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Research Article

An iterative oscillation test for delay difference equations

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Abstract

In this paper, we advance a recent oscillation test for the oscillation of the delay difference equation

$$x(n+1) - x(n) + p(n)x(n-\tau) = 0 \quad \text{for } n = 0, 1, \ldots$$

where $\{p(n)\}$ is a nonnegative sequence of reals and τ is a nonnegative integer. We also present a numerical example emphasizing the significance of our new result in the literature of delay difference equations.

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1. Introduction

Consider the difference equation

$$x(n+1) - x(n) + p(n)x(n-\tau) = 0 \quad \text{for } n = 0, 1, \ldots \tag{1.1}$$

where $\{p(n)\} \subset [0, \infty)$ and $\tau \in \{0, 1, \ldots\}$. The readers are referred to [1–17] for the development of the subject.

By a solution of (1.1), we mean a sequence $\{x(n)\}$ for which $x(n+1) = x(n) - p(n)x(n-\tau)$ is satisfied for $n = 0, 1, \cdots$. It is known that for prescribed values $\varphi_0, \varphi_1, \cdots, \varphi_\tau$, (1.1) admits a unique solution $\{x(n)\}$ satisfying $x(-j) = \varphi_j$ for $j = 0, 1, \cdots, \tau$. A solution $\{x(n)\}$ of (1.1) is said to be eventually positive if $\sup\{n : x(n) \leq 0\} < \infty$. Similarly, if $\sup\{n : x(n) \geq 0\} < \infty$, then $\{x(n)\}$ is said to be eventually negative. A solution $\{x(n)\}$ of (1.1), which is neither eventually positive nor eventually negative is said to be oscillatory.

Here, for the sake of convenience, we will quote some related results on the oscillation and nonoscillation of solutions to (1.1). One of the first results on equations of the form (1.1) is given by Erbe, L. H. and Zhang, B. G. in 1989.

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Theorem 1.1 ([8, Theorems 2.2 and 2.3]). (i) *Assume that*

$$\liminf_{n\to\infty} p(n) > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}.$$

Then, every solution of (1.1) *oscillates.*

(ii) *Assume that*

$$p(n) \le \frac{\tau^\tau}{(\tau+1)^{\tau+1}} \quad \textit{for all large } n.$$

Then, (1.1) *has an eventually positive solution.*

Theorem 1.1 (i) is improved by Ladas, G., Philos, Ch. G. and Sficas, Y. G. in 1989 by replacing the point-wise condition with the mean of consecutive τ-terms.

Theorem 1.2 ([12, Theorem 1]). *Assume that*

$$\liminf_{n\to\infty} \sum_{j=n-\tau}^{n-1} p(j) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}.$$

Then, every solution of (1.1) *oscillates.*

Next, Yu, J. S., Zhang, B. G. and Wang, Z. C. in 1994 explored a very important approach, which improves the above result by replacing the sum with a product. Their approach also allowed to prove a new nonoscillation test, which improves Theorem 1.1 (ii).

Theorem 1.3 ([17, Theorem 1]). (i) *Assume that*

$$\liminf_{n\to\infty} \inf_{\lambda\in\Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1-\lambda p(j)]} \right\} > 1,$$

where

$$\Lambda := \{\lambda > 0 : \ 1-\lambda p(n) > 0 \textit{ for all large } n\}. \tag{1.2}$$

Then, every solution of (1.1) *oscillates.*

(ii) *Assume that there exists* $\lambda_0 \in \Lambda$ *such that*

$$\frac{1}{\lambda_0 \prod_{j=n-\tau}^{n-1}[1-\lambda_0 p(j)]} \le 1 \quad \textit{for all large } n.$$

Then, (1.1) *has an eventually positive solution.*

Finally, we would like to quote the following results by the first author from [10]. We will be confine our attention on the oscillation part of this recent result.

Theorem 1.4 ([10, Theorems 1 and 2]). (i) *Assume that*

$$\liminf_{n\to\infty} \inf_{\lambda\ge 1} \left\{ \frac{1}{\lambda} \prod_{j=n-\tau}^{n} [1+\lambda p(j)] \right\} > 1.$$

Then, every solution of (1.1) *oscillates.*

(ii) *Assume that there exists* $\lambda_0 \ge 1$ *such that*

$$\frac{1}{\lambda_0} \prod_{j=n-\tau}^{n} [1+\lambda_0 p(j)] \le 1 \quad \textit{for all large } n.$$

Then, (1.1) *has an eventually positive solution.*

At this point, we find it valuable to draw readers attention to the so-called "Ladas' conjecture" [11], which has been mentioned in the papers [10, 16, 17], for complete discussion on this subject.

This paper is organized in the following setting. We state our main result in Section 2. In Section 3, we present a numerical example to show the applicability and significance of the main result. In Section 4, we fill the background for the proof of the main result. The proof of the main result is given in Section 5. Finally, Section 6 includes our last words to finalize the discission of the paper.

2. The main result

Here, we state our main result, which advances the oscillation test Theorem 1.4 (i). Recall that its proof is postponed to Section 5.

Theorem 2.1. *Assume that there exists $\ell \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \beta_\ell(n) > 1, \tag{2.1}$$

where

$$\beta_k(n) := \begin{cases} 1, & k = 0, \\ \inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-\tau}^{n}[1+\lambda\beta_{k-1}(j)p(j)]\right\}, & k\in\mathbb{N}. \end{cases} \tag{2.2$_k$}$$

Then, every solution of (1.1) *oscillates.*

Remark 2.2. With $\ell = 1$, Theorem 2.1 reduces to Theorem 1.4 (i).

3. A numerical example

In the example below, we will show that Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 cannot deliver an answer on the oscillatory behavior of solutions but Theorem 2.1 applies and gives a positive answer.

Example 3.1. Consider the equation

$$x(n+1) - x(n) + \begin{Bmatrix} \frac{15}{100}, & 0 = n \pmod 4 \\ \frac{17}{100}, & 1 = n \pmod 4 \\ \frac{14}{100}, & 2 = n \pmod 4 \\ \frac{15}{100}, & 3 = n \pmod 4 \end{Bmatrix} x(n-2) = 0 \quad \text{for } n = 0, 1, \cdots. \tag{3.1}$$

- We have $\Lambda := (0, \frac{100}{17})$, which is defined in (1.2), and

$$\frac{1}{\lambda\prod_{j=n-2}^{n-1}[1-\lambda p(j)]} = \begin{cases} \frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}, & 0 = n \pmod 4 \\ \frac{1}{\lambda(1-\lambda\frac{15}{100})^2}, & 1 = n \pmod 4 \\ \frac{1}{\lambda(1-\lambda\frac{17}{100})(1-\lambda\frac{15}{100})}, & 2 = n \pmod 4 \\ \frac{1}{\lambda(1-\lambda\frac{14}{100})(1-\lambda\frac{17}{100})}, & 3 = n \pmod 4 \end{cases}$$

for $n = 0, 1, \cdots$. Simply, we compute

$$\begin{aligned} \inf_{\lambda\in(0,\frac{100}{17})}\left\{\frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}\right\} &= \frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}\Bigg|_{\lambda\to\frac{10}{63}(29-\sqrt{211})} \\ &= \frac{1}{50}(211\sqrt{211}-3016) \\ &\approx \frac{98}{100} < 1, \end{aligned}$$

which shows that Theorem 1.3 (i) fails. This implies that Theorem 1.1 (i) and Theorem 1.2 also cannot apply.

- We have

$$\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)]=\begin{cases}\frac{1}{\lambda}(1+\lambda\frac{15}{100})^2(1+\lambda\frac{14}{100}), & 0=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})^2, & 1=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100}), & 2=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{15}{100})(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100}), & 3=n\,(\mathrm{mod}\,4)\end{cases}$$

for $n=0,1,\cdots$. Simply, we compute

$$\begin{aligned}\inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\left(1+\lambda\frac{15}{100}\right)^2\left(1+\lambda\frac{14}{100}\right)\right\}&\approx\frac{1}{\lambda}\left(1+\lambda\frac{15}{100}\right)^2\left(1+\lambda\frac{14}{100}\right)\Big|_{\lambda\to\frac{341}{100}}\\&\approx\frac{99}{100}<1,\end{aligned}$$

which shows that Theorem 1.4 fails too.

- First, we compute

$$\begin{aligned}\beta_1(n)=&\inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)]\right\}\\ \approx&\begin{cases}\frac{1}{\lambda}(1+\lambda\frac{15}{100})^2(1+\lambda\frac{14}{100})|_{\lambda\to\frac{341}{100}}, & 0=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})^2|_{\lambda\to\frac{319}{100}}, & 1=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})|_{\lambda\to\frac{327}{100}}, & 2=n\,(\mathrm{mod}\,4)\\ \frac{1}{\lambda}(1+\lambda\frac{15}{100})(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})|_{\lambda\to\frac{327}{100}}, & 3=n\,(\mathrm{mod}\,4)\end{cases}\\ \approx&\begin{cases}\frac{99}{100}, & 0=n\,(\mathrm{mod}\,4)\\ \frac{106}{100}, & 1=n\,(\mathrm{mod}\,4)\\ \frac{103}{100}, & 2=n\,(\mathrm{mod}\,4)\\ \frac{103}{100}, & 3=n\,(\mathrm{mod}\,4)\end{cases}\end{aligned}$$

for $n=0,1,\cdots$. Next, we compute

$$\begin{aligned}\beta_2(n)=&\inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-\tau}^{n}[1+\lambda\beta_1(j)p(j)]\right\}\\ \approx&\begin{cases}\inf_{\lambda\geq 1}\{\frac{1}{\lambda}(1+\lambda\frac{99}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{14}{100})\}, & 0=n\,(\mathrm{mod}\,4)\\ \inf_{\lambda\geq 1}\{\frac{1}{\lambda}(1+\lambda\frac{106}{100}\frac{17}{100})(1+\lambda\frac{99}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{15}{100})\}, & 1=n\,(\mathrm{mod}\,4)\\ \inf_{\lambda\geq 1}\{\frac{1}{\lambda}(1+\lambda\frac{103}{100}\frac{14}{100})(1+\lambda\frac{106}{100}\frac{17}{100})(1+\lambda\frac{99}{100}\frac{15}{100})\}, & 2=n\,(\mathrm{mod}\,4)\\ \inf_{\lambda\geq 1}\{\frac{1}{\lambda}(1+\lambda\frac{103}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{14}{100})(1+\lambda\frac{106}{100}\frac{17}{100})\}, & 3=n\,(\mathrm{mod}\,4)\end{cases}\\ \approx&\begin{cases}\frac{101}{100}, & 0=n\,(\mathrm{mod}\,4)\\ \frac{109}{100}, & 1=n\,(\mathrm{mod}\,4)\\ \frac{106}{100}, & 2=n\,(\mathrm{mod}\,4)\\ \frac{108}{100}, & 3=n\,(\mathrm{mod}\,4)\end{cases}\end{aligned}$$

for $n=0,1,\cdots$. This yields $\liminf_{n\to\infty}\beta_2(n)>1$, i.e., Theorem 2.1 applies with $\ell=2$. Therefore, every solution of (3.1) is oscillatory.

4. Preparatory results

This section consists of three lemmas, which are required in the proof of the main result Theorem 2.1. The connection between these three lemmas are interesting on their own.

Lemma 4.1. *If* (1.1) *has a nonoscillatory solution, then*

$$\sum_{j=n-\tau}^{n} p(j) < 1 \quad \textit{for all large } n. \tag{4.1}$$

Proof. The claim follows from the proof of [8, Theorem 2.5]. □

Lemma 4.2. *Assume*

$$\limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) < \infty \tag{4.2}$$

and

$$\lim_{n\to\infty}\left(p(n)\sum_{j=n-\tau}^{n-1} p(j)\right) = 0. \tag{4.3}$$

Then,

$$\limsup_{n\to\infty} \beta_k(n) \leq \left(\limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j)\right)^k \quad \textit{for } k \in \mathbb{N}. \tag{4.4}$$

Proof. It follows from (4.3) that

$$\lim_{n\to\infty}(p(n)p(n-j)) = 0 \quad \text{for } j = 1, 2, \cdots, \tau. \tag{4.5}$$

By expanding the product in (2.2_k), we write

$$\begin{aligned}\beta_k(n) = \inf_{\lambda\geq 1}\Bigg\{\frac{1}{\lambda}\Bigg(1 + \lambda \sum_{j=n-\tau}^{n} \beta_{k-1}(j)p(j) \\ + \lambda^2 \sum_{j=n-\tau}^{n-1} \beta_{k-1}(j)p(j) \sum_{i=j+1}^{n} \beta_{k-1}(i)p(i) \\ + \cdots + \lambda^{\tau+1} \prod_{j=n-\tau}^{n} \beta_{k-1}(j)p(j)\Bigg)\Bigg\}\end{aligned} \tag{4.6$_k$}$$

for all large n. From (4.5) and (4.6_1), we see that

$$\beta_1(n) = \inf_{\lambda\geq 1}\left\{\frac{1}{\lambda} + \sum_{j=n-\tau}^{n} p(j) + o(1)(\lambda + \cdots + \lambda^\tau)\right\} \quad \text{for all large } n, \tag{4.7}$$

where o is the so-called “little-o notation” meaning that the coefficients of $\lambda, \lambda^2, \cdots, \lambda^\tau$ tend to 0 as $n \to \infty$. It follows from (4.2) and (4.7) that

$$\limsup_{n\to\infty} \beta_1(n) = \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) =: M, \tag{4.8}$$

i.e., $\{\beta_1(n)\}$ is bounded. From (4.5), (4.6_2) and (4.8), we see that

$$\beta_2(n) = \inf_{\lambda\geq 1}\left\{\frac{1}{\lambda} + \sum_{j=n-\tau}^{n} \beta_1(j)p(j) + o(1)(\lambda + \cdots + \lambda^\tau)\right\} \quad \text{for all large } n. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\limsup_{n\to\infty} \beta_2(n) = \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} \beta_1(j)p(j) \leq M \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) \leq M^2, \tag{4.10}$$

i.e., $\{\beta_2(n)\}$ is bounded. By induction, we obtain $\limsup_{n\to\infty} \beta_k(n) \leq M^k$ for $k \in \mathbb{N}$, which proves (4.4). □

Lemma 4.3. *Let $\{x(n)\}$ be a nonoscillatory solution of* (1.1). *If*

$$\limsup_{n\to\infty}\Bigg(p(n)\sum_{j=n-\tau}^{n-1}p(j)\Bigg)>0, \tag{4.11}$$

then

$$\liminf_{n\to\infty}\frac{x(n-\tau)}{x(n+1)}<\infty.$$

Proof. In view of (4.11), there exist an increasing divergent sequence $\{n_k\}$ and a constant $\varepsilon>0$ such that

$$p(n_k)\sum_{j=n_k-\tau}^{n_k-1}p(j)\geq\varepsilon\quad\text{for all } k\in\mathbb{N}. \tag{4.12}$$

It follows from (1.1) that

$$x(n_k)>x(n_k)-x(n_k+1)=p(n_k)x(n_k-\tau)\quad\text{for all } k\in\mathbb{N}. \tag{4.13}$$

Also, from (1.1), we have

$$\begin{aligned}x(n_k-\tau)>x(n_k-\tau)-x(n_k)&=-\sum_{j=n_k-\tau}^{n_k-1}[x(j+1)-x(j)]\\&=\sum_{j=n_k-\tau}^{n_k-1}p(j)x(j-\tau)\\&\geq\sum_{j=n_k-\tau}^{n_k-1}p(j)x(n_k-1-\tau)\end{aligned} \tag{4.14}$$

for all $k\in\mathbb{N}$. Combining (4.12), (4.13) and (4.14) yields

$$\frac{x(n_k-1-\tau)}{x(n_k)}<\frac{1}{p(n_k)\sum_{j=n_k-\tau}^{n_k-1}p(j)}\leq\frac{1}{\varepsilon}\quad\text{for all } k\in\mathbb{N},$$

which completes the proof. □

5. The proof

Proof of Theorem 2.1. Assume the contrary that $\{x(n)\}$ is an nonoscillatory solution of (1.1). Without loss of generality, we suppose that $\{x(n)\}$ is eventually positive. Then, there exists $n_1\in\mathbb{N}$ such that $x(n+1), x(n)$ and $x(n-\tau)$ are positive for $n=n_1,n_1+1,\cdots$. By (1.1), $\{x(n)\}$ is a nonincreasing sequence on $\{n_1,n_1+1,\cdots\}$. Define $w(n):=\frac{x(n-\tau)}{x(n+1)}$ for $n=n_1,n_1+1,\cdots$. Note that $w(n)\geq 1$ for $n=n_1,n_1+1,\cdots$. From (1.1), we write

$$x(n+1)-x(n)+w(n)p(n)x(n+1)=0\quad\text{for } n=n_1,n_1+1,\cdots,$$

which yields

$$w(n)=\prod_{j=n-\tau}^{n}[1+w(j)p(j)]\quad\text{for } n=n_2,n_2+1,\cdots, \tag{5.1}$$

where $n_2:=n_1+\tau$. Now, we define

$$z_k(n):=\begin{cases}w(n), & k=0\\ \min\{z_{k-1}(j):\ j=n-\tau,n-\tau+1,\cdots,n\}, & k=1,2,\cdots,\ell\end{cases} \tag{5.2$_k$}$$

for $n = n_2, n_2+1, \cdots$. By (5.1), (5.2$_1$) and $w(n) \geq 1$ for $n = n_2, n_2+1, \cdots$, it follows that $z_1(n) \geq 1$ for $n = n_3, n_3+1, \cdots$, where $n_3 := n_2 + \tau$. By (2.2$_1$),

$$\begin{aligned} w(n) &\geq \prod_{j=n-\tau}^{n} [1 + z_1(n)p(j)] = \left(\frac{1}{z_1(n)} \prod_{j=n-\tau}^{n} [1 + z_1(n)p(j)] \right) z_1(n) \\ &\geq \beta_1(n) z_1(n) \end{aligned}$$

for $n = n_3, n_3+1, \cdots$. From (5.1) and (5.2$_2$), we know that $z_2(n) \leq z_1(n)$ for $n = n_4, n_4+1, \cdots$, and by definition $z_2(n) \geq 1$ for $n = n_4, n_4+1, \cdots$, where $n_4 := n_3 + \tau$. By (2.2$_2$),

$$\begin{aligned} w(n) &\geq \prod_{j=n-\tau}^{n} [1 + z_1(j)\beta_1(j)p(j)] \geq \prod_{j=n-\tau}^{n} [1 + z_2(n)\beta_1(j)p(j)] \\ &= \left(\frac{1}{z_2(n)} \prod_{j=n-\tau}^{n} [1 + z_2(n)\beta_1(j)p(j)] \right) z_2(n) \\ &\geq \beta_2(n) z_2(n) \end{aligned}$$

for $n = n_4, n_4+1, \cdots$. By induction, it follows from (2.2$_\ell$), (5.1) and (5.2$_\ell$) that

$$w(n) \geq \beta_\ell(n) z_\ell(n) \quad \text{for } n = n_5, n_5+1, \cdots. \tag{5.3}$$

where $n_5 := n_4 + \ell\tau$. By Lemma 4.1, Lemma 4.2 and Lemma 4.3, we obtain $\omega_* := \liminf_{n\to\infty} w(n) < \infty$. Note that $\liminf_{n\to\infty} z_\ell(n) = \omega_*$. Thus, taking inferior limits on both sides of (5.3), we get

$$\begin{aligned} w_* &\geq \liminf_{n\to\infty} \beta_\ell(n) \liminf_{n\to\infty} z_\ell(n) \\ &= \liminf_{n\to\infty} \beta_\ell(n) w_*, \end{aligned}$$

which yields $\liminf_{n\to\infty} \beta_\ell(n) \leq 1$ contradicting (2.1). This completes the proof. □

6. Final comments

In the literature, our main result Theorem 2.1 is not the only iterative test for the oscillation of solutions of delay difference equations. In this direction, one of the first important results by Tang, X. H. and Yu, J. S. is quoted below.

Theorem 6.1 ([15, Corollary 1]). *Assume that there exists $\ell \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} p_\ell(n) > \left(\frac{\tau}{\tau+1} \right)^{\ell(\tau+1)},$$

where

$$p_k(n) := \begin{cases} 1, & k = 0 \\ \displaystyle\sum_{j=n+1}^{n+\tau} p_{k-1}(j)p(j), & k \in \mathbb{N}. \end{cases}$$

Then, every solution of (1.1) *oscillates.*

Remark 6.2. Recall that Theorem 6.1 includes Theorem 1.2 with $\ell = 1$.

Next, we quote a special case of another iterative result by Bohner, M., Karpuz, B. and Öcalan, Ö., which is extracted from [3] for the discrete time scale nonnegative integers.

Theorem 6.3 (cf. [3, Theorem 2.3]). *Assume that there exists $\ell \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \alpha_\ell(n) > 1,$$

where

$$\alpha_k(n) := \begin{cases} 1, & k = 0 \\ \inf\limits_{\lambda\in\Lambda_k} \left\{ \dfrac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1-\lambda\alpha_{k-1}(j)p(j)]} \right\}, & k \in \mathbb{N} \end{cases}$$

and

$$\Lambda_k := \{\lambda > 0 : \ 1 - \lambda\alpha_{k-1}(n)p(n) > 0 \text{ for all large } n\} \quad \text{for } k \in \mathbb{N}.$$

Then, every solution of (1.1) *oscillates.*

Remark 6.4. Note that Theorem 6.3 includes Theorem 1.3 with $\ell = 1$.

Remark 6.5. For fixed $\ell \in \mathbb{N}$, Theorem 6.1 and Theorem 6.3 are not comparable.

As the final sentence, we would like to mention that our main result Theorem 2.1 complements Theorem 6.3 in a similar manner that Theorem 1.4 complements Theorem 1.3.

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