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ON SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE nTH DERIVATIVES ARE (η_1, η_2)-STRONGLY CONVEX

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Abstract. The aim of this paper we establish some new inequalities of Hermite-Hadamard type by using (η_1, η_2) −strongly convex function whose nth derivatives in absolute value at certain powers. Moreover, we also consider their relevances for other related known results.

1. Introduction

In the following integral inequalities which are well known in the literature as the Hermite-Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

Many authors have studied and generalized the Hermite-Hadamard inequality in several ways via different classes functions. For some recent result related to the Hermite-Hadamard inequality, we refer the interested reader to the papers. $[4-15]$. Convex functions have played an important role in the development of various fields in pure and applied sciences. A significant class of convex functions is strongly convex functions. The strongly convex functions also play an important role in optimization theory and mathematical economics.

Now let's state the definitions necessary for our work.

Definition 1.1. [11]A set $I \subseteq \mathbb{R}$ is invex with respect to a real bifunction $\eta : I \times I \to \mathbb{R}$, if

$$x, y \in I,\ \lambda \in [0,1] \Longrightarrow y + \lambda\eta(x,y) \in I. \tag{1.2}$$

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If I is an invex set with respect to η, then a function $f : I \to \mathbb{R}$ is called preinvex , if $x, y \in I$ and $\lambda \in [0, 1]$.

$$f\left(y + \lambda\eta\left(x, y\right)\right) \leq \lambda f\left(x\right) + \left(1 - \lambda\right) f(y). \tag{1.3}$$

In 2016,Gordji et al. [11] introduced the concept $\eta-$convexity as follows:

Definition 1.2. A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta-$convex, if

$$f\left(tx + (1 - t)y\right) \leq f\left(y\right) + t\eta\left(f(x), f(y)\right) \tag{1.4}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.3. [24] Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f\left(I\right) \times f\left(I\right) \to \mathbb{R}$. The function f is said to be $\left(\eta_1, \eta_2\right)-$ convex, if

$$f\left(x + \lambda\eta_1\left(y, x\right)\right) \leq f\left(x\right) + \lambda\eta_2\left(f(y), f(x)\right) \tag{1.5}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.4. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f\left(I\right) \times f\left(I\right) \to \mathbb{R}$. The function f is said to be $\left(\eta_1, \eta_2\right)-$ strongly convex, if $c \geq 0$,

$$\begin{aligned} &f\left(x + \lambda\eta_1\left(y, x\right)\right) \\ &\leq f\left(x\right) + \lambda\eta_2\left(f(y), f(x)\right) - c\lambda\left(1 - \lambda\right)\eta_1\left(y, x\right)\eta_2\left(y, x\right) \end{aligned} \tag{1.6}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.5. An $\left(\eta_1, \eta_2\right)-$ strongly convex function reduces to

Remark 1.6. **(i)** If we choose $c = 0$ in definition 1.4 we obtain $\left(\eta_1, \eta_2\right)-$ convex function.

(ii) If we choose $c = 0$ and $\eta_1\left(x, y\right) = x - y$ for all $x, y \in I$ in definition 1.4 we obtain $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2\left(x, y\right) = x - y$ for all $x, y \in f\left(I\ \right)$ in definition 1.4 we obtain preinvex function.

(iv) If we choose $c = 0$ and $\eta_1\left(x, y\right) = \eta_2\left(x, y\right) = x - y$ in definition 1.4 we obtain classical convex function.

(v) If we choose $\eta_1\left(x, y\right) = \eta_2\left(x, y\right) = x - y$ in definition 1.4 we obtain strongly convex function.

(vi) If we choose $\eta_1\left(x, y\right) = x - y$ for all $x, y \in I$ in definition 1.4 we obtain $\eta - strongly$ convex function.

2. Main Results

In this section, we establish some new inequalities of Hermite-Hadamard type by using $\left(\eta_1, \eta_2\right)-$strongly convex function whose n th derivatives in absolute value at certain powers. Moreover, we also consider their relevances for other related known results.

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that for all $x \in I$ and $t \in [0, 1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable functions on*

I° with $a < b$, and $n \in N^{+}$. For any $a, b \in I^{\circ}$ with $\eta_1(b,a) > 0$, suppose that $f^n \in L_1[a, a+\eta_1(b,a)]$. Then for $\alpha > 0$, the following equality holds;

$$\begin{aligned}
&\tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx \\
&- \textstyle\sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k \left[f^{(k-1)}(a+\eta_1(b,a)) + (-1)^k f^{(k-1)}(a)\right]}{2(k!)} \\
&= \tfrac{\eta_1(b,a)^{nthis}}{2(n!)} \int_0^1 t^n f^{(n)}(a+t\eta_1(b,a))\,dt
\end{aligned} \tag{2.1}$$

Proof. By integration by parts, it follows that

$$\begin{aligned}
&\tfrac{\eta_1(b,a)^{n+1}}{2(n!)} \int_0^1 t^n f^{(n)}(a+t\eta_1(b,a))\,dt \\
&= -\tfrac{\eta_1(b,a)^n}{2(n!)} f^{(n-1)}(a+\eta_1(b,a)) + \tfrac{\eta_1(b,a)^n}{2[(n-1)!]} \int_0^1 t^{n-1} f^{(n-1)}(a+t\eta_1(b,a))\,dt \\
&= -\tfrac{\eta_1(b,a)^n}{2(n!)} f^{(n-1)}(a+\eta_1(b,a)) - \tfrac{\eta_1(b,a)^{n-1}}{2[(n-1)!]} f^{(n-2)}(a+\eta_1(b,a)) \\
&+ \tfrac{\eta_1(b,a)^{n-1}}{2[(n-2)!]} \int_0^1 t^{n-2} f^{(n-2)}(a+t\eta_1(b,a))\,dt \\
&= -\textstyle\sum_{k=1}^{n-1} \tfrac{\eta_1(b,a)^{k+1} f^{(k)}(a+\eta_1(b,a))}{2(k!)} + \tfrac{\eta_1(b,a)^2}{2} \int_0^1 t f'(a+t\eta_1(b,a))\,dt \\
&= -\textstyle\sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k f^{(k-1)}(a+\eta_1(b,a))}{2(k!)} + \tfrac{1}{2} \int_a^{a+\eta_1(b,a)} f(x)\,dx.
\end{aligned} \tag{2.2}$$

with the same argument as the above we have

$$\begin{aligned}
&\tfrac{\eta_1(b,a)^{n+1}}{2(n!)} \int_0^1 (t-1)^n f^{(n)}(a+t\eta_1(b,a))\,dt \\
&= -\tfrac{\eta_1(b,a)^n}{2(n!)} (-1)^n f^{(n-1)}(a) + \tfrac{\eta_1(b,a)^n}{2[(n-1)!]} \int_0^1 (t-1)^{n-1} f^{(n-1)}(a+t\eta_1(b,a))\,dt \\
&= -\tfrac{\eta_1(b,a)^n}{2(n!)} (-1)^n f^{(n-1)}(a) - \tfrac{\eta_1(b,a)^{n-1}}{2[(n-1)!]} (-1)^n f^{(n-2)}(a) \\
&+ \tfrac{\eta_1(b,a)^{n-1}}{2[(n-2)!]} \int_0^1 (t-1)^{n-2} f^{(n-2)}(a+t\eta_1(b,a))\,dt \\
&= -\textstyle\sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k (-1)^k f^{(k-1)}(a)}{2(k!)} + \tfrac{1}{2} \int_a^{a+\eta_1(b,a)} f(x)\,dx.
\end{aligned} \tag{2.3}$$

Adding these two equations leads to Lemma 2.1. □

Lemma 2.2. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that for all $x \in I$ and $t \in [0,1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable functions on I° with $a < b$, and $n \in N^{+}$. For any $a, b \in I^{\circ}$ with $\eta_1(b,a) > 0$, suppose that $f^n \in L_1[a, a+\eta_1(b,a)]$ .Then for $\alpha > 0$, the following equality holds;*

$$\begin{aligned}
&\tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \tfrac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \\
&- \textstyle\sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k \left[1+(-1)^k\right]}{2^k(k!)} \\
&\times \left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right] \\
&= \tfrac{\eta_1(b,a)^n}{2(n!)} \left[\int_0^{\frac{1}{2}} (-t)^n f^{(n)}(a+t\eta_1(b,a))\,dt\right. \\
&\left.+ \int_{\frac{1}{2}}^1 (1-t)^n f^{(n)}(b+t\eta_1(a,b))\,dt\right].
\end{aligned} \tag{2.4}$$

Proof. This follows from integration by parts immediately. □

Theorem 2.3. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that for all $x \in I$ and $t \in [0,1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable functions on I° with $a < b$, and $n \in N^{+}$ $(\eta_1, \eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I) \times f(I)$ with modulus $c \geq 0$. For any $a, b \in I^{\circ}$ with $\eta_1(b,a) > 0$, suppose that $f^n \in L_1[a, a+\eta_1(b,a)]$ and $|f^n|^q$ for $q \geq 1$.Then for*

$\alpha > 0$, *the following inequality holds;*

$$\begin{aligned} &\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^n \frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right| \\ &\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}} \\ &\times\left(\frac{2}{n+1}\left(|f^n(a)|^q\right)+\frac{1}{n+1}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{2c\eta_1(b,a)\eta_2(b,a)}{(n+2)(n+3)}\right)^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

Proof. By using Lemma 1, the power mean inequality and the (η_1, η_2) −strongly convex function of $|f^n|^q$, we have

(2.6)

$$\begin{aligned} &\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^n \frac{\eta_1(b,a)^k\left[f^{(k-1)}(a+\eta_1(b,a))+(-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right| \\ &\le \frac{\eta_1(b,a)^n}{2(n!)}\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(a+t\eta_1(b,a))\right|dt \\ &\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(a+t\eta_1(b,a))\right|dt\right)^{\frac{1}{q}} \\ &\le \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1\left[t^n+(1-t)^n\right]\left[|f^n(a)|^q+t\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}} \\ &= \frac{\eta_1(b,a)^n}{2(n!)}\left(\int_0^1\left[t^n+(1-t)^n\right]\right)^{1-\frac{1}{q}} \\ &\times\left(\left(|f^n(a)|^q\right)\int_0^1\left[t^n+(1-t)^n\right]dt+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\left(\int_0^1 t\left[t^n+(1-t)^n\right]dt\right)\right. \\ &\left.-c\eta_1(b,a)\eta_2(b,a)\int_0^1 t(1-t)\left[t^n+(1-t)^n\right]dt\right)^{\frac{1}{q}} \\ &= \frac{\eta_1(b,a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{2}{n+1}\left(|f^n(a)|^q\right)+\frac{1}{n+1}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{2c\eta_1(b,a)\eta_2(b,a)}{(n+2)(n+3)}\right)^{\frac{1}{q}} \end{aligned}$$

where

$$\int_0^1\left[t^n+(1-t)^n\right]dt=\frac{2}{n+1} \tag{2.7}$$

$$\int_0^1 t\left[t^n+(1-t)^n\right]dt=\frac{1}{n+1} \tag{2.8}$$

and

$$\int_0^1 t(1-t)\left[t^n+(1-t)^n\right]dt=\frac{2}{(n+2)(n+3)} \tag{2.9}$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.3 which show that our result generalize several results obtained previous works.

Remark 2.4. As can be seen from the special elections below, our results are more general.

(i) If we choose $c = 0$ in *Theorem* 2.3 the results are we obtain also provided for (η_1, η_2) −convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c = 0$ and $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.3 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$ in *Theorem* 2.3 the results are we obtain also provided for preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.3 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x,y) = \eta_2(x,y) = x - y$ in *Theorem* 2.3 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x,y) = x - y$ for all $x, y \in I$ in *Theorem* 2.3 we obtain $\eta-$*strongly* convex function.

Theorem 2.5. *Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1$such that for all $x \in I$ and $t \in [0,1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differantiable functions on I° with $a < b$, and $n \in N^+$ $(\eta_1, \eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I) \times f(I)$ with modulus $c \geq 0$. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$, suppose that $f^n \in L_1[a, a + \eta_1(b,a)]$ and $|f^n|^q$ for $q \geq 1$.Then for $\alpha > 0$, the following inequality holds;*

(2.10)
$$\left|\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^n \frac{\eta_1(b,a)^k \left[f^{(k-1)}(a+\eta_1(b,a)) + (-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right|$$
$$\leq \frac{\eta_1(b,a)^n}{2(n!)} \left(\int_0^1 [t^n + (1-t)^n]^p\,dt\right)^{\frac{1}{p}}$$
$$\times \left(|f^n(a)|^q + \frac{1}{2}\eta_2(|f^n(b)|^q, |f^n(a)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 1, the Hölder's inequality and the $(\eta_1, \eta_2)-$strongly convexity of $|f^n|^q$, we have

(2.11)
$$\left|\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx - \sum_{k=1}^n \frac{\eta_1(b,a)^k \left[f^{(k-1)}(a+\eta_1(b,a)) + (-1)^k f^{(k-1)}(a)\right]}{2(k!)}\right|$$
$$\leq \frac{\eta_1(b,a)^n}{2(n!)} \int_0^1 [t^n + (1-t)^n] \left|f^{(n)}(a + t\eta_1(b,a))\right| dt$$
$$\leq \frac{\eta_1(b,a)^n}{2(n!)} \left(\int_0^1 [t^n + (1-t)^n]^p\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(n)}(a + t\eta_1(b,a))\right|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta_1(b,a)^n}{2(n!)} \left(\int_0^1 [t^n + (1-t)^n]^p\,dt\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 [|f^n(a)|^q + t\eta_2(|f^n(b)|^q, |f^n(a)|^q) - ct(1-t)\eta_1(b,a)\eta_2(b,a)]\,dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta_1(b,a)^n}{2(n!)} \left(\int_0^1 [t^n + (1-t)^n]^p\,dt\right)^{\frac{1}{p}}$$
$$\times \left(|f^n(a)|^q \int_0^1 1dt + \eta_2(|f^n(b)|^q, |f^n(a)|^q) \int_0^1 tdt - c\eta_1(b,a)\eta_2(b,a) \int_0^1 t(1-t)\,dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta_1(b,a)^n}{2(n!)} \left(\int_0^1 [t^n + (1-t)^n]^p\,dt\right)^{\frac{1}{p}} \left(|f^n(a)|^q + \frac{1}{2}\eta_2(|f^n(b)|^q, |f^n(a)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}.$$

It can easily be verified that $t^n + (1-t)^n \leq 1$ for $t \in [0,1]$. So, it follows that

$$\int_0^1 [t^n + (1-t)^n]^p\,dt \leq \int_0^1 [t^n + (1-t)^n]\,dt = \frac{2}{n+1} \tag{2.12}$$

Hence, the desired inequality follows from 2.11 and 2.12. This completes the proof of the theorem. □

We will give some special cases of Theorem 2.5 which show that our result generalize several results obtained previous works.

Remark 2.6. As can be seen from the special elections below, our results are more general.

(i) If we choose $c = 0$ in *Theorem* 2.5 the results are we obtain also provided for $(\eta_1, \eta_2)-$convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c = 0$ and $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.5 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$ in *Theorem* 2.5 the results are we obtain also provided for preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.5 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x, y) = \eta_2(x, y) = x - y$ in *Theorem* 2.5 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x, y) = x - y$ for all $x, y \in I$ in *Theorem* 2.5 we obtain $\eta - strongly$ convex function.

Theorem 2.7. *Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1$such that for all $x \in I$ and $t \in [0, 1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differantiable functions on I° with $a < b$, and $n \in N^+$ $(\eta_1, \eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I) \times f(I)$ with modulus $c \geq 0$. For any $a, b \in I^\circ$ with $\eta_1(b, a) > 0$, suppose that $f^n \in L_1[a, a + \eta_1(b, a)]$ and $|f^n|^q$ for $q \geq 1$.Then for $\alpha > 0$, the following inequality holds;*

$$\Big| \tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \tfrac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx$$
$$- \sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k \left[1+(-1)^k\right]}{2^k (k!)}$$
$$\Big| \times \left[f^{(k-1)}\left(a + \tfrac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b + \tfrac{1}{2}\eta_1(a,b)\right) \right] \Big|$$

$$\begin{aligned}
&\leq \tfrac{\eta_1(b,a)^n}{(n!)} \left(\tfrac{1}{2^{n+1}(n+1)} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\Bigg(\tfrac{1}{2^{n+1}(n+1)} |f^n(a)|^q + \tfrac{1}{2^{n+2}(n+2)} \eta_2(|f^n(b)|^q, |f^n(a)|^q) \\
&- \tfrac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)} \Bigg)^{\frac{1}{q}} \Bigg] \\
&+ \tfrac{\eta_1(b,a)^n}{(n!)} \left(\tfrac{1}{2^{n+1}(n+1)} \right)^{1-\frac{1}{q}} \\
&\Bigg[\Bigg(\tfrac{1}{2^{n+1}(n+1)} |f^n(b)|^q + \tfrac{n+3}{2^{n+2}(n+2)(n+1)} \eta_2(|f^n(a)|^q, |f^n(b)|^q) \\
&- \tfrac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)} \Bigg)^{\frac{1}{q}} \Bigg].
\end{aligned} \tag{2.13}$$

Proof. By using Lemma 2, the Power mean inequality and the $(\eta_1, \eta_2)-$strongly convexity of $|f^n|^q$, we have

$$\begin{aligned}
&\Big| \tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \tfrac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \\
&- \sum_{k=1}^{n} \tfrac{\eta_1(b,a)^k \left[1+(-1)^k\right]}{2^k (k!)} \left[f^{(k-1)}\left(a + \tfrac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b + \tfrac{1}{2}\eta_1(a,b)\right) \right] \Big| \\
&\leq \tfrac{\eta_1(b,a)^n}{(n!)} \left[\int_0^{\frac{1}{2}} (t)^n \left| f^{(n)}(a + t\eta_1(b,a)) \right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left| f^{(n)}(b + t\eta_1(a,b)) \right| dt \right] \\
&\leq \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_0^{\frac{1}{2}} (t)^n\,dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} (t)^n \left| f^{(n)}(a + t\eta_1(b,a)) \right|^q \right)^{\frac{1}{q}} dt \right] \\
&+ \tfrac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^n\,dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 (1-t)^n \left| f^{(n)}(b + t\eta_1(a,b)) \right|^q dt \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^n dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left.\left(\int_0^{\frac{1}{2}}(t)^n\left[|f^n(a)|^q+t\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left.\left(\int_{\frac{1}{2}}^1(1-t)^n\left[|f^n(b)|^q+t\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-ct(1-t)\eta_1(b,a)\eta_2(b,a)\right]dt\right)^{\frac{1}{q}}\right]$$

(2.14)

$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^n dt\right)^{1-\frac{1}{q}}\left(\left(|f^n(a)|^q\int_0^{\frac{1}{2}}(t)^n dt\right)+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\int_0^{\frac{1}{2}}t^{n+1}dt\right.\right.$$
$$\left.\left.-c\eta_1(b,a)\eta_2(b,a)\int_0^{\frac{1}{2}}t^{n+1}(1-t)\,dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\left(|f^n(b)|^q\int_{\frac{1}{2}}^1(1-t)^n\right)+\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\int_{\frac{1}{2}}^1 t(1-t)^n\right.\right.$$
$$\left.\left.-c\eta_1(b,a)\eta_2(b,a)\int_{\frac{1}{2}}^1 t(1-t)^{n+1}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^n(a)|^q+\frac{1}{2^{n+2}(n+2)}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\right.\right.$$
$$\left.\left.-\frac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^n(b)|^q+\frac{n+3}{2^{n+2}(n+2)(n+1)}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\right.\right.$$
$$\left.\left.-\frac{c\eta_1(b,a)\eta_2(b,a)(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}}\right].$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.7 which show that our result generalize several results obtained previous works.

Remark 2.8. As can be seen from the special elections below, our results are more general.

(i) If we choose $c = 0$ in $Theorem$ 2.7 the results are we obtain also provided for $(\eta_1, \eta_2)-$convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c = 0$ and $\eta_1(x, y) = x - y$ for all $x, y \in I$ in $Theorem$ 2.7 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c = 0$ and $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$ in $Theorem$ 2.7 the results are we obtain also provided for preinvex function.

(iv) If we choose $c = 0$ and $\eta_1(x, y) = \eta_2(x, y) = x - y$ in $Theorem$ 2.7 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x, y) = \eta_2(x, y) = x - y$ in $Theorem$ 2.7 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x, y) = x - y$ for all $x, y \in I$ in $Theorem$ 2.7we obtain $\eta - strongly$ convex function.

Theorem 2.9. *Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1$such that for all $x \in I$ and $t \in [0, 1]$. Also let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differantiable functions on I° with $a < b$, and $n \in N^+$ $(\eta_1, \eta_2)-$strongly convex function where η_2 is an integrable bi functionon $f(I) \times f(I)$ with modulus $c \geq 0$. For any $a, b \in I^\circ$ with*

$\eta_1(b,a) > 0$, *suppose that* $f^n \in L_1[a, a+\eta_1(b,a)]$ *and* $|f^n|^q$ *for* $q \geq 1$. *Then for* $\alpha > 0$, *the following inequality holds;*

$$\left| \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \frac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \right.$$
$$- \sum_{k=1}^{n} \frac{\eta_1(b,a)^k \left[1+(-1)^k\right]}{2^k(k!)}$$
$$\left. \times \left[f^{(k-1)}\left(a+\frac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b+\frac{1}{2}\eta_1(a,b)\right)\right] \right|$$

$$\leq \frac{\eta_1(b,a)^n}{2(n!)} \left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}} \tag{2.15}$$
$$\times \left[\left(|f^n(a)|^q + \frac{1}{4}\eta_2(|f^n(b)|^q, |f^n(a)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6} \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(|f^n(b)|^q + \frac{3}{4}\eta_2(|f^n(a)|^q, |f^n(b)|^q) - \frac{c\eta_1(b,a)\eta_2(b,a)}{6} \right)^{\frac{1}{q}} \right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Again, using Lemma 2, the Hölder's inequality and the $(\eta_1, \eta_2)-$strongly convexity of $|f^n|^q$, we have

$$\left| \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)\,dx + \frac{1}{\eta_1(b,a)} \int_{b+\frac{1}{2}\eta_1(a,b)}^{b+\eta_1(a,b)} f(x)\,dx \right.$$
$$\left. - \sum_{k=1}^{n} \frac{\eta_1(b,a)^k\left[1+(-1)^k\right]}{2^k(k!)} \left[f^{(k-1)}\left(a+\tfrac{1}{2}\eta_1(b,a)\right) + f^{(k-1)}\left(b+\tfrac{1}{2}\eta_1(a,b)\right)\right] \right|$$
$$\leq \frac{\eta_1(b,a)^n}{(n!)} \left[\int_0^{\frac{1}{2}} (t)^n \left|f^{(n)}(a+t\eta_1(b,a))\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(b+t\eta_1(a,b))\right| dt \right]$$
$$\leq \frac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_0^{\frac{1}{2}} (t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta_1(b,a))\right|^q dt\right)^{\frac{1}{q}} \right]$$
$$+ \frac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f^{(n)}(b+t\eta_1(a,b))\right| dt\right)^{\frac{1}{q}} \right]$$

$$\leq \frac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_0^{\frac{1}{2}} (t)^{np}\,dt\right)^{\frac{1}{p}} \right.$$
$$\left. \times \left(\int_0^{\frac{1}{2}} \left[|f^n(a)|^q + t\eta_2(|f^n(b)|^q, |f^n(a)|^q) - ct(1-t)\eta_1(b,a)\eta_2(b,a)\right] dt\right)^{\frac{1}{q}} \right]$$
$$+ \frac{\eta_1(b,a)^n}{(n!)} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \right.$$
$$\left. \times \left(\int_{\frac{1}{2}}^1 \left[|f^n(b)|^q + t\eta_2(|f^n(a)|^q, |f^n(b)|^q) - ct(1-t)\eta_1(b,a)\eta_2(b,a)\right] dt\right)^{\frac{1}{q}} \right]$$

(2.16)

$$\leq \frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_0^{\frac{1}{2}}(t)^{np}\,dt\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(|f^n(a)|^q\int_0^{\frac{1}{2}}1dt+\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)\int_0^{\frac{1}{2}}tdt-c\eta_1(b,a)\,\eta_2(b,a)\int_0^{\frac{1}{2}}t\,(1-t)\,dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left[\left(\int_{\frac{1}{2}}^1(1-t)^{np}\,dt\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(|f^n(b)|^q\int_{\frac{1}{2}}^1 1dt+\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)\int_{\frac{1}{2}}^1 tdt-c\eta_1(b,a)\,\eta_2(b,a)\int_{\frac{1}{2}}^1 t\,(1-t)\,dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}|f^n(a)|^q+\frac{1}{8}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{c\eta_1(b,a)\eta_2(b,a)}{12}\right)^{\frac{1}{q}}\right]$$
$$+\frac{\eta_1(b,a)^n}{(n!)}\left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}|f^n(b)|^q+\frac{3}{8}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-\frac{c\eta_1(b,a)\eta_2(b,a)}{12}\right)^{\frac{1}{q}}\right]$$
$$=\frac{\eta_1(b,a)^n}{2(n!)}\left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\left[\left(|f^n(a)|^q+\frac{1}{4}\eta_2\left(|f^n(b)|^q,|f^n(a)|^q\right)-\frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f^n(b)|^q+\frac{3}{4}\eta_2\left(|f^n(a)|^q,|f^n(b)|^q\right)-\frac{c\eta_1(b,a)\eta_2(b,a)}{6}\right)^{\frac{1}{q}}\right]$$

This completes the proof of the theorem. □

We will give some special cases of Theorem 2.9 which show that our result generalize several results obtained previous works.

Remark 2.10. As can be seen from the special elections below, our results are more general.

(i) If we choose $c=0$ in *Theorem* 2.9 the results are we obtain also provided for (η_1,η_2) −convex functions, is proved by S. Kermausuor et. al. [25].

(ii) If we choose $c=0$ and $\eta_1(x,y)=x-y$ for all $x,y\in I$ in *Theorem* 2.9 the results are we obtain also provided for $\eta-$ convex function.

(iii) If we choose $c=0$ and $\eta_2(x,y)=x-y$ for all $x,y\in f(I)$ in *Theorem* 2.9 the results are we obtain also provided for preinvex function.

(iv) If we choose $c=0$ and $\eta_1(x,y)=\eta_2(x,y)=x-y$ in *Theorem* 2.9 the results are we obtain also provided for classical convex function.

(v) If we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$ in *Theorem* 2.9 the results are we obtain also provided for strongly convex function.

(vi) If we choose $\eta_1(x,y)=x-y$ for all $x,y\in I$ in *Theorem* 2.9 we obtain $\eta-$ *strongly* convex function.

3. Conclusion

In this study, we present some inequalities for (η_1,η_2) −strongly convex functions involving whose nth derivatives in absolute value at certain powers. It is also shown that the results proved here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of strongly convexity in their future work.

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