi\\ +2(n-1)\,\{\eta(U)\,(\Psi h^2X-\Psi hX)-\mu\eta(X)\Psi hU\}+X(\breve{\lambda})U\\ +\,(2(n-1)-n\mu))\,\{-\,(g(U,\Psi X)-g(U,\Psi hX))\,\xi-\eta(U)\,(\Psi X+\Psi hX)\}=0,\end{gathered}$$

which implies that

$$\begin{gathered}(\kappa+1)g(X,U)\xi-2(\kappa+1)\eta(X)\eta(U)\xi-g(hX,U)\xi+(\kappa+1)\eta(U)X-\eta(U)hX\\+\mu\eta(X)hU-(2\kappa(n-1)+n\mu)g(X,\Psi U)\xi-n\mu g(X,\Psi hU)\xi\\+(2\kappa(n-1)+n\mu)\eta(U)\Psi X-n\mu\eta(U)\Psi hX\\-2\mu(n-1)\eta(X)\Psi hU+X(\breve{\lambda})U=0.\end{gathered}\tag{6.7}$$

By contracting X in (6.7), we obtain

$$2n\,(\kappa+1)\,\eta(U)=-U(\breve{\lambda})\tag{6.8}$$

On the other hand, by taking $U=\xi$ in (6.5), we obtain

$$\breve{\lambda}=0.\tag{6.9}$$

Using (6.9) in (6.8), we conclude that $\kappa=-1$, which contradicts with the assumption $\kappa>-1$.

Case 2. Assume that $\kappa<-1$. By using (3.6), (3.3), (3.4) and (2.2) in (6.6), we get

$$\begin{gathered}(1+\kappa)g(X,U)\xi-g(hX,U)\xi+\eta(U)(h^2X-hX)-\mu\eta(X)hU\\+(2\kappa(1-n)-n\mu)\,g(X,\Psi U)\xi-n\mu g(X,\Psi hU)\xi\\-(2\kappa(1-n)-n\mu)\,\eta(U)\Psi X-n\mu\eta(U)\Psi hX\\-\mu\eta(X)\Psi hU+X(\breve{\lambda})U=0.\end{gathered}\tag{6.10}$$

By contracting X in (6.10), we have

$$(2n+1)\,(\kappa+1)\eta(U)=-U(\breve{\lambda}).\tag{6.11}$$

On the other hand, by taking $U=\xi$ in (6.5), we get

$$\breve{\lambda}=0.$$

Using the last equation in (6.11), we conclude that $\kappa=-1$, which contradicts with the assumption $\kappa>-1$.

Hence we give the following:

Theorem 6.6. *There does not exist an almost Ricci soliton $(\xi,\breve{\lambda},g)$ in a $(2n+1)$-dimensional pcm (κ,μ)-manifold (M,g) wrt the S-vK connection with $\kappa>-1$ or $\kappa<-1$.*

Now, we consider $\kappa=-1$. In this case we give the following:

Theorem 6.7. *If a $(2n+1)$-dimensional pcm (κ,μ)-manifold (M,g) wrt the S-vK connection admits an almost Ricci soliton $(\xi,\breve{\lambda},g)$, then the almost Ricci soliton is steady.*

Proof. By putting $U=\xi$ in (6.5), we get $Q\xi=2\kappa n\xi-\breve{\lambda}$. On the other hand, from (3.8) we have $Q\xi=2\kappa n\xi$. Thefore we obtain $\breve{\lambda}=0$, which completes the proof. □

7. Almost gradient Ricci solitons on pcm (κ,μ)-manifolds with respect to the Schouten-van Kampen connection

If the vector field V is the gradient of a potential function $-f$, that is $V=-gradf$, then g is called an almost gradient Ricci soliton. In this case equation (1.2) becomes

$$\nabla gradf=Ric+\lambda g,\tag{7.1}$$

where ∇ is the LC connection.

Now assume that M is a $(2n+1)$-dimensional $(n>1)$ pcm (κ,μ)-manifold $(\kappa\neq-1)$ wrt the S-vK connection. If we take $V=-gradf$ in (6.1), we write

$$(\breve{\mathcal{L}}_{gradf}\,g)(U,T)=(\mathcal{L}_{gradf}\,g)(U,T)=g(\nabla_U gradf,T)+g(U,\nabla_T gradf).\tag{7.2}$$

We can easily see that

$$g(\nabla_U gradf,T)=g(U,\nabla_T gradf),$$

which implies that

$$\breve{\mathcal{L}}_{gradf}\, g + 2\breve{R}ic + 2\breve{\lambda} g = 0, \tag{7.3}$$

that is

$$g(\nabla_U gradf, T) = \breve{R}ic(U,T) + \breve{\lambda} g(U,T). \tag{7.4}$$

This reduces to

$$\nabla_U gradf = \breve{Q}U + \breve{\lambda}U. \tag{7.5}$$

Now from (7.5), we write

$$\begin{aligned} R_{cur}(U,T)gradf &= \nabla_U \nabla_T gradf - \nabla_U \nabla_T gradf - \nabla_{[U,T]} gradf \\ &= \nabla_U \breve{Q}T + U(\breve{\lambda})T - \breve{\lambda}\nabla_U T \\ &\quad -\nabla_T \breve{Q}U - T(\breve{\lambda})U - \breve{\lambda}\nabla_T U \\ &\quad -\breve{Q}[U,T] - \breve{\lambda}[U,T] \end{aligned}$$

which implies that

$$\begin{aligned} R_{cur}(U,T)gradf &= (\nabla_U Q)T - (\nabla_T Q)U - 2n\kappa(2g(U,\Psi T) \\ &\quad +\eta(T)\nabla_U \xi - \eta(U)\nabla_T \xi) \\ &\quad -\mu((\nabla_U h)T + (\nabla_T h)U) + U(\breve{\lambda})T - T(\breve{\lambda})U. \end{aligned} \tag{7.6}$$

Taking covariant derivative of Q given by (3.7), we have

$$\begin{aligned} (\nabla_U Q)\, T &= (2(n-1) + n(2\kappa - \mu)) \left[\begin{array}{c} g(U,\Psi T)\xi + g(\Psi hU, T)\xi \\ -\eta(T)\,(\Psi U - \Psi hU) \end{array} \right] \\ &\quad +(2(n-1)+\mu)(\nabla_U h)T. \end{aligned} \tag{7.7}$$

Using (7.7) and (2.2) in (7.6), we obtain

$$\begin{aligned} R_{cur}(U,T)gradf &= 2(2\kappa - n^2)g(U,\Psi T)\xi \\ &\quad +(n^2 + 2\kappa n - 2\kappa)(\eta(T)\Psi U - \eta(U)\Psi T) \\ &\quad -(n^2 - 2\mu n + 2\mu)(\eta(T)\Psi hU - \eta(U)\Psi hT) \\ &\quad +U(\breve{\lambda})T - T(\breve{\lambda})U, \end{aligned} \tag{7.8}$$

which implies that

$$g(R_{cur}(U,T)gradf, \xi) = 2(2\kappa - n^2)g(U,\Psi T) + U(\breve{\lambda})\eta(T) - T(\breve{\lambda})\eta(U). \tag{7.9}$$

If we put $U = \xi$ in the last equation, we get

$$g(R_{cur}(\xi,T)gradf, \xi) = \xi(\breve{\lambda})\eta(T) - T(\breve{\lambda}). \tag{7.10}$$

On the other hand, from (1.1) we have

$$g(R_{cur}(\xi,T)gradf, \xi) = \kappa g(T, gradf - \xi(f)\xi) + \mu g(hT, gradf). \tag{7.11}$$

Using (7.10) and (7.11), it follows that

$$\kappa(gradf) - \kappa\xi(f)\xi + \mu h(gradf) - \xi(\breve{\lambda})\xi + \; grad\breve{\lambda} = 0. \tag{7.12}$$

From (7.8), we get

$$Q(gradf) = -2n(gradf),$$

which infers

$$2n\kappa(gradf) + 2n\mu h(gradf) = Q(gradf) + 2n\left(\kappa\xi(f) + \xi(\breve{\lambda})\right)\xi, \tag{7.13}$$

via (7.12). Then, by using (3.8) and taking inner product of the last equation with ξ, we obtain

$$\kappa\xi(f) + \xi(\breve{\lambda}) = 0.$$

If we put this equation in (7.13), we get

$$2n\kappa(gradf) + 2n\mu h(gradf) = Q(gradf). \tag{7.14}$$

Taking $U=\xi$ in (7.5) and using (3.18), we obtain

$$\nabla_\xi gradf = \breve{\lambda}\xi.$$

By differentiating (7.14) with respect to ξ and using the last equation we have

$$\mu\left(\mu\left(1-2n\right)+2(n-1)\right)h\Psi gradf = 0,$$

which is equal to

$$\mu\left(\mu\left(1-2n\right)+2(n-1)\right)\Psi gradf = 0, \tag{7.15}$$

via (3.2). Also taking ΨU and ΨT instead of U and T, respectively, in (7.9) we write

$$g(R_{cur}(\Psi U,\Psi T)gradf,\xi) = (4\kappa-2n^2)g(\Psi U,T). \tag{7.16}$$

In a pcm (κ,μ)-manifold it is well known that$R_{cur}(\Psi U,\Psi T)\xi=0$. Then we obtain

$$(4\kappa-2n^2)g(\Psi U,T)=0.$$

Because of $d\eta$ is being non-zero, one gets

$$\kappa=\frac{n^2}{2}. \tag{7.17}$$

Hence, considering (7.15) and (7.17) we assume the following three cases:

Case 1. If $\mu=0$, then we can state that that the manifold is a $N(\kappa)$-pcm manifold.

Case 2. If $\Psi gradf=0$ and $\mu\neq 0$, then we write

$$\Psi^2 gradf = gradf-\eta(gradf)\xi=0,$$

that is

$$gradf=\xi(f)\xi. \tag{7.18}$$

By taking covariant derivative of the above equation along U, we have

$$\nabla_U gradf = U(\xi(f))\xi+\xi(f)\left(-\Psi U+\Psi hU\right). \tag{7.19}$$

If we replace U with ΨU and take inner product with ΨT in (7.19), we obtain

$$g(\nabla_{\Psi U} gradf,\Psi T) = -\xi(f)\left(g(U,\Psi T)+g(hU,\Psi T)\right), \tag{7.20}$$

which implies

$$g(\nabla_{\Psi T} gradf,\Psi U) = -\xi(f)\left(g(T,\Psi U)+g(hT,\Psi U)\right). \tag{7.21}$$

We know that $d^2f=0$ and so, for any vector fields U and T, we have $UT(f)-TU(f)-[U,T]f=0$. It follows that

$$Ug(gradf,T)-Tg(gradf,U)-g(gradf,[U,T])=0,$$

that is

$$\nabla_U(gradf,T)-g(gradf,\nabla_U T)-\nabla_T(gradf,U)-g(gradf,\nabla_T U)=0.$$

Since g is a metric connection then we have

$$g(\nabla_U gradf,T)=g(U,\nabla_T gradf). \tag{7.22}$$

By taking $U=\Psi U$ and $T=\Psi T$ in (7.22), we write

$$g(\nabla_{\Psi U} gradf,\Psi T)=g(\Psi U,\nabla_{\Psi T} gradf).$$

Then, from (7.20), (7.21) and the last equation above, we obtain

$$\xi(f)g(U,\Psi T)=0,$$

which infer $\xi(f)=0$, since $d\eta\neq 0$. From (7.18) we obtain $gradf=0$, that is, f is a constant. Therefore, from (7.5), we get $\breve{Ric}(U,T)=-\breve{\lambda}g(U,T)$, which implies that the manifold is an Einstein manifold with respect to the the S-vK connection. Furthermore, by using (3.17), we have

$$Ric(U,T) = -\breve{\lambda}g(U,T) + 2n\kappa\eta(U)\eta(T) + \mu g(hU,T). \tag{7.23}$$

By using (3.7) in (7.23), we have

$$Ric(U,T) = ag(U,T) + b\eta(U)\eta(T),$$

where $a = -\frac{\breve{\lambda}(2(n-1)+\mu)+\mu(2(1-n)+n\mu)}{2(n-1)}$ and $b = \frac{2n\kappa(2(n-1)+\mu)-\mu(2(n-1)+n(2\kappa-n))}{2(n-1)}$, which implies that the manifold is an η-Einstein manifold wrt the LC connection.

Case 3. If $\mu(1-2n) + 2(n-1) = 0$, then we obtain

$$\mu = \frac{2(n-1)}{2n-1}. \tag{7.24}$$

Using (7.14) and (3.7), we get

$$(2(1-n) + n\mu - 2n\kappa))\,(gradf - \xi(f)\xi) + (2(n-1) + \mu - 2n\mu)hgradf = 0. \tag{7.25}$$

Using (7.24) and (7.17) in (7.25), we conclude that $gradf = \xi(f)\xi$. So, we get the similar results given in Case 1.

Hence we give the following:

Theorem 7.1. *Let (M,g) be a $(2n+1)$-dimensional $(n>1)$ pcm (κ,μ)-manifold ($\kappa \neq -1$) bearing an almost gradient Ricci soliton wrt the S-vK connection. Then either the manifold is a $N(\kappa)$-pcm manifold, or it is an Einstein manifold wrt the S-vK connection (equivalently, it is an η-Einstein manifold wrt the LC connection).*

8. Almost Yamabe solitons on pcm (κ,μ)-manifolds with respect to the Schouten-van Kampen connection

In this section we study almost Yamabe solitons on a pcm (κ,μ)-manifold $(\kappa \neq -1)$ wrt the S-vK connection. Assume that $(M, V, \breve{\delta}, g)$ is an almost Yamabe soliton on a pcm (κ,μ)-manifold wrt the S-vK connection. Then we write

$$\frac{1}{2}(\mathcal{L}_V g)(U,T) = (\breve{r} - \breve{\delta})g(U,T). \tag{8.1}$$

From (3.19), we write

$$\frac{1}{2}(\mathcal{L}_V g)(U,T) = (r - 2n\kappa - \breve{\delta})g(U,T). \tag{8.2}$$

Hence, we state the following:

Theorem 8.1. *An almost Yamabe soliton (M,V,δ,g) on a $(2n+1)$-dimensional pcm (κ,μ)-manifold with $\kappa \neq -1$ is invariant under the S-vK connection if and only if the manifold is a para-Sasakian manifold.*

For $V = \xi$ in (8.2), we get

$$g(U, \Psi hT) = (r - 2n\kappa - \breve{\delta})g(U,T). \tag{8.3}$$

So we give the followings:

Theorem 8.2. *Let M be a $(2n+1)$-dimensional pcm (κ,μ)-manifold $(\kappa \neq -1)$ bearing a Yamabe soliton $(\xi, \breve{\delta}, g)$ wrt the S-vK connection. Then, M is of constant scalar curvature $2n\kappa + \breve{\delta}$ wrt the LC connection.*

Corollary 8.3. *An almost Yamabe soliton $(\xi, \breve{\delta}, g)$ on a $(2n+1)$-dimensional pcm (κ,μ)-manifold $(\kappa \neq -1)$ wrt the S-vK connection is steady if $r = 2n\kappa$.*

We conclude with an example of pcm (κ,μ)-manifold wrt the S-vK connection such that $\kappa < -1$.

Example 8.4. Let g be the Lie algebra endowed with a basis $\{E_1, E_2, E_3, E_4, E_5\}$ and non-zero Lie brackets

$$\begin{array}{lcll} [E_1,E_5] &=& \alpha\beta E_1+\alpha\beta E_2, & [E_2,E_5]=\alpha\beta E_1+\alpha\beta E_2,\\ [E_3,E_5] &=& -\alpha\beta E_3+\alpha\beta E_4, & [E_4,E_5]=\alpha\beta E_3-\alpha\beta E_4,\\ [E_1,E_2] &=& \alpha E_1+\alpha E_2, & [E_1,E_3]=\beta E_2+\alpha E_4-2E_5,\\ [E_1,E_4] &=& \beta E_2+\alpha E_3, & [E_2,E_3]=\beta E_1-\alpha E_4,\\ [E_2,E_4] &=& \beta E_1-\alpha E_3+2E_5, & [E_3,E_4]=-\beta E_3+\beta E_4, \end{array} \tag{8.4}$$

where α, β are non-zero real numbers such that $\alpha\beta > 0$. Let G be a Lie group whose Lie algebra is g. Define on G a left invariant pcm structure (Ψ, ξ, η, g) by imposing that, at the identity, $g(E_1, E_1) = g(E_4, E_4) = -g(E_2, E_2) = -g(E_3, E_3) = g(E_5, E_5) = 1$, $g(E_i, E_j) = 0$, for any $i \neq j$, and $\Psi E_1 = E_3, \Psi E_2 = E_4, \Psi E_3 = E_1, \Psi E_4 = E_2, \Psi E_5 = 0, \xi = E_5$ and $\eta = g(\cdot, E_5)$. A very long but straightforward computation shows that

$$\begin{array}{rcll} \nabla_{E_1}\xi &=& \alpha\beta E_1-\Psi E_1, & \nabla_{E_2}\xi=\alpha\beta E_2-\Psi E_2,\\ \nabla_{\Psi E_1}\xi &=& -E_1-\alpha\beta\Psi E_1, & \nabla_{\Psi E_2}\xi=-E_2-\alpha\beta\Psi E_2,\\ \nabla_{\xi}E_1 &=& -\alpha\beta E_2-\Psi E_1, & \nabla_{\xi}E_2=-\alpha\beta E_1-\Psi E_2,\\ \nabla_{\xi}\Psi E_1 &=& -E_1-\alpha\beta\Psi E_2, & \nabla_{\xi}\Psi E_2=-E_2-\alpha\beta\Psi E_1,\\ \nabla_{E_1}E_1 &=& \alpha E_2-\alpha\beta E_5, & \nabla_{E_1}E_2=\alpha E_1,\\ \nabla_{E_1}\Psi E_1 &=& \alpha\Psi E_2-E_5, & \nabla_{E_1}\Psi E_2=\alpha\Psi E_1,\\ \nabla_{E_2}E_1 &=& \alpha E_2, & \nabla_{E_2}E_2=-\alpha E_1+\alpha\beta E_5,\\ \nabla_{E_2}\Psi E_1 &=& -\alpha\Psi E_2, & \nabla_{E_2}\Psi E_2=-\alpha\Psi E_1+E_5,\\ \nabla_{\Psi E_1}E_1 &=& -\beta E_2+E_5, & \nabla_{\Psi E_1}E_2=-\beta E_1,\\ \nabla_{\Psi E_1}\Psi E_1 &=& -\beta\Psi E_2-\alpha\beta E_5, & \nabla_{\Psi E_1}\Psi E_2=-\beta\Psi E_1,\\ \nabla_{\Psi E_2}E_1 &=& -\beta E_2, & \nabla_{\Psi E_2}E_2=-\beta E_1-E_5,\\ \nabla_{\Psi E_2}\Psi E_1 &=& -\beta\Psi E_2, & \nabla_{\Psi E_2}\Psi E_2=-\beta\Psi E_1+\alpha\beta E_5, \end{array} \tag{8.5}$$

where $\lambda = \alpha\beta$ and $\mu = 2$. Then one can prove that the curvature tensor field of the LC connection of (G, g) satisfies that (κ, μ)-nullity condition (1.1), with $\kappa = -1 - (\alpha\beta)^2$ and $\mu = 2$, which implies that (G, Ψ, ξ, η, g) is a 5-dimensional pcm (κ, μ)-manifold [6]. Now we shall construct the S-vK connection on (G, Ψ, ξ, η, g). Using (8.5), we get

$$\begin{array}{rcllll} \breve{\nabla}_{E_1}E_1 &=& \alpha E_2, & \breve{\nabla}_{E_1}E_2=\alpha E_1, & \breve{\nabla}_{E_1}E_3=\alpha E_4, &\\ \breve{\nabla}_{E_1}E_4 &=& \alpha E_3, & \breve{\nabla}_{E_2}E_1=-\alpha E_2, & \breve{\nabla}_{E_2}E_2=-\alpha E_1, &\\ \breve{\nabla}_{E_2}E_3 &=& -\alpha E_4, & \breve{\nabla}_{E_2}E_4=-\alpha E_3, & \breve{\nabla}_{E_3}E_1=-\beta E_2, &\\ \breve{\nabla}_{E_3}E_2 &=& -\beta E_1, & \breve{\nabla}_{E_3}E_3=-\beta E_4, & \breve{\nabla}_{E_3}E_4=-\beta E_3, &\\ \breve{\nabla}_{E_4}E_1 &=& -\beta E_2, & \breve{\nabla}_{E_4}E_2=-\beta E_1, & \breve{\nabla}_{E_4}E_3=-\beta E_4, &\\ \breve{\nabla}_{E_4}E_4 &=& -\beta E_3, & \breve{\nabla}_{E_5}E_1=-\alpha\beta E_2-E_3, & &\\ \breve{\nabla}_{E_5}E_2 &=& -\alpha\beta E_1-E_4, & \breve{\nabla}_{E_5}E_3=-E_1-\alpha\beta E_4, & \breve{\nabla}_{E_5}E_4=-E_2-\alpha\beta E_3. & \end{array} \tag{8.6}$$

Now using (8.6), we can calculate the non-zero components of its curvature tensor wrt the S-vK connection as follows:

$$\begin{aligned}
\breve{R}_{cur}(E_1, E_3)E_1 &= -2E_3, & \breve{R}_{cur}(E_1, E_3)E_2 &= -2E_4,\\
\breve{R}_{cur}(E_1, E_3)E_3 &= -2E_1, & \breve{R}_{cur}(E_1, E_3)E_4 &= -2E_2,\\
\breve{R}_{cur}(E_1, E_4)E_1 &= 2\alpha\beta E_2, & \breve{R}_{cur}(E_1, E_4)E_2 &= 2\alpha\beta E_1,\\
\breve{R}_{cur}(E_1, E_4)E_3 &= 2\alpha\beta E_4, & \breve{R}_{cur}(E_1, E_4)E_4 &= 2\alpha\beta E_3,\\
\breve{R}_{cur}(E_2, E_3)E_1 &= -2\alpha\beta E_2, & \breve{R}_{cur}(E_2, E_3)E_2 &= -2\alpha\beta E_1,\\
\breve{R}_{cur}(E_2, E_3)E_3 &= -2\alpha\beta E_4, & \breve{R}_{cur}(E_2, E_3)E_4 &= -2\alpha\beta E_3,\\
\breve{R}_{cur}(E_2, E_4)E_1 &= 2E_3, & \breve{R}_{cur}(E_2, E_4)E_2 &= 2E_4,\\
\breve{R}_{cur}(E_2, E_4)E_3 &= 2E_1, & \breve{R}_{cur}(E_2, E_4)E_4 &= 2E_2,
\end{aligned} \tag{8.7}$$

which imply that the non-zero components of its Ricci tensor wrt the S-vK connection as follows:

$$\breve{R}ic(E_1, E_1) = \breve{R}ic(E_4, E_4) = 2, \quad \breve{R}ic(E_2, E_2) = \breve{R}ic(E_3, E_3) = -2. \tag{8.8}$$

From (8.8), (6.2) and (6.9), one can see that there does not exist an almost Ricci soliton on such a 5-dimensional pcm (κ, μ)-manifold with $\kappa < -1$.

Furthermore, for $U = u_1E_1 + u_2E_2 + u_3E_3 + u_4E_4 + u_5E_5$, $T = t_1E_1 + t_2E_2 + t_3E_3 + t_4E_4 + t_5E_5 \in \chi(G)$, we have

$$g(U, \Psi hT) = \alpha\beta(u_1t_1 - u_2t_2 + u_3t_3 - u_4t_4).$$

By using the last equation in (8.3), we say that the 5-dimensional pcm (κ, μ)-manifold $\breve{G}$ admits a Yamabe soliton $(\xi, 8 - \alpha\beta, g)$ wrt the S-vK connection. Such a Yamabe soliton is expanding if $\alpha\beta > 8$, steady if $\alpha\beta = 8$ and shrinking if $\alpha\beta < 8$.

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