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k-FREE NUMBERS AND INTEGER PARTS OF αp

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Abstract. In this note, we obtain asymptotic results on integer parts of αp that are free of kth powers of primes, where p is a prime number and α is a positive real number.

1. Introduction and Statement of Results

Let α and β be real numbers such that $\alpha > 0$. Let $\lfloor x \rfloor$ denote the largest integer not greater than x. Sequences of the form $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ are called Beatty sequences. A Beatty sequence is said to be homogeneous if $\beta = 0$. Beatty sequences have been attracting a lot of attention since they can be viewed as analogues of arithmetic progressions, therefore they show up in a broad context. The interested reader is referred to [1, 2, 4–6, 8–11, 14–16, 19, 24].

Let $k \geqslant 2$ be an integer. An integer is said to be *k-free* if it is not divisible by a kth power of a prime. Very recently in [3], an asymptotic formula with an explicit error term is obtained for k-free values of homogeneous Beatty sequences at prime arguments (i.e. sequences of the form $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$) provided that α is of finite type (see Definition 1). This result can be viewed as a natural analogue of the result of Mirsky [20]. In this article, we pursue this result and obtain two asymptotic formulas that are of the same flavour. The results we present here are well applicable to non-homogeneous Beatty sequences.

Theorem 1. *Let $k \geq 2$ be an integer. Let $\{\alpha_i\}_{i=1}^{\ell}$ be a finite type subset of irrational numbers each greater than one. Assume that $\{\alpha_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau > 0$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ and*

$$\pi(x, k, \boldsymbol{\alpha}) = \#\{p \leqslant x : \lfloor \alpha_i p \rfloor \text{ is } k\text{-free for each } i = 1, \dots, \ell\}.$$

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Then the following asymptotic is satisfied:

$$\pi(x,k,\boldsymbol{\alpha}) = \frac{\pi(x)}{\zeta^{\ell}(k)} + O\left(x^{1-\frac{k-1}{(k-1+\ell)(3\tau+2)+k(\ell-1)\tau+k\ell}} e^{\frac{C\log x}{\log\log x}}\right)$$

for some constant $C = C(\alpha_1,\dots,\alpha_\ell)$ *and every large* x.

A nested version of Theorem 1 is given below.

Theorem 2. *Let* $k \geqslant 2$ *be an integer. Let* $\{\alpha_1\alpha_2, \alpha_2\}$ *be a finite type subset of irrational numbers each greater than zero. Then the following asymptotic is satisfied:*

$$\#\{p \leqslant x : \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \text{ is } k\text{-free}\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon})$$

for some $\varepsilon > 0$.

Here, the interested reader is invited to investigate the following problem: Let $\{\alpha_i\}_{i=1}^n$ be positive real numbers. Define

$$a_j = \prod_{i=1}^{j} \alpha_{n+1-i}.$$

Assuming that $\{a_1, a_2, \dots, a_n\}$ is of finite type (see Definition 1), show that

$$\#\{p \leqslant x : \lfloor a_n \lfloor a_{n-1} \cdots \lfloor a_1 p \rfloor \rfloor \rfloor \text{ is } k\text{-free}\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon})$$

for some $\varepsilon > 0$. It might also be fruitful to investigate the possible power saving in the error term above.

1.1. Preliminaries and Notation.

1.1.1. *Notation.* We recall that for functions F and G where G is real non-negative, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. Further we use $F \sim G$ to indicate $(F/G)(x)$ tends to 1 as $x \to \infty$.

Given a real number x, we use the notation $\{x\}$ for the fractional part of x, the notation $\lfloor x \rfloor$ for the greatest integer not exceeding x and $e(x) = e^{2\pi i x}$.

We use $\|x\|$ to denote the distance from the real number x to the nearest integer. $\Lambda(n) = \log p$ if $n = p^r$ where p is a prime number (here and hereafter). Otherwise, $\Lambda(n) = 0$. $\mu(n)$ denotes the Mobius function. $\phi(n)$ denotes the Euler's totient function. $\tau(n)$ denotes the number of positive divisors of n. We also use $\pi(x)$ to denote the number of primes not more than x.

1.1.2. *Preliminaries.*

Definition 1. *An irrational number α is called of finite type τ, if*

$$\tau = \sup\left\{\beta : \liminf_{\substack{q\to\infty\\ q\in\mathbb{N}}} q^{\beta}||\alpha q|| = 0\right\} < \infty.$$

If α is an irrational number of finite type τ, then by Dirichlet's approximation theorem (Lemma 2.1 of [25]) one has $\tau \geqslant 1$. The celebrated theorems of Khinchin [17] and of Roth [21,22] state that $\tau = 1$ for almost all (in the sense of the Lebesque measure) real numbers and for all irrational algebraic numbers respectively.

Definition 2. *A finite subset of real numbers $\{\beta_1, \beta_2, \ldots, \beta_\ell\}$ is said to be of finite type if there is $\tau > 0$ such that the inequality*

$$||h_1\beta_1 + h_2\beta_2 + \cdots + h_\ell\beta_\ell|| < (\max\{1, |h_1|, \ldots, |h_\ell|\})^{-\tau} \tag{1}$$

has only finitely many solutions for $h_i \in \mathbb{Z}$.

If $\{\beta_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau > 0$, then it follows from Dirichlet's theorem on rational approximations that $\tau \geqslant 1$. Furthermore, such a set is linearly independent over $\mathbb{Q}$.

Throughout this paper, we shall mostly use the weak form of the prime number theorem, that is

$$\pi(x) \sim \frac{x}{\log x}.$$

Lemma 1. *For every positive integer $n \geq 1$,*

$$\tau(n) < e^{\frac{C\log 5n}{\log\log 5n}}$$

for some constant $C > 0$.

Proof. Follows from [23, Theorem 2.11]. □

Lemma 2. *If*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$$

for some integers a and q such that $(a, q) = 1$, then

$$\sum_{p\leqslant x} e(\alpha p) \ll x\log^3 x\left(q^{-\frac{1}{2}} + x^{-\frac{1}{5}} + q^{\frac{1}{2}}x^{-\frac{1}{2}}\right).$$

Proof. This follows in a standard way using the main result of [12, §25]. □

Lemma 3 (Erdős-Turán-Koksma Inequality)**.** *If $\{\boldsymbol{x}_i\}_{i=1}^{N}$ is a finite sequence in $\mathbb{R}^{\ell}$, then for any $J \subseteq [0,1)^{\ell}$ that is a Cartesian product of subintervals of $[0,1)$ and any $H \geqslant 1$, we have*

$$\#\{1 \leqslant i \leqslant N : \ \boldsymbol{x}_i \in J \mod 1\} - |J|N \ll \frac{N}{H} + \sum_{0<||\boldsymbol{h}||\leqslant H} \frac{1}{r(\boldsymbol{h})}\left|\sum_{1\leqslant i\leqslant N} e(\langle \boldsymbol{h}, \boldsymbol{x}_i\rangle)\right|.$$

Here $|J|$ denotes the ℓ-dimensional Lebesgue measure of J, $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^\ell$ and we set $||\boldsymbol{h}|| = \max_{1\leqslant i\leqslant \ell}\{|h_i|\}$ and

$$r(\boldsymbol{h}) = \prod_{i=1}^{\ell} \max\{|h_i|, 1\} \tag{2}$$

for all $\boldsymbol{h} = (h_1, h_2, \ldots, h_\ell) \in \mathbb{Z}^\ell$. Moreover, the implied constant depends only on ℓ.

Proof. For the proof see [18]. □

The following lemma is a classical result due to Vinogradov [26, Lemma 12].

Lemma 4. *Let α, β and Δ be real numbers such that*

$$0 < \Delta < \frac{1}{2} \quad \textit{and} \quad \Delta \leqslant \beta - \alpha \leqslant 1 - \Delta.$$

Then there exists a periodic function $\Psi(x)$, with period 1, satisfying

(i) $\Psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \leqslant x \leqslant \beta - \frac{1}{2}\Delta$,

(ii) $\Psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \leqslant x \leqslant 1 + \alpha - \frac{1}{2}\Delta$,

(iii) $0 \leqslant \Psi(x) \leqslant 1$ *in the remainder of the interval* $\alpha - \frac{1}{2}\Delta \leqslant x \leqslant 1 + \alpha - \frac{1}{2}\Delta$,

(iv) $\Psi(x)$ *has a Fourier expansion of the form*

$$\Psi(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),$$

where

$$|a_h| \leqslant c \cdot \min\left\{|h|^{-1}, |h|^{-2}\Delta^{-1}\right\}$$

for every $|h| \geqslant 1$ and some c fixed. Furthermore, $a_0 = \beta - \alpha$.

2. Proof of The Main Results

2.1. **Proof of Theorem 1.** Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and

$$\pi(x, k, \boldsymbol{\alpha}) = \#\{p \leqslant x : \lfloor \alpha_i p \rfloor \text{ is } k\text{-free for each } i = 1, \ldots, \ell\}.$$

Let $\mathcal{I}_k$ denote the characteristic function of k-free integers. Since

$$\mathcal{I}_k(n) = \sum_{d^k | n} \mu(d), \tag{3}$$

we have

$$
\begin{aligned}
&\pi(x,k,\boldsymbol{\alpha})\\
&\quad= \sum_{p\leqslant x}\mathcal{I}_k(\lfloor\alpha_1 p\rfloor)\cdots\mathcal{I}_k(\lfloor\alpha_\ell p\rfloor)\\
&\quad= \sum_{p\leqslant x}\left(\sum_{d_1^k\mid\lfloor\alpha_1 p\rfloor}\mu(d_1)\right)\cdots\left(\sum_{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}\mu(d_\ell)\right)\\
&\quad= \sum_{p\leqslant x}\sum_{\substack{(d_1,\dots,d_\ell)\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\\
&\quad= \sum_{(d_1,\dots,d_\ell)}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1\\
&\quad= \sum_{\substack{(d_1,\dots,d_\ell)\\ d_i\leqslant z\\ i=1,\dots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1 \;+\; \sum_{\substack{(d_1,\dots,d_\ell)\\ d_i>z\\ \text{for some } i=1,\dots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1,
\end{aligned}
$$

where $z \leqslant x^{1/k}$ will be chosen later. It follows from Lemma 1 that for all $i = 1, 2, \dots, \ell$ there exists a positive constant $c_i = c_i(\alpha_i)$ depending on α_i such that

$$\tau(\lfloor\alpha_i p\rfloor) \ll e^{\frac{c_i \log x}{\log\log x}}$$

for every $p \leqslant x$. Then, for all $i = 1, 2, \dots, \ell$ and $p \le x$

$$\tau(\lfloor\alpha_i p\rfloor) \ll e^{\frac{c \log x}{\log\log x}}, \tag{4}$$

where $c = \max\{c_1, \dots, c_\ell\}$. Set $C = c(\ell - 1)$. Then, by (4) and using partial summation in the last step, we get

$$
\begin{aligned}
&\sum_{\substack{(d_1,\dots,d_\ell)\\ d_i>z\\ \text{for some } i=1,\dots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1\\
&\quad< \sum_{\substack{(d_1,\dots,d_\ell)\\ d_1>z}}\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1+\cdots+\sum_{\substack{(d_1,\dots,d_\ell)\\ d_\ell>z}}\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1\\
&\quad= \sum_{p\leqslant x}\left(\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}}1\right)\cdots\left(\sum_{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}1\right)+\cdots+\sum_{p\leqslant x}\left(\sum_{d_1^k\mid\lfloor\alpha_1 p\rfloor}1\right)\cdots\left(\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}}1\right)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \sum_{p\leqslant x}\left(\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}} 1\right)\left(\prod_{i=2}^{\ell}\tau(\lfloor\alpha_i p\rfloor)\right)+\cdots+\sum_{p\leqslant x}\left(\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}} 1\right)\left(\prod_{i=1}^{\ell-1}\tau(\lfloor\alpha_i p\rfloor)\right)\\
&\ll e^{\frac{C\log x}{\log\log x}}\left(\sum_{p\leqslant x}\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}} 1+\cdots+\sum_{p\leqslant x}\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}} 1\right)\\
&\ll e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\sum_{\substack{p\leqslant x\\ d_1^k\mid\lfloor\alpha_1 p\rfloor}} 1+\cdots+\sum_{d_\ell>z}\sum_{\substack{p\leqslant x\\ d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}} 1\right)\\
&\leqslant e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\sum_{\substack{m\leqslant\alpha_1 x\\ d_1^k\mid m}} 1+\cdots+\sum_{d_\ell>z}\sum_{\substack{m\leqslant\alpha_\ell x\\ d_\ell^k\mid m}} 1\right)\\
&\leqslant e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\frac{\alpha_1 x}{d_1^k}+\cdots+\sum_{d_\ell>z}\frac{\alpha_\ell x}{d_\ell^k}\right)\ll\frac{e^{\frac{C\log x}{\log\log x}}x}{z^{k-1}}.
\end{aligned}$$

Therefore,

$$\pi(x,k,\boldsymbol{\alpha})=\sum_{\substack{(d_1,\dots,d_\ell)\\ d_i\leqslant z\\ i=1,\dots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}} 1+O\left(\frac{e^{\frac{C\log x}{\log\log x}}x}{z^{k-1}}\right). \tag{5}$$

Next, we will study the sum above appearing in (5) which runs over all tuples $(d_1,\dots,d_\ell)$ of positive integers where $d_i\leqslant z$ for all $i=1,\dots,\ell$.
So, let $\mathbf{d}=(d_1,\dots,d_\ell)$ be such a tuple and set

$$D=\prod_{j=1}^{\ell}d_j^k,\qquad D_i=\prod_{\substack{j=1\\ j\neq i}}^{\ell}d_j^k\quad\text{and}\quad\mathcal{I}_{\mathbf{d}}=\left[0,\frac{1}{d_1^k}\right)\times\cdots\times\left[0,\frac{1}{d_\ell^k}\right) \tag{6}$$

for all $i=1,\dots,\ell$. For a positive integer i, let p_i denote the ith prime. Observing that

$$\lfloor\alpha p\rfloor\equiv 0\pmod d\text{ if and only if }\left\{\frac{\alpha p}{d}\right\}<\frac{1}{d}, \tag{7}$$

we have

$$\sum_{\substack{p\leqslant x\\ d_i^k \mid \lfloor \alpha_i p\rfloor\\ i=1,\dots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \lfloor \alpha_i p\rfloor \equiv 0 \pmod{d_i^k}\\ i=1,\dots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \left\{\frac{\alpha_i p}{d_i^k}\right\}<\frac{1}{d_i^k}\\ i=1,\dots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \left(\left\{\frac{\alpha_1 p}{d_1^k}\right\},\dots,\left\{\frac{\alpha_\ell p}{d_\ell^k}\right\}\right)\in\mathcal{I}_{\mathbf{d}}}} 1 \tag{8}$$
$$= \#\{i\leqslant \pi(x) : \mathbf{t}_i\in\mathcal{I}_{\mathbf{d}}\},$$

where

$$\mathbf{t}_i = \left(\left\{\frac{\alpha_1 p_i}{d_1^k}\right\},\dots,\left\{\frac{\alpha_\ell p_i}{d_\ell^k}\right\}\right).$$

It follows from Erdős-Turán-Koksma Inequality that for all $H\geqslant 1$,

$$\begin{aligned}
&\#\{i\leqslant \pi(x) : \mathbf{t}_i\in\mathcal{I}_{\mathbf{d}}\} - \frac{\pi(x)}{d_1^k\cdots d_\ell^k}\\
&\ll \frac{\pi(x)}{H} + \sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\left|\sum_{i\leqslant\pi(x)} e\left(\langle \mathbf{h},\mathbf{t}_i\rangle\right)\right| \\
&\ll \frac{\pi(x)}{H} + \sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\left|\sum_{p\leqslant x} e\left(\frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}\cdot p\right)\right|.
\end{aligned}\tag{9}$$

Next, we shall prove the following lemma.

Lemma 5.

$$\begin{aligned}
&\sum_{p\leqslant x} e\left(\frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}\cdot p\right)\\
&\qquad \ll x\log^3 x\left(x^{-\frac{1}{2(\tau+1)}}\left(\max\{|h_1|D_1,\dots,|h_\ell|D_\ell\}\right)^{\frac{\tau}{2(\tau+1)}}D^{\frac{1}{2(\tau+1)}} + x^{-\frac15}\right)
\end{aligned}$$

uniformly for all $\mathbf{h}=(h_1,\dots,h_\ell)\in\mathbb{Z}^\ell$ *such that* $||\mathbf{h}||>0$*, where* D_i *and* D *are defined in* (6)*.*

Proof. Since $\{\alpha_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau>0$, there exists a positive constant $A\geq 1$ such that

$$(\max\{|h_1|,\dots,|h_\ell|\})^{-\tau}\leqslant A\ ||h_1\alpha_1+h_2\alpha_2+\cdots+h_\ell\alpha_\ell|| \tag{10}$$

for all $(h_1,\dots,h_\ell)\in\mathbb{Z}^\ell$ such that $\max_{1\leqslant i\leqslant \ell}\{|h_i|\}>0$. Let $\mathbf{h}=(h_1,\dots,h_\ell)\in\mathbb{Z}^\ell$ be such a tuple and set

$$m_{\mathbf{h}} = \frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}.$$

Let $1\leqslant Q<x/2$ to be determined later. By Dirichlet's rational approximation theorem, there exists $\dfrac{r}{q}\in\mathbb{Q}$ such that $1\leqslant q\leqslant \dfrac{x}{Q}$ and

$$\left|m_{\mathbf{h}}-\frac{r}{q}\right|<\frac{Q}{qx}.$$

So,

$$\|q(h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell)\|<\frac{QD}{x}. \tag{11}$$

On the other hand, it follows from (10) that

$$\|q(h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell)\|\geqslant A^{-1}q^{-\tau}(\max\{|h_1D_1|,\dots,|h_\ell D_\ell|\})^{-\tau}. \tag{12}$$

Combining (11) and (12), we get

$$q\geqslant\frac{x^{\frac{1}{\tau}}}{\max\{|h_1D_1|,\dots,|h_\ell D_\ell|\}A^{\frac{1}{\tau}}D^{\frac{1}{\tau}}Q^{\frac{1}{\tau}}}. \tag{13}$$

Then it follows from Lemma 2 that

$$\sum_{p\leqslant x}e\left(m_{\mathbf{h}}\cdot p\right)\ll x\log^3 x\left(x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}Q^{\frac{1}{2\tau}}+x^{-\frac{1}{5}}+Q^{-\frac{1}{2}}\right), \tag{14}$$

where for the sake of brevity we set $M=\max\{|h_1D_1|,\dots,|h_\ell D_\ell|\}$. By [13, Lemma 2.4], there exists $1\leqslant Q<x/2$ such that the left hand side of (14) is

$$\ll x\log^3 x\left(x^{-\frac{1}{2(\tau+1)}}M^{\frac{\tau}{2(\tau+1)}}D^{\frac{1}{2(\tau+1)}}+x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}+x^{-\frac{1}{5}}\right).$$

At this point, we can assume that $x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}<1$, because otherwise the required upper bound holds trivially. Therefore, the second term is beaten by the first term giving the proof of Lemma 5. $\square$

We next proceed by plugging this upper bound into (9). We also use the upper bound $|h_i|\leqslant H$ together with the upper bounds $D\leqslant z^{k\ell}$ and $D_i\leqslant z^{k(\ell-1)}$. Then the difference in the first line of (9) is

$$\ll\frac{\pi(x)}{H}+\left(x^{1-\frac{1}{2(\tau+1)}}H^{\frac{\tau}{2(\tau+1)}}z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}}\log^3 x+x^{\frac{4}{5}}\log^3 x\right)\left(\sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\right). \tag{15}$$

Now, by (2)

$$\sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\leqslant\sum_{0\leqslant\|\mathbf{h}\|\leqslant H}\frac{1}{\prod_{i=1}^{\ell}\left(\max\{|h_i|,1\}\right)}\leqslant\left(1+2\sum_{1\leqslant h\leqslant H}\frac{1}{h}\right)^{\ell}\ll\log^\ell H, \tag{16}$$

where in the last step we use integral test. Here we note that the implied constant depends on ℓ. Coupling (8), (9), (15) and (16), we arrive at

$$\left(\sum_{\substack{p\leqslant x\\ d_i^k|\lfloor\alpha_i p\rfloor\\ i=1,\dots,\ell}}1\right)-\frac{\pi(x)}{d_1^k\cdots d_\ell^k}$$

$$\ll \frac{\pi(x)}{H} + x^{1-\frac{1}{2(\tau+1)}} H^{\frac{\tau}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}} \log^{\ell} H \log^3 x + x^{\frac{4}{5}} \log^{\ell} H \log^3 x \quad (17)$$

for every $H \geqslant 1$ and every $(d_1, \ldots, d_\ell)$ such that $d_i \leqslant z \leqslant x^{1/k}$ for each i. Noting $\pi(x) \ll x$ and choosing $1 \leqslant H \leqslant x$ by [13, Lemma 2.4], the left hand side of (17) is

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}} + x^{\frac{4}{5}} \right).$$

On summing this over all tuples $(d_1, \ldots, d_\ell)$ of positive integers where $d_i \leqslant z$ for all $i = 1, \ldots, \ell$, we observe from (5) that for all $1 \leqslant z \leqslant x^{1/k}$,

$$\pi(x, k, \boldsymbol{\alpha}) - \pi(x) \sum_{\substack{(d_1,\ldots,d_\ell) \\ d_i \leqslant z \\ i=1,\ldots,\ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{d_1^k \cdots d_\ell^k}$$

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}+\ell} + x^{\frac{4}{5}} z^{\ell} \right) + \frac{e^{\frac{C \log x}{\log\log x}} x}{z^{k-1}}.$$

Here,

$$\sum_{\substack{(d_1,\ldots,d_\ell) \\ d_i \leqslant z \\ i=1,\ldots,\ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{d_1^k \cdots d_\ell^k} = \left(\sum_{d \leqslant z} \frac{\mu(d)}{d^k} \right)^{\ell}$$

and using the following inequality

$$\left| \sum_{d \leqslant z} \frac{\mu(d)}{d^k} - \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \right| \leqslant \sum_{d > z} \frac{1}{d^k} \ll \frac{1}{z^{k-1}},$$

it follows by the mean value theorem that

$$\left(\sum_{d \leqslant z} \frac{\mu(d)}{d^k} \right)^{\ell} - \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \right)^{\ell} \ll \frac{1}{z^{k-1}}.$$

Therefore, the contribution of the sums running over $d_i \leqslant z$ for all $i = 1, \ldots, \ell$ is

$$\frac{\pi(x)}{\zeta^{\ell}(k)} + O\left(\frac{\pi(x)}{z^{k-1}} \right)$$

yielding for all $1 \leqslant z \leqslant x^{1/k}$

$$\pi(x, k, \boldsymbol{\alpha}) - \frac{\pi(x)}{\zeta^{\ell}(k)}$$

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}+\ell} + x^{\frac{4}{5}} z^{\ell} \right) + \frac{e^{\frac{C \log x}{\log\log x}} x}{z^{k-1}}, \quad (18)$$

where $C = C(\ell, \boldsymbol{\alpha})$ is positive. On the right hand side of (18), the first term beats the third term as $\tau \geq 1$ and the second term whenever

$$z \leqslant x^{\frac{1}{k(\ell-1)\tau+k\ell}}$$

which one can assume since otherwise (18) holds trivially. Using now [13, Lemma 2.4] to choose optimal $z \leqslant x^{1/k}$, the left hand side of (18) is

$$\begin{aligned}
&\ll e^{\frac{C' \log x}{\log\log x}} \left(x^{1-\frac{1}{3\tau+2}} + x^{\frac{1}{k}} + x^{\frac{(k-1)(3\tau+1)+k(\ell-1)\tau+k\ell+\ell(3\tau+2)}{(k-1)(3\tau+2)+k(\ell-1)\tau+k\ell+\ell(3\tau+2)}} \right) \\
&\ll x^{1-\frac{k-1}{(k-1+\ell)(3\tau+2)+k(\ell-1)\tau+k\ell}} e^{\frac{C' \log x}{\log\log x}}
\end{aligned}$$

for some constant C' depending on ℓ and $\boldsymbol{\alpha}$, therefore the claim follows.

2.2. **Proof of Theorem 2.** The proof will be similar to that of Theorem 1. We shall therefore be brief. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and define

$$\pi_{\boldsymbol{\alpha}}(x,k) = \#\{p \leqslant x : \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \text{ is } k\text{-free}\}.$$

Let $1 \leqslant z \leqslant x^{1/k}$ be a number to be determined. Using (3), it follows that

$$\pi_{\boldsymbol{\alpha}}(x,k) = \sum_{p \leqslant x} \sum_{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor} \mu(d) = \sum_{p \leqslant x} \sum_{\substack{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) + \sum_{p \leqslant x} \sum_{\substack{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d > z}} \mu(d).$$

As we did before, we have

$$\sum_{p \leqslant x} \sum_{\substack{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d > z}} \mu(d) \ll \frac{x}{z^{k-1}},$$

where the implied constant depends only on α_1 and α_2. This yields

$$\pi_{\boldsymbol{\alpha}}(x,k) = \sum_{p \leqslant x} \sum_{\substack{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) + O\left(\frac{x}{z^{k-1}}\right).$$

We now proceed to derive the main term. Writing

$$\sum_{p \leqslant x} \sum_{\substack{d^k | \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) = \sum_{d \leqslant z} \mu(d) \left(\left(\sum_{\substack{p \leqslant x \\ \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \equiv 0 \pmod{d^k}}} 1 \right) - \frac{\pi(x)}{d^k} \right) + \pi(x) \sum_{d \leqslant z} \frac{\mu(d)}{d^k},$$

and using partial summation to get

$$\sum_{d \leqslant z} \frac{\mu(d)}{d^k} = \frac{1}{\zeta(k)} + O\left(\frac{1}{z^{k-1}}\right),$$

one arrives at

$$\pi_{\boldsymbol{\alpha}}(x,k) = \frac{\pi(x)}{\zeta(k)} + O\left(\frac{x}{z^{k-1}} + \sum_{d\leqslant z}\left|\left(\sum_{\substack{p\leqslant x\\ \lfloor\alpha_1\lfloor\alpha_2 p\rfloor\rfloor\equiv 0 \pmod{d^k}}} 1\right) - \frac{\pi(x)}{d^k}\right|\right) \tag{19}$$

for any $1\leqslant z\leqslant x^{1/k}$. Let us now concentrate on the error term and proceed to show that it is $\ll x^{1-\varepsilon}$ for some $\varepsilon>0$. Using observation (7), together with Lemma 3 one ends up with

$$\left(\sum_{\substack{p\leqslant x\\ \lfloor\alpha_1\lfloor\alpha_2 p\rfloor\rfloor\equiv 0 \pmod{d^k}}} 1\right) - \frac{\pi(x)}{d^k} \ll \frac{\pi(x)}{H_1} + \sum_{1\leqslant|h_1|\leqslant H_1}\frac{1}{|h_1|}\left|\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right)\right|, \tag{20}$$

where H_1 is a positive number to be determined. So, it boils down to estimate the exponential sum above. To do this, we let K be a sufficiently large number and we write

$$\lfloor\alpha_2 p\rfloor = \alpha_2 p - \{\alpha_2 p\},$$

yielding

$$\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right) = \sum_{0\leqslant i\leqslant K-1}\sum_{p\in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1\{\alpha_2 p\}}{d^k}\right), \tag{21}$$

where $I_i(x) = \left\{p\leqslant x : \frac{i}{K}\leqslant\{\alpha_2 p\}<\frac{i+1}{K}\right\}$. Since

$$e(t) = 1 + O(|t|)$$

uniformly for all $t\in\mathbb{R}$, we have

$$e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1\{\alpha_2 p\}}{d^k}\right) = e\left(-\frac{\alpha_1 h_1 i}{Kd^k}\right)\left(e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\frac{|h_1|}{Kd^k}\right)\right)$$

if $p\in I_i(x)$. Therefore, the left hand side of (21) is

$$\ll \frac{|h_1|\pi(x)}{Kd^k} + \sum_{0\leqslant i\leqslant K-1}\left|\sum_{p\in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right)\right|. \tag{22}$$

Given $0\leqslant i\leqslant K-1$, let $\beta_i = i/K$, $\gamma_i = (i+1)/K$ and $0<\Delta<1/K$ be a number to be chosen. By Lemma 4, there exists a periodic function $\Psi_i(x)$, with period 1, satisfying

(i) $\Psi_i(x) = 1$ in the interval $\beta_i + \frac{1}{2}\Delta \leqslant x \leqslant \gamma_i - \frac{1}{2}\Delta$,

(ii) $\Psi_i(x) = 0$ in the interval $\gamma_i + \frac{1}{2}\Delta \leqslant x \leqslant 1 + \beta_i - \frac{1}{2}\Delta$,

(iii) $0 \leqslant \Psi_i(x) \leqslant 1$ in the remainder of the interval $\beta_i - \frac{1}{2}\Delta \leqslant x \leqslant 1 + \beta_i - \frac{1}{2}\Delta$,

(iv) $\Psi_i(x)$ has a Fourier expansion of the form

$$\Psi_i(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),$$

where $a_0 = 1/K$ and

$$|a_h| \leqslant c \cdot \min\left\{|h|^{-1}, |h|^{-2}\Delta^{-1}\right\}$$

for every $|h| \geqslant 1$ and some c fixed.

Let $\psi_i(x)$ be 1 if $\beta_i \leqslant \{x\} \leqslant \gamma_i$ and $\psi_i(x) = 0$ otherwise. It follows that $\Psi_i(x)$ and $\psi_i(x)$ agree on $[0,1]$ except possibly for two subintervals of $[0,1]$ of length $\leqslant \Delta$. Therefore,

$$\sum_{p \in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) = \sum_{p \leqslant x} \Psi_i(\alpha_2 p) e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\sum_{\substack{p \leqslant x \\ \{\alpha_2 p\} \in I}} 1\right) \tag{23}$$

where I is a union of two intervals and is of length Δ. Since α_2 is of finite type, following the proof of Theorem 5.1 in [8] together with a partial summation argument, it follows that for some $0 < \varepsilon'' < 1/5$, one has

$$\sum_{\substack{p \leqslant x \\ \{\alpha_2 p\} \in I}} 1 = \Delta\pi(x) + O\left(x^{1-\varepsilon''}\right), \tag{24}$$

uniformly for all $0 < \Delta < 1/K$. Therefore, we see that the left hand side of (23) is

$$= \frac{1}{K}\sum_{p \leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right)$$
$$+ O\left(\sum_{|h_2|>0} |a_{h_2}| \left|\sum_{p \leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1 + \alpha_2 h_2 d^k)p}{d^k}\right)\right| + \Delta\pi(x) + x^{1-\varepsilon''}\right).$$

Letting H_2 be a positive integer to be determined, we split the sum running over h_2 at H_2. For $|h_2| > H_2$, estimating the innermost exponential sum by $\pi(x)$, and using the upper bounds $a_h \ll 1/(\Delta h^2)$ and $a_h \ll 1/|h|$, we obtain that the left hand side of (23) is

$$= \frac{1}{K}\sum_{p \leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\sum_{0<|h_2| \leqslant H_2} \frac{1}{|h_2|} \left|\sum_{p \leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1 + \alpha_2 h_2 d^k)p}{d^k}\right)\right|\right)$$
$$+ O\left(\frac{\pi(x)}{\Delta H_2} + \Delta\pi(x) + x^{1-\varepsilon''}\right).$$

Plugging this upper bound into (22) yields that

$$\begin{aligned}\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1 \lfloor \alpha_2 p\rfloor}{d^k}\right) &\\ \ll \left|\sum_{p\leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right)\right| + \sum_{i\leqslant K}\sum_{0<|h_2|\leqslant H_2}\frac{1}{|h_2|}\left|\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right)\right| & \quad (25)\\ + \frac{\pi(x)K}{\Delta H_2} + \Delta K\pi(x) + Kx^{1-\varepsilon''} + \frac{|h_1|\pi(x)}{Kd^k}. &\end{aligned}$$

We are therefore left with the estimation of

$$\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right), \tag{26}$$

whenever $\max\{|h_1|,|h_2|\} > 0$. To estimate the exponential sum, by Dirichlet's theorem we pick up a rational number a/q satisfying

$$\left|\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)}{d^k} - \frac{a}{q}\right| < \frac{1}{qx^{1-\kappa}}$$

with $1 \leqslant q \leqslant x^{1-\kappa}$, where $0 < \kappa < 1$ is to be determined. Since $\{\alpha_1\alpha_2, \alpha_2\}$ is of finite type, similar to how we obtain (13)

$$\frac{x^{\frac{1-\kappa}{\tau}}}{d^{\frac{k}{\tau}}\max\{|h_1|,|h_2 d^k|\}} \ll q \leqslant x^{1-\kappa}$$

for some $\tau \geqslant 1$. Then by Lemma 2, the exponential sum (26) is

$$\ll x\log^3 x\left((\max\{|h_1|,|h_2 d^k|\})^{\frac12} d^{\frac{k}{2\tau}} x^{-\frac{1-\kappa}{2\tau}} + x^{-\frac15} + x^{-\frac{\kappa}{2}}\right).$$

At this point, we assume that $0 < \max\{|h_1|,|h_2|\} \leqslant x^{\varepsilon'}$ where ε' is a sufficiently small number to be determined in terms of κ. Then,

$$\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right) \ll \left(d^{\frac{k\tau+k}{2\tau}} x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}} + x^{\frac45} + x^{1-\frac{\kappa}{2}}\right)\log^3 x, \tag{27}$$

uniformly for

$$0 < \max\{|h_1|,|h_2|\} \leqslant x^{\varepsilon'}.$$

Plugging the upper bound (27) into (25), we arrive at

$$\begin{aligned}\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1 \lfloor \alpha_2 p\rfloor}{d^k}\right) \ll K\left(d^{\frac{k\tau+k}{2\tau}} x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}} + x^{\frac45} + x^{1-\frac{\kappa}{2}}\right)\log^4 x &\\ + \frac{\pi(x)K}{\Delta H_2} + \Delta K\pi(x) + Kx^{1-\varepsilon''} + \frac{H_1\pi(x)}{Kd^k}, &\end{aligned}$$

uniformly for $|h_1| \leqslant x^{\varepsilon'}$, provided that $H_2 \leqslant x^{\varepsilon'}$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and K is sufficiently large. Plugging this upper bound into (20) and summing over $d \leqslant z$, we see that the error term in (19) is

$$\ll \frac{xz}{H_1} + K\left(z^{1+\frac{k\tau+k}{2\tau}} x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}} + zx^{\frac{4}{5}} + zx^{1-\frac{\kappa}{2}}\right)\log^5 x \\ + \left(\frac{zxK}{\Delta H_2} + z\Delta K x + zKx^{1-\varepsilon''} + \frac{H_1 x}{K}\right)\log x + \frac{x}{z^{k-1}} \quad (28)$$

provided that $0 < H_1, H_2 \leqslant x^{\varepsilon'}$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and K is sufficiently large. We now make all unspecified constants explicit. For $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 < 1$ to be determined, we set

$$K = x^{\varepsilon_1}, H_1 = x^{\varepsilon_2}, H_2 = x^{\varepsilon_3}, \Delta = x^{-\varepsilon_4} \text{ and } z = x^{\varepsilon_5},$$

where $0 < \varepsilon_5 \leqslant 1/k$ (this assumption is from the beginning of the proof). Examining each term in (28), the right hand side of (28) is $\ll x^{1-\varepsilon}$ for some $\varepsilon > 0$, if the following inequalities are satisfied:

(1) $\varepsilon_5 < 1/k$,
(2) $\varepsilon_2, \varepsilon_3 < \varepsilon'$,
(3) $\varepsilon_5 < \varepsilon_2 < \varepsilon_1$,
(4) $\varepsilon_1 + \varepsilon_5 < \min\{\varepsilon_4, \varepsilon'', \kappa/2\}$,
(5) $\varepsilon_1 + \varepsilon_4 + \varepsilon_5 < \varepsilon_3$,
(6) $\varepsilon_1 + \varepsilon_5(1 + \frac{k\tau+k}{2\tau}) + \frac{\varepsilon'}{2} < \frac{1-\kappa}{2\tau}$,

where $\varepsilon'' < 1/5$ is a fixed positive number defined in (24), $\tau \geqslant 1$ is a fixed number and $0 < \kappa < 1$ and $0 < \varepsilon' < 1$ are to be chosen. We choose $\kappa = 2/5$ and $\varepsilon' = (1-\kappa)/(2\tau)$. Then since $\varepsilon'' < 1/5$, we assume that $\varepsilon_4 < \varepsilon''$ so that the fourth inequality becomes equivalent to $\varepsilon_1 + \varepsilon_5 < \varepsilon_4$. We next choose $\varepsilon_3 < \varepsilon'$ and $\varepsilon_4 < \min\left\{\varepsilon_3, \varepsilon''\right\}$ and $\varepsilon_1 < \min\left\{\varepsilon_4, \varepsilon_3 - \varepsilon_4, (1-\kappa)/(4\tau)\right\}$. Finally, we choose $\varepsilon_2 < \min\left\{\varepsilon_1, \varepsilon'\right\}$ and

$$\varepsilon_5 < \min\left\{\varepsilon_2, \varepsilon_4 - \varepsilon_1, \varepsilon_3 - \varepsilon_1 - \varepsilon_4, \frac{1}{k}, \frac{2\tau}{(k+2)\tau + k}\left(\frac{1-\kappa}{4\tau} - \varepsilon_1\right)\right\},$$

completing the proof.

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References

[1] Abercrombie, A. G., Beatty sequences and multiplicative number theory, *Acta Arith.*, 70(3) (1995), 195–207. http://doi.org/10.4064/aa-70-3-195-207

[2] Abercrombie, A. G., Banks, W. D., Shparlinski, I. E., Arithmetic functions on Beatty sequences, *Acta Arith.* 136(1) (2009), 81–89. http://doi.org/10.4064/aa136-1-6

[3] Akbal, Y., A short note on some arithmetical properties of the integer part of αp, *Turkish Journal of Mathematics*, 43(3) (2019), 1253–1262. http://doi.org/10.3906/mat-1809-43

[4] Akbal, Y., A note on values of Beatty sequences that are free of large prime factors, *Colloquium Mathematicum*, 160(1) (2020), 53–64. http://doi.org/10.4064/cm7715-2-2019

[5] Kumchev, A. V., On sums of primes from Beatty sequences, *Integers* 8 (2008), A8, 12 pp.

[6] Banks, W. D., Shparlinski, I. E., Non-residues and primitive roots in Beatty sequences, *Bull. Austral. Math. Soc.*, 73(3) (2006), 433–443. http://doi.org/10.1017/S0004972700035449

[7] Banks, W. D., Shparlinski, I. E., Short character sums with Beatty sequences, *Math. Res. Lett.* 13 (2006), 539–547. http://doi.org/10.4310/MRL.2006.v13.n4.a4

[8] Banks, W. D., Shparlinski, I. E., Prime numbers with Beatty sequences, *Colloq. Math.* 115(2) (2009), 147–157. http://doi.org/10.4064/cm115-2-1

[9] Banks, W. D., Shparlinski, I. E., Prime divisors in Beatty sequences, *J. Number Theory* 123(2) (2007), 413–425. http://doi.org/10.1016/j.jnt.2006.07.011

[10] Banks, W. D., Güloğlu, A., Vaughan, R. C., Waring's problem for Beatty sequences and a local to global principle, *J. Theor. Nombres Bordeaux* 26(1) (2014), 1–16. http://doi.org/10.5802/jtnb.855

[11] Banks, W. D., Güloğlu, A., Nevans, C. W., Representations of integers as sums of primes from a Beatty sequence, *Acta Arith.*, 130(3) (2007), 255–275. http://doi.org/10.4064/aa130-3-4

[12] Davenport, H., Multiplicative number theory, Third edition, Revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, 74, Springer-Verlag, New York, 2000.

[13] Graham, S. W., Kolesnik, G., Van der Corput's method of exponential sums, London Mathematical Society Lecture Note Series, 126, Cambridge University Press, Cambridge, 1991.

[14] Güloğlu, A., Nevans, C. W., Sums of multiplicative functions over a Beatty sequence, *Bull. Aust.Math. Soc.* 78(2) (2008), 327–334. http://doi.org/10.1017/S0004972708000853

[15] Harman, G., Primes in Beatty sequences in short intervals, *Mathematika*, 62(2) (2016), 572–586. http://doi.org/10.1112/S0025579315000376

[16] Harman, G., Primes in intersections of Beatty sequences, *J. Integer Seq.*, 18(7) (2015), 12 p.

[17] Khinchin, A. Y., Zur metrischen Theorie der diophantischen Approximationen, *Math. Z.* 24(4) (1926), 706–714. http://doi.org/10.1007/BF01216806

[18] Koksma, J. F., Some Theorems on Diophantine Inequalities, Scriptum no. 5, Math. Centrmn Amsterdam, 1950.

[19] Li, H., Pan, H., Primes of the form $\lfloor \alpha p + \beta \rfloor$, *J. Number Theory*, 129(10) (2009), 2328–2334.

[20] Mirsky, L., The number of representations of an integer as the sum of a prime and a k-free integer, *Amer.Math. Monthly*, 56 (1949), 17–19. http://doi.org/10.1080/00029890.1949.11990233

[21] Roth, K. F., Rational approximations to algebraic numbers, *Mathematika*, 2 (1955), 1–20. http://doi.org/10.1112/S0025579300000644

[22] Roth, K. F., Corrigendum to "Rational approximations to algebraic numbers", *Mathematika*, 2 (1955), 168.

[23] Montgomery, H. L., Vaughan, R. C., Multiplicative Number Theory I. Classical Theory, Cambridge University Press, Cambridge, 2007.

[24] Vaughan, R. C., The general Goldbach problem with Beatty primes, *Ramanujan J.*, 34(3) (2014), 347–359. http://doi.org/10.1007/s11139-013-9501-3

[25] Vaughan, R. C., The Hardy-Littlewood Method, 2nd ed., Cambridge Univ. Press, 1997.

[26] Vinogradov, I. M. The Method of Trigonometrical Sums in the Theory of Numbers, Dover Publications, Inc., Mineola, NY, 2004.