MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
https://doi.org/10.36753/mathenot.944392
9 (4) 151-157 (2021) - Research Article
ISSN: 2147-6268

Multiplication Operators on Second Order Cesàro-Orlicz Sequence Spaces

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Abstract
The main purpose of this paper is to characterize the compact, invertible, Fredholm and closed range multiplication operators on second Cesàro-Orlicz sequence spaces.

Keywords: *Compact operator; Fredholm multiplication operator; Invertible operator; Multiplication operator; Orlicz function; Second order Cesàro sequence space.*

AMS Subject Classification (2020): *47B38, 46A06*

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1. Preliminaries, background and notation

Over years, the interest on properties of multipliers between functional Banach spaces have increased. Let X and Y be Banach spaces consisting of sequences with real or complex terms. A numeric sequence $u = (u_n)$ such that $uf = (u_n f_n) \in Y$ for all $f \in X$ is called a multiplier for X and Y. Each multiplier $u = (u_n)$ induces a linear operator $M_u : X \to Y$ by $M_u(f) = uf$. If M_u is continuous, it is called the *multiplication operator* with symbol u.

Several studies on multiplication operators have been carried out. Mostly, multipliers of spaces of measurable functions have been thoroughly examined. In Halmos's monograph [1], one can find important knowledge about multiplication operators on the Hilbert space of square integrable measurable functions with respect to a given measure. In [2, 3], Singh and Kumar present good works on properties of multiplication operators on spaces of measurable functions and they study compactness and closedness of the range of multiplication operators on certain Hilbert spaces. Mursaleen et al. [4], İlkhan et al. [5] have studied multiplication operators on Cesàro function spaces. Further, Castillo et al. [6–8], obtained significant results and modified the techniques used by the others to study multiplication operators on Orlicz-Lorentz spaces, weak L_p spaces and variable Lebesgue spaces.

The Cesàro sequence space Ces_p was firstly introduced by Shiue [9] as the set of all real sequences $x = (x_n)$ satisfying

$$\|x\|_{Ces_p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty,$$

Received : *28-05-2021,* ***Accepted :*** *31-12-2021*

where $1 \leq p < \infty$. Some topological and geometrical properties of Cesàro spaces were studied by Shiue [9] , Leibowitz [10], Jagers [11], Cui and Pluciennik [12], Cui and Hudzik [13], Altay and Kama [14], Kama [15].

A continuous, non-decreasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it satisfies the following conditions:

- $\varphi(0) = 0$,
- $\varphi(x) > 0$ for $x > 0$,
- $\varphi(x) \to \infty$ as $x \to \infty$.

Additionally, if there exists $K > 0$ such that $\varphi(Lx) \leq KL\varphi(x)$ for all $x \geq 0$ and for $L > 1$, then we say that Orlicz function satisfies the δ_2-condition. We write $e = (e_k)$ and $e^n = (e^n_k)$ for the sequences with $e_k = 1$ for all k, and $e^n_n = 1$ and $e^n_k = 0$ for $k \neq n$.

Lindenstrauss and Tzafriri [16] define the Orlicz sequence space

$$\ell_\varphi = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \varphi\left(\frac{|x_k|}{\lambda}\right) < \infty, \text{ for some } \lambda > 0 \right\}$$

using the idea of Orlicz function. Here and what follows, the space of all complex sequences is denoted by ω. The Orlicz space ℓ_φ with the norm

$$\|x\| = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \varphi\left(\frac{|x_k|}{\lambda}\right) \leq 1 \right\}$$

is a Banach space.

The space

$$Ces_\varphi(\mathbb{N}) = \left\{ x = (x_k) \in \omega : \sum_{m=1}^{\infty} \varphi\left(\frac{1}{m} \sum_{k=1}^{m} |\lambda x_k|\right) < \infty \right\}$$

is called the Cesàro-Orlicz sequence space which is a Banach space with the norm

$$\|x\|_{Ces_\varphi} = \inf \left\{ \lambda > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |x_k|}{\lambda}\right) \leq 1 \right\}$$

(see [17]). If $\varphi(x) = |x|^p \quad (p > 1)$, then the Cesàro-Orlicz sequence space $Ces_\varphi(\mathbb{N})$ reduces to the Cesàro sequence space Ces_p.

After Lim and Lee [18] found the dual spaces of Cesàro-Orlicz sequence spaces $Ces_\varphi(\mathbb{N})$, Cui et al. [19] and Damian [20] investigated some properties of these spaces. Later, the authors in [21] studied the multiplication operators on Cesàro-Orlicz sequence spaces.

In 2016, N. Braha [22] defined the second-order Cesàro sequence space as

$$Ces^2(p) = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (n+1-k)|x_k| \right)^p < \infty \right\}$$

for $1 \leq p < \infty$ and he examined some topological and geometrical properties of the space $Ces^2(p)$.

Now, we define the second-order Cesàro-Orlicz sequence space by

$$Ces^2_\varphi(\mathbb{N}) = \left\{ x = (x_k) \in \omega : \sum_{m=1}^{\infty} \varphi\left(\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|\lambda x_k| \right) < \infty \right\}.$$

It is clear that the sequence space $Ces^2_\varphi(\mathbb{N})$ is a Banach space with the norm

$$\|x\|_{Ces^2_\varphi} = \inf \left\{ \lambda > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|x_k|}{\lambda} \right) \leq 1 \right\}.$$

In this paper, we give the characterization of the boundedness, compactness, closed range and Fredholmness for the multiplication operators $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ defined by $M_u f = uf$ for any $u \in \omega$.

2. Boundedness of Multiplication Operators

In this section, we will prove the theorems related to isometry and boundedness of multiplication operators.

Theorem 2.1. *Given any sequence $u \in \omega$, the multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is bounded if and only if the sequence u is bounded.*

Proof. Let M_u be a bounded operator. On the contrary, assume that u is not a bounded sequence. Then, given any $n \in \mathbb{N}$, there exists some $k_n \in \mathbb{N}$ such that $|u_{k_n}| > n$. It is clear that $\|e^{k_n}\|_{Ces^2_\varphi} = \sum_{m=k_n}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)}$. Set $\widehat{e}^{k_n} = \frac{e^{k_n}}{\|e^{k_n}\|_{Ces^2_\varphi}}$. Then, we have $\|\widehat{e}^{k_n}\|_{Ces^2_\varphi} = 1$. It follows that

$$\begin{aligned}
\|M_u\widehat{e}^{k_n}\|_{Ces^2_\varphi} &= \frac{\|M_u e^{k_n}\|_{Ces^2_\varphi}}{\|e^{k_n}\|_{Ces^2_\varphi}} \\
&= \frac{\sum_{m=k_n}^{\infty} \frac{(m+1-k)|u_{k_n}|}{(m+1)(m+2)\lambda\varphi^{-1}(1)}}{\|e^{k_n}\|_{Ces^2_\varphi}} \\
&= |u_{k_n}| > n.
\end{aligned}$$

This contradicts the fact that M_u is a bounded operator. Hence, we conclude that u is bounded.

Conversely, let u be a bounded sequence. Then, there exists $K > 0$ such that $|u_n| \leq K$ for all $n \in \mathbb{N}$. Given any $x \in Ces^2_\varphi(\mathbb{N})$, we obtain that

$$\begin{aligned}
\|M_u x\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|(ux)_k|}{\lambda}\right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_k||x_k|}{\lambda}\right) \\
&\leq K\sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right) \\
&= K\|x\|_{Ces^2_\varphi}
\end{aligned}$$

which implies that M_u is a bounded operator. □

Theorem 2.2. *The multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is an isometry if and only if $|u_n| = 1$ for all $n \in \mathbb{N}$.*

Proof. On the contrary, assume that $|u_{n_0}| \neq 1$ for some $n_0 \in \mathbb{N}$. Clearly, we have $\|e^{n_0}\|_{Ces^2_\varphi} = \sum_{m=n_0}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)}$. Let $|u_{n_0}| > 1$. Then,

$$\begin{aligned}
\|M_u e^{n_0}\|_{Ces^2_\varphi} &= \left(\sum_{m=n_0}^{\infty} \frac{(m+1-k)|u_{n_0}|}{(m+1)(m+2)\lambda\varphi^{-1}(1)}\right) \\
&> \sum_{m=n_0}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\
&= \|e^{n_0}\|_{Ces^2_\varphi}
\end{aligned}$$

holds. Similarly, if $|u_{n_0}| < 1$, $\|M_u e^{n_0}\|_{Ces^2_\varphi} < \|e^{n_0}\|_{Ces^2_\varphi}$ holds. Thus, we obtain a contradiction. Hence, we conclude that $|u_n| = 1$ for all $n \in \mathbb{N}$.

Now, suppose that $|u_n| = 1$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\|M_u x\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_k x_k|}{\lambda}\right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right) \\
&= \|x\|_{Ces^2_\varphi}.
\end{aligned}$$

Therefore, $\|M_u x\|_{Ces^2_\varphi} = \|x\|_{Ces^2_\varphi}$ for all $x \in Ces^2_\varphi(\mathbb{N})$ and hence M_u is an isometry. □

3. Compactness of Multiplication Operators

Before we prove our main result in this section, remember the definition of a compact operator.

Let X be a Banach space and B_1 be the closed unit ball in X. If the closure of the set $T(B_1)$ is compact, then the bounded linear operator $T : X \to X$ is said to be *compact*.

By $B(Ces^2_\varphi(\mathbb{N}))$ we denote the set of all bounded linear operators from $Ces^2_\varphi(\mathbb{N})$ into itself. Now, we give our main results about the compactness of the multiplication operator.

Theorem 3.1. *A bounded linear multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is compact if and only if $u_n \to 0$ as $n \to \infty$.*

Proof. Firstly, let M_u be a compact operator. On the contrary, assume that $u_n \nrightarrow 0$ as $n \to \infty$. Then, there exists $\varepsilon_0 > 0$ such that the set $N_{\varepsilon_0} = \{k \in \mathbb{N} : |u_k| \geq \varepsilon_0\}$ is an infinite set and we can write $N_{\varepsilon_0} = \{p_1, p_2, ..., p_n, ...\}$. Then, the set $\{e^{p_n} : p_n \in N_{\varepsilon_0}\}$ is bounded in $Ces^2_\varphi(\mathbb{N})$. It follows that

$$\begin{aligned}
&\|M_u e^{p_n} - M_u e^{p_s}\|_{Ces^2_\varphi}\\
=\;& \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u(k)e^{p_n}(k) - u(k)e^{p_s}(k)|}{\lambda}\right)\\
=\;& \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u(k)||e^{p_n}(k) - e^{p_s}(k)|}{\lambda}\right)\\
\geq\;& \varepsilon_0 \|e^{p_n} - e^{p_s}\|_{Ces^2_\varphi}
\end{aligned}$$

for all $p_n, p_s \in N_{\varepsilon_0}$. This shows that $\{M_u e^{p_n} : p_n \in N_{\varepsilon_0}\}$ cannot have a convergent subsequence. This contradicts the fact that M_u is a compact operator. Thus, $u_n \to 0$ as $n \to \infty$ holds.

Conversely, let $u_n \to 0$ as $n \to \infty$. Then, for every $\varepsilon > 0$, the set $N_\varepsilon = \{n \in \mathbb{N} : |u_n| \geq \varepsilon\}$ is a finite set. Hence, the space $Ces^2_\varphi(N_\varepsilon)$ is finite dimensional and so $M_u|Ces^2_\varphi(N_\varepsilon)$ is a compact operator. Let $u_n \in \omega$ be defined by

$$u_n(m) = \begin{cases} u(m) & , \quad \forall m \in N_{\frac{1}{n}} \\ 0 & , \quad \forall m \notin N_{\frac{1}{n}} \end{cases}$$

for each $n \in \mathbb{N}$. M_{u_n} is a compact operator since the space $Ces^2_\varphi(N_{\frac{1}{n}})$ is finite dimensional for each $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
&\|(M_{u_n} - M_u)x\|_{Ces^2_\varphi}\\
=\;& \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right)\\
=\;& \sum_{m\in N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right)\\
+\;& \sum_{m\notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right)\\
=\;& \sum_{m\notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u(k)x_k|}{\lambda}\right)\\
<\;& \frac{1}{n}\sum_{m\notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right)\\
\leq\;& \frac{1}{n}\|x\|_{Ces^2_\varphi}.
\end{aligned}$$

Hence, we have $\|(M_{u_n} - M_u)\|_{Ces^2_\varphi} \leq \frac{1}{n}$ and so M_u is a compact operator. □

Theorem 3.2. *A bounded linear multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ has closed range if and only if u is bounded away from zero on $S = \{k \in \mathbb{N} : u_k \neq 0\}$.*

Proof. If the range of M_u is closed, then M_u is bounded away from zero on $(kerM_u)^\perp = Ces^2_\varphi(S)$. This means that there exists $\varepsilon > 0$ such that

$$\|M_u x\|_{Ces^2_\varphi} \geq \varepsilon \|x\|_{Ces^2_\varphi} \tag{3.1}$$

for all $x \in Ces^2_\varphi(S)$. Set $H = \{k \in S : |u_k| < \frac{\varepsilon}{2}\}$. If $H \neq \emptyset$, then for $n_0 \in H$, we have

$$\begin{aligned}
\|M_u e^{n_0}\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|u(k)e^{n_0}(k)|}{\lambda}\right) \\
&= \sum_{m=n_0}^{\infty} \frac{(m+1-k)|u(n_0)|}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\
&< \varepsilon \sum_{m=n_0}^{\infty} \frac{(m+1-k)}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\
&= \varepsilon \|e^{n_0}\|_{Ces^2_\varphi}.
\end{aligned}$$

That is, $\|M_u e^{n_0}\|_{Ces^2_\varphi} < \|e^{n_0}\|_{Ces^2_\varphi}$ which contradicts (3.1). Hence, $H = \emptyset$ so that $|u_k| \geq \varepsilon$ for all $k \in S$.

For the converse, let u be bounded away from zero on S. Then, there exists $\varepsilon > 0$ such that $|u_n| \geq \varepsilon$ for all $n \in S$. Choose a limit point z in range of M_u. Then there exists a sequence $(M_u x^n)$ which converges to z. Clearly, the sequence $\{M_u x^n\}$ is a Cauchy sequence. We obtain that

$$\begin{aligned}
\|M_u x^n - M_u x^m\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|u_k x^n_k - u_k x^m_k|}{\lambda}\right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{\substack{k=0 \\ k\in S}}^{m} (m+1-k)|u_k||x^n_k - x^m_k|}{\lambda}\right) \\
&\geq \varepsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{\substack{k=0 \\ k\in S}}^{m} (m+1-k)|x^n_k - x^m_k|}{\lambda}\right) \\
&= \varepsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|\widetilde{x^n_k} - \widetilde{x^m_k}|}{\lambda}\right) \\
&= \varepsilon \|\widetilde{x^n} - \widetilde{x^m}\|_{Ces^2_\varphi},
\end{aligned}$$

where

$$\widetilde{x^n_k} = \begin{cases} x^n_k & , \quad k \in S \\ 0 & , \quad k \notin S. \end{cases}$$

Hence, $\{\widetilde{x_n}\}$ is a Cauchy sequence in $Ces^2_\varphi(\mathbb{N})$. Since $Ces^2_\varphi(\mathbb{N})$ is a complete space, the sequence $\{\widetilde{x_n}\}$ converges to a point $x \in Ces^2_\varphi(\mathbb{N})$. By continuity of M_u, $M_u\widetilde{x_n} \to M_u x$. Also, we have $M_u x^n = M_u \widetilde{x_n} \to z$ and so $M_u x = z$. Hence, $z \in ranM_u$ which means that the range of M_u is closed. □

4. Invertible and Fredholm Multiplication Operators

Before we prove our main results in this section, remember the definition of the Fredholm operator.

If T has closed range, $\dim(kerT)$ and co-$\dim(ranT)$ are finite, then the bounded linear operator $T : X \to X$ is said to be a Fredholm operator.

Theorem 4.1. *Given any sequence $u \in \omega$, the multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is invertible if and only if there exist $K_1 > 0$ and $K_2 > 0$ such that $K_1 < u_n < K_2$ for all $n \in \mathbb{N}$.*

Proof. Let M_u be an invertible operator. Then, the range of M_u is $Ces^2_\varphi(\mathbb{N})$ and so it is closed. From Theorem 3.2, there exists $\varepsilon > 0$ such that $|u_n| \geq \varepsilon$ for all $n \in S$. If $u_k = 0$, for some $k \in \mathbb{N}$, we have $e^k \in kerM_u$ which is a contradiction, since $ker M_u$ is trivial. Hence, we have $|u_n| \geq \varepsilon$ for all $n \in \mathbb{N}$. By boundedness of M_u and Theorem 2.1, there exists $K > 0$ such that $|u_n| \leq K$ for all $n \in \mathbb{N}$. Thus, we conclude that $\varepsilon \leq |u_n| \leq K$ for all $n \in \mathbb{N}$.

For the converse, define a sequence $\gamma \in \omega$ as $\gamma_n = \frac{1}{u_n}$. Theorem 2.1 implies that M_u and M_γ are bounded linear operators. Also $M_u \cdot M_\gamma = M_\gamma \cdot M_u = I$ which means M_u is invertible and M_γ is its inverse. □

Theorem 4.2. *A bounded multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is a Fredholm operator if and only if*

(i) the set $\{k \in \mathbb{N} : u_k = 0\}$ is finite,

(ii) $|u_n| \geq \varepsilon$, for all $n \in S$.

Proof. Let M_u be a Fredholm operator. If the set $\{k \in \mathbb{N} : u_k = 0\}$ is infinite, then $M_u e^n = (0, 0, ..., 0, ...)$ for all $n \in \mathbb{N}$ with $u_n = 0$. Since e^n's are linearly independent, the space $\{x \in Ces^2_\varphi(\mathbb{N}) : M_u x = (0, 0, ..., 0, ...)\}$ is infinite dimensional. This is a contradiction. Thus, we conclude that (i) holds. Also, from Theorem 3.2, (ii) holds.

Conversely, let the conditions (i) and (ii) hold. By Theorem 3.2 and the condition (ii), we obtain that the range of M_u is closed. The condition (i) implies that $ker M_u$ and $ker M_u^*$ are finite dimensional. Hence, we conclude that M_u is Fredholm. □

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] P. R. Halmos, A Hilbert space problem book, New York-Basel-Honkong, (1991).

[2] R. K. Singh and A. Kumar, Multiplication operators and composition operators with closed ranges, Bulletin of the Australian Mathematical Society, **16**(1977), 247-252.

[3] R. K. Singh and A. Kumar, Compact composition operators, Journal of the Australian Mathematical Society: Pure Mathematics and Statistics, Series A, **28**(1979), 309-314.

[4] M. Mursaleen, A. Aghajani and K. Raj, Multiplication operators on Cesàro function spaces, Filomat, **30**(5) (2016), 1175-1184.

[5] M. İlkhan, S. Demiriz and E. E. Kara, Multiplication operators on Cesàro second order function spaces, Positivity, **24**(3) (2020), 605-614.

[6] R. E. Castillo, H. C. Chaparro and J. C. Ramos-Fernández, Orlicz-Lorentz spaces and their multiplication operators, Hacettepe Journal of Mathematics and Statistics, **44**(2015), 991-1009.

[7] R. E. Castillo, J. C. Ramos-Fernández and H. Rafeiro, Multiplication operators in variable Lebesgue spaces, Revista Colombiana de Matemáticas, **49**(2015), 293-305.

[8] R. E. Castillo, F. A. Vallejo Narvaez and J. C. Ramos-Fernández, Multiplication and composition operators on weak L_p spaces, Bulletin of the Malaysian Mathematical Sciences Society, **38**(2015), 927-973.

[9] J. S. Shiue, A note on Cesàro function spaces, Tamkang J. Math., **1**(1970), 91-95.

[10] G. M. Leibowitz, A note on Cesàro sequence spaces, Tamkang J. Math., **2**(1971), 151-157.

[11] A. A. Jagers, A note on Cesàro sequence spaces, Nieuw Arch. Wiskund, **22**(1974), 113-124.

[12] Y. Cui and R. Pluciennik, Banach-Saks property and property β in Cesàro sequence spaces, Southeast Asian Bull. Math., **24**(2000), 201-210.

[13] Y. Cui and H. Hudzik, Some geometric properties related to fixed point theory in Cesàro spaces, Collect. Math., **50**(1999), 277-288.

[14] B. Altay and R. Kama, On Cesàro summability of vector valued multiplier spaces and operator valued series, Positivity, **22**(2) (2018), 575-586.

[15] R. Kama, On some vector valued multiplier spaces with statistical Cesàro summability. Filomat, **33**(16) (2019), 5135-5147.

[16] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., **10**(1971), 379-390.

[17] N. Petrot and S. Suantai, Some geometric properties in Orlicz-Cesàro spaces, Science Asia, **31**(2005), 173-177.

[18] S. K. Lim and P. Y. Lee, An Orlicz extansion of Cesàro sequence spaces, Comment. Math. Prace. Mat., **28**(1988), 117-128.

[19] Y. Cui, H. Hudzik, N. Petrot and A. Szymaszkiewicz, Basic topological and geometric properties of Cesàro-Orlicz spaces, Proc. Indian Acad. Sci. Math., **115**(2005), 461-476.

[20] D. M. Kubiak, A note on Cesàro-Orlicz spaces, J. Math. Anal. Appl., **349**(2009), 291-296.

[21] K. Raj, C. Sharma and S. Pandoh, Multiplication operators on Cesàro-Orlicz sequence spaces, Fasciculi Mathematici, **57**(2016), 137-145.

[22] N. L. Braha, Geometric properties of the second-order Cesàro spaces, Banach J. Math. Anal., **10**(2016), 1-14.

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