Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (3) (2022), 775 – 786
DOI : 10.15672/hujms.946069

Research Article

# Hermite–Hadamard-type inequalities for conformable integrals

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## Abstract

In this study, some inequalities of Hermite–Hadamard type for integrals arising in conformable fractional calculus are presented. In fact, the obtained inequalities are not only valid for those integrals arising in conformable fractional calculus, but for more general integrals as well. Numerous known versions are recovered as special cases. We also illustrate our findings via applications to modified Bessel functions, special means, and midpoint approximations.

**Mathematics Subject Classification (2020).** 26D07, 26D15, 26D10, 26A33

**Keywords.** Hermite–Hadamard inequality, convex function, conformable integral

## 1. Introduction

In [1], a so-called conformable fractional integral of the form

$$\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x = \int_{\rho_1}^{\rho_2} x^{\omega-1} f(x)\mathrm{d}x \tag{1.1}$$

(if it exists and is finite) was introduced for $\omega \in (0,1]$. In view of recent results in the theories of differential, integral, and fractional differential equations, it becomes clear that certain integral inequalities are very useful in determining bounds of unknown functions; see, e.g., [2, 6–9, 11, 14, 15, 19, 25, 29]. Also, there are various integral inequalities in the literature, and research in this area is very active. One main inequality is the Hermite–Hadamard integral inequality, due to Hadamard (1893), which says that for convex functions $f$, we have

$$f\left(\frac{\rho_1+\rho_2}{2}\right) \le \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}x \le \frac{f(\rho_1)+f(\rho_2)}{2}. \tag{1.2}$$

Inequality (1.2) has been extended and applied to time scales [4, 5] and to many models of fractional calculus, such as Riemann–Liouville [18, 21, 26], $\psi$-Riemann–Liouville [17],

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Received: 31.05.2021; Accepted: 02.12.2021

conformable fractional [20], generalized fractional [22, 24], time scales fractional [3, 28], and tempered fractional [23].

In this article, after proving some auxiliary results in Section 2, we devote Section 3 to obtain three distinct inequalities of Hermite–Hadamard type for integrals of the form (1.1). In each case, by choosing $\omega = 1$, we recover known results from the literature. In Section 4, we apply our results to integer values of $\omega$, and in Section 4, we demonstrate the usefulness of our inequalities by offering three different applications. We emphasize that our Hermite–Hadamard inequalities for conformable integrals (1.1) collapse to known Hermite–Hadamard inequalities for integrals when $\omega = 1$.

## 2. Auxiliary results

At first, we recall the definition of convexity. We say that $f : I \to \mathbb{R}$ is convex on the interval $I$ if

$$f(\eta x + (1-\eta)y) \leq \eta f(x) + (1-\eta) f(y) \quad \text{for all} \quad x, y \in I \quad \text{and} \quad \eta \in [0,1].$$

In the first auxiliary result, we collect some formulas that will be needed in the proofs of our main results.

**Lemma 2.1.** *Let $\omega > 0$ and $z_1 \neq z_2$. Denote*

$$\begin{aligned} z_* &:= \frac{z_1 + z_2}{2}, \\ G(z_1, z_2) &:= \int_{z_1}^{z_*} (x^\omega - z_1^\omega)\, \mathrm{d}x, \\ F_1(z_1, z_2) &:= \frac{1}{z_2 - z_1} \int_{z_1}^{z_*} (x^\omega - z_1^\omega)(x - z_1)\, \mathrm{d}x, \\ F_2(z_1, z_2) &:= \frac{1}{z_2 - z_1} \int_{z_1}^{z_*} (x^\omega - z_1^\omega)(z_2 - x)\, \mathrm{d}x. \end{aligned}$$

*Then we have*

$$G(z_1, z_2) = \frac{z_*^{\omega+1} - z_1^{\omega+1}}{\omega + 1} - \frac{z_1^\omega (z_2 - z_1)}{2}, \tag{2.1}$$

$$\int_{z_*}^{z_2} (z_2^\omega - x^\omega)\, \mathrm{d}x = G(z_2, z_1), \tag{2.2}$$

$$F_1(z_1, z_2) = \frac{z_1^{\omega+2} - z_*^{\omega+2}}{(\omega+1)(\omega+2)(z_2 - z_1)} + \frac{z_*^{\omega+1}}{2(\omega+1)} - z_1^\omega \frac{z_2 - z_1}{8}, \tag{2.3}$$

$$F_2(z_1, z_2) = \frac{z_*^{\omega+2} - z_1^{\omega+2}}{(\omega+1)(\omega+2)(z_2 - z_1)} + \frac{\frac{z_*^{\omega+1}}{2} - z_1^{\omega+1}}{\omega + 1} - 3 z_1^\omega \frac{z_2 - z_1}{8}, \tag{2.4}$$

$$\frac{1}{z_2 - z_1} \int_{z_*}^{z_2} (z_2^\omega - x^\omega)(x - z_1)\, \mathrm{d}x = F_2(z_2, z_1), \tag{2.5}$$

$$\frac{1}{z_2 - z_1} \int_{z_*}^{z_2} (z_2^\omega - x^\omega)(z_2 - x)\, \mathrm{d}x = F_1(z_2, z_1). \tag{2.6}$$

***Proof.*** A simple integration verifies (2.1). The definition of $G$ shows (2.2). Now we verify the second equal sign in (2.3). We use (2.1) three times (with replacements of $\omega$ by $\omega + 1, \omega, 1$) as well as twice the formula

$$z_* - z_1 = \frac{z_2 - z_1}{2} \tag{2.7}$$

to find

$$F_1(z_1, z_2) = \frac{1}{z_2 - z_1} \int_{z_1}^{z_*} \left[ x^{\omega+1} - z_1^{\omega+1} - z_1 (x^\omega - z_1^\omega) - z_1^\omega (x - z_1) \right] \mathrm{d}x$$

$$= \frac{z_*^{\omega+2} - z_1^{\omega+2}}{(\omega+2)(z_2 - z_1)} - \frac{z_1^{\omega+1}}{2} - z_1\left(\frac{z_*^{\omega+1} - z_1^{\omega+1}}{(\omega+1)(z_2 - z_1)} - \frac{z_1^{\omega}}{2}\right) - z_1^{\omega}\left(\frac{z_*^2 - z_1^2}{2(z_2 - z_1)} - \frac{z_1}{2}\right)$$

$$= \frac{z_1^{\omega+2} - z_*^{\omega+2} + (\omega+2)\left(z_*^{\omega+2} - z_1^{\omega+2}\right)}{(\omega+1)(\omega+2)(z_2 - z_1)} + \frac{z_1^{\omega+2} - z_1 z_*^{\omega+1}}{(\omega+1)(z_2 - z_1)} - z_1^{\omega}\left(\frac{z_* + z_1}{4} - \frac{z_1}{2}\right).$$

Since $F_1 + F_2 = G$, (2.4) is just a simple calculation using (2.1) and (2.3). Finally, (2.5) and (2.6) follow from the definitions of $F_2$ and $F_1$, respectively. □

**Remark 2.2.** If we let $\omega = 1$ in Lemma 2.1 and use (2.7) several times, then we have

$$G(z_1, z_2) = \int_{z_1}^{z_*} (x - z_1)\,\mathrm{d}x = \frac{(z_* - z_1)^2}{2} = \frac{(z_2 - z_1)^2}{8},$$

$$F_1(z_1, z_2) = \frac{1}{z_2 - z_1}\int_{z_1}^{z_*} (x - z_1)^2\,\mathrm{d}x = \frac{(z_* - z_1)^3}{3(z_2 - z_1)} = \frac{(z_2 - z_1)^2}{24},$$

$$F_2(z_1, z_2) = G(z_1, z_2) - F_1(z_1, z_2) = \frac{(z_2 - z_1)^2}{12}.$$

Now, the following lemma is a key to obtain our main results. In the sequel, we assume

($\mathrm{H}_\omega$) $\omega > 0$, $0 \le \rho_1 < \rho_2$, and $f : [\rho_1, \rho_2] \to \mathbb{R}$ is such that

$$\wp_1 := \int_{\rho_1}^{\rho_*} \left(x^\omega - \rho_1^\omega\right) f'(x)\mathrm{d}x \quad \text{and} \quad \wp_2 := \int_{\rho_*}^{\rho_2} \left(\rho_2^\omega - x^\omega\right) f'(x)\mathrm{d}x$$

are well defined. Recall $\rho_* = (\rho_1 + \rho_2)/2$.

**Lemma 2.3.** *If ($H_\omega$) holds, then*

$$\frac{\omega}{\rho_2^\omega - \rho_1^\omega}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x - f(\rho_*) = \frac{\wp_2 - \wp_1}{\rho_2^\omega - \rho_1^\omega}. \tag{2.8}$$

***Proof.*** Using integration by parts and (1.1), we get

$$\begin{aligned}\wp_1 &= \int_{\rho_1}^{\rho_*} \left(x^\omega - \rho_1^\omega\right) f'(x)\mathrm{d}x \\ &= \left(x^\omega - \rho_1^\omega\right) f(x)\Big|_{\rho_1}^{\rho_*} - \int_{\rho_1}^{\rho_*} \omega x^{\omega-1} f(x)\mathrm{d}x \\ &= \left(\rho_*^\omega - \rho_1^\omega\right) f(\rho_*) - \omega\int_{\rho_1}^{\rho_*} f(x)\mathrm{d}_\omega x.\end{aligned}$$

Similarly,

$$\begin{aligned}\wp_2 &= \int_{\rho_*}^{\rho_2} \left(\rho_2^\omega - x^\omega\right) f'(x)\mathrm{d}x \\ &= \left(\rho_2^\omega - x^\omega\right) f(x)\Big|_{\rho_*}^{\rho_2} + \int_{\rho_*}^{\rho_2} \omega x^{\omega-1} f(x)\mathrm{d}x \\ &= \omega\int_{\rho_*}^{\rho_2} f(x)\mathrm{d}_\omega x - \left(\rho_2^\omega - \rho_*^\omega\right) f(\rho_*).\end{aligned}$$

Altogether, we have

$$\wp_2 - \wp_1 = \omega\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x - \left(\rho_2^\omega - \rho_1^\omega\right) f(\rho_*),$$

from which we obtain the desired identity (2.8). This completes the proof. □

**Remark 2.4.** If we take absolute values in (2.8), then we obtain the inequality

$$\left|\frac{\omega}{\rho_2^\omega - \rho_1^\omega}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x - f(\rho_*)\right| \le \frac{|\wp_1| + |\wp_2|}{\rho_2^\omega - \rho_1^\omega}. \tag{2.9}$$

## 3. Main results

Now we present the three main results of this paper.

**Theorem 3.1.** *If* $(H_\omega)$ *holds and* $|f'|$ *is convex, then*

$$\left|\frac{\omega}{\rho_2^\omega-\rho_1^\omega}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x - f(\rho_*)\right| \le \frac{F(\rho_1,\rho_2)\,|f'(\rho_1)| + F(\rho_2,\rho_1)\,|f'(\rho_2)|}{\rho_2^\omega-\rho_1^\omega},$$

*where*

$$F(z_1,z_2) := \frac{2z_*^{\omega+2}-z_1^{\omega+2}-z_2^{\omega+2}}{(\omega+1)(\omega+2)\,(z_2-z_1)} + \frac{z_*^{\omega+1}-z_1^{\omega+1}}{\omega+1} + \left(z_2^\omega-3z_1^\omega\right)\frac{z_2-z_1}{8}, \quad z_1\neq z_2.$$

***Proof.*** By the convexity of $|f'|$, we get

$$|f'(x)| = \left|f'\left(\frac{\rho_2-x}{\rho_2-\rho_1}\rho_1+\frac{x-\rho_1}{\rho_2-\rho_1}\rho_2\right)\right| \le \frac{\rho_2-x}{\rho_2-\rho_1}\,|f'(\rho_1)| + \frac{x-\rho_1}{\rho_2-\rho_1}\,|f'(\rho_2)|,$$

and thus

$$\begin{aligned}|\wp_1| &\le \int_{\rho_1}^{\rho_*}\left(x^\omega-\rho_1^\omega\right)|f'(x)|\,\mathrm{d}x\\ &\le \int_{\rho_1}^{\rho_*}\left(x^\omega-\rho_1^\omega\right)\left[\frac{\rho_2-x}{\rho_2-\rho_1}\,|f'(\rho_1)|+\frac{x-\rho_1}{\rho_2-\rho_1}\,|f'(\rho_2)|\right]\mathrm{d}x\\ &= F_2(\rho_1,\rho_2)\,|f'(\rho_1)| + F_1(\rho_1,\rho_2)\,|f'(\rho_2)|.\end{aligned}$$

Similarly,

$$\begin{aligned}|\wp_2| &\le \int_{\rho_*}^{\rho_2}\left(\rho_2^\omega-x^\omega\right)|f'(x)|\,\mathrm{d}x\\ &\le \int_{\rho_*}^{\rho_2}\left(\rho_2^\omega-x^\omega\right)\left[\frac{\rho_2-x}{\rho_2-\rho_1}\,|f'(\rho_1)|+\frac{x-\rho_1}{\rho_2-\rho_1}\,|f'(\rho_2)|\right]\mathrm{d}x\\ &= F_1(\rho_2,\rho_1)\,|f'(\rho_1)| + F_2(\rho_2,\rho_1)\,|f'(\rho_2)|,\end{aligned}$$

where we have also used (2.5) and (2.6). From (2.3) and (2.4), it is easy to see that

$$F_1(z_2,z_1)+F_2(z_1,z_2) = F(z_1,z_2). \tag{3.1}$$

Employing now (2.9) completes the proof. □

When we choose $\omega=1$ in Theorem 3.1, we get the following result.

**Corollary 3.2.** *If* $(H_1)$ *holds and* $|f'|$ *is convex, then we have*

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}x - f(\rho_*)\right| \le \frac{\rho_2-\rho_1}{8}\left(|f'(\rho_1)|+|f'(\rho_2)|\right).$$

***Proof.*** If $\omega=1$, then, due to (3.1) and Remark 2.2, we get

$$F(z_1,z_2) = F_1(z_2,z_1)+F_2(z_1,z_2) = \frac{(z_2-z_1)^2}{24}+\frac{(z_2-z_1)^2}{12} = \frac{(z_2-z_1)^2}{8},$$

and thus the statement follows from Theorem 3.1. □

**Remark 3.3.** Note that Corollary 3.2 was proved first in [13, Theorem 2.1]. Hence, our Theorem 3.1 is a generalization of [13, Theorem 2.1].

**Theorem 3.4.** *If* $(H_\omega)$ *holds and* $|f'|^q$ *is convex for some* $q\ge 1$*, then*

$$\begin{aligned}&\left|\frac{\omega}{\rho_2^\omega-\rho_1^\omega}\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x - f(\rho_*)\right|\\ &\quad\le \frac{\left(G(\rho_1,\rho_2)\right)^{1-\frac{1}{q}}}{\rho_2^\omega-\rho_1^\omega}\left(F_2(\rho_1,\rho_2)\,|f'(\rho_1)|^q + F_1(\rho_1,\rho_2)\,|f'(\rho_2)|^q\right)^{\frac{1}{q}}\end{aligned}$$

$$+\frac{(G(\rho_2,\rho_1))^{1-\frac{1}{q}}}{\rho_2^\omega-\rho_1^\omega}\left(F_1(\rho_2,\rho_1)\left|f'(\rho_1)\right|^q+F_2(\rho_2,\rho_1)\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}$$

*where $F_1$, $F_2$, and $G$ are given in Lemma 2.1.*

***Proof.*** By the convexity of $|f'|^q$, we get

$$|f'(x)|^q=\left|f'\left(\frac{\rho_2-x}{\rho_2-\rho_1}\rho_1+\frac{x-\rho_1}{\rho_2-\rho_1}\rho_2\right)\right|^q\le\frac{\rho_2-x}{\rho_2-\rho_1}\left|f'(\rho_1)\right|^q+\frac{x-\rho_1}{\rho_2-\rho_1}\left|f'(\rho_2)\right|^q, \quad (3.2)$$

and thus, with the help of the power-mean inequality,

$$\begin{aligned}|\wp_1|&\le\int_{\rho_1}^{\rho_*}(x^\omega-\rho_1^\omega)\left|f'(x)\right|\mathrm{d}x\\&\le\left(\int_{\rho_1}^{\rho_*}(x^\omega-\rho_1^\omega)\,\mathrm{d}x\right)^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_*}(x^\omega-\rho_1^\omega)\left|f'(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\&=(G(\rho_1,\rho_2))^{1-\frac{1}{q}}\left(\int_{\rho_1}^{\rho_*}(x^\omega-\rho_1^\omega)\left|f'(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\&\le(G(\rho_1,\rho_2))^{1-\frac{1}{q}}\left(F_2(\rho_1,\rho_2)\left|f'(\rho_1)\right|^q+F_1(\rho_1,\rho_2)\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly,

$$\begin{aligned}|\wp_2|&\le\int_{\rho_*}^{\rho_2}(\rho_2^\omega-x^\omega)\left|f'(x)\right|\mathrm{d}x\\&\le\left(\int_{\rho_*}^{\rho_2}(\rho_2^\omega-x^\omega)\,\mathrm{d}x\right)^{1-\frac{1}{q}}\left(\int_{\rho_*}^{\rho_2}(\rho_2^\omega-x^\omega)\left|f'(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\&=(G(\rho_2,\rho_1))^{1-\frac{1}{q}}\left(\int_{\rho_*}^{\rho_2}(\rho_2^\omega-x^\omega)\left|f'(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\&\le(G(\rho_2,\rho_1))^{1-\frac{1}{q}}\left(F_1(\rho_2,\rho_1)\left|f'(\rho_1)\right|^q+F_2(\rho_2,\rho_1)\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}},\end{aligned}$$

where we have also used (2.2), (2.5), and (2.6). Employing now (2.9) completes the proof. □

**Remark 3.5.** Theorem 3.4 with $q=1$ becomes Theorem 3.1.

When we choose $\omega=1$ in Theorem 3.4, we get the following result.

**Corollary 3.6.** *If $(H_1)$ holds and $|f'|^q$ is convex for some $q\ge1$, then*

$$\begin{aligned}&\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}f(x)\mathrm{d}x-f(\rho_*)\right|\\&\qquad\le\frac{\rho_2-\rho_1}{8\cdot3^{\frac{1}{q}}}\left[\left(2\left|f'(\rho_1)\right|^q+\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(\rho_1)\right|^q+2\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

***Proof.*** This follows directly from Theorem 3.4 and Remark 2.2. □

**Remark 3.7.** Note that Corollary 3.6 was proved first in [10, Proof of Corollary 1]. Hence, our Theorem 3.4 is a generalization of [10, Corollary 1].

**Theorem 3.8.** *If $(H_\omega)$ holds and $|f'|^q$ is convex for some $q>1$, then*

$$\begin{aligned}\left|\frac{\omega}{\rho_2^\omega-\rho_1^\omega}\int_{\rho_1}^{\rho_2}f(x)\mathrm{d}_\omega x-f(\rho_*)\right|\le&\frac{(\rho_2-\rho_1)^{\frac{1}{q}}}{8^{\frac{1}{q}}(\rho_2^\omega-\rho_1^\omega)}\left\{(G_p(\rho_1,\rho_2))^{\frac{1}{p}}\left(3\left|f'(\rho_1)\right|^q+\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.+\,(G_p(\rho_2,\rho_1))^{\frac{1}{p}}\left(\left|f'(\rho_1)\right|^q+3\left|f'(\rho_2)\right|^q\right)^{\frac{1}{q}}\right\},\end{aligned}$$

*where*

$$p=\frac{q}{q-1}\quad\text{and}\quad G_p(z_1,z_2):=\left|\int_{z_1}^{z_*}\left|x^\omega-z_1^\omega\right|^p\mathrm{d}x\right|.$$

***Proof.*** By the convexity of $|f'|^q$, we get (3.2), and thus, with the help of Hölder's inequality,

$$\begin{aligned}
|\wp_1| &\leq \int_{\rho_1}^{\rho_*} (x^\omega - \rho_1^\omega) |f'(x)| \,\mathrm{d}x \\
&\leq \left(\int_{\rho_1}^{\rho_*} (x^\omega - \rho_1^\omega)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_*} |f'(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= (G_p(\rho_1, \rho_2))^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_*} |f'(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq (G_p(\rho_1, \rho_2))^{\frac{1}{p}} \left(|f'(\rho_1)|^q \int_{\rho_1}^{\rho_*} \frac{\rho_2 - x}{\rho_2 - \rho_1} \mathrm{d}x + |f'(\rho_2)|^q \int_{\rho_1}^{\rho_*} \frac{x - \rho_1}{\rho_2 - \rho_1} \mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\frac{\rho_2 - \rho_1}{8}\right)^{\frac{1}{q}} (G_p(\rho_1, \rho_2))^{\frac{1}{p}} \left(3 |f'(\rho_1)|^q + |f'(\rho_2)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\wp_2| &\leq \int_{\rho_*}^{\rho_2} (\rho_2^\omega - x^\omega) |f'(x)| \,\mathrm{d}x \\
&\leq \left(\int_{\rho_*}^{\rho_2} (\rho_2^\omega - x^\omega)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\rho_*}^{\rho_2} |f'(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= (G_p(\rho_2, \rho_1))^{\frac{1}{p}} \left(\int_{\rho_*}^{\rho_2} |f'(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq (G_p(\rho_2, \rho_1))^{\frac{1}{p}} \left(|f'(\rho_1)|^q \int_{\rho_*}^{\rho_2} \frac{\rho_2 - x}{\rho_2 - \rho_1} \mathrm{d}x + |f'(\rho_2)|^q \int_{\rho_*}^{\rho_2} \frac{x - \rho_1}{\rho_2 - \rho_1} \mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\frac{\rho_2 - \rho_1}{8}\right)^{\frac{1}{q}} (G_p(\rho_2, \rho_1))^{\frac{1}{p}} \left(|f'(\rho_1)|^q + 3 |f'(\rho_2)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Employing now (2.9) completes the proof. □

When we choose $\omega = 1$ in Theorem 3.8, we get the following result.

**Corollary 3.9.** *If* $(H_1)$ *holds and* $|f'|^q$ *is convex for some* $q > 1$, *and* $p = q/(q-1)$, *then*

$$\begin{aligned}
&\left|\frac{1}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} f(x)\mathrm{d}x - f(\rho_*)\right| \\
&\quad \leq \frac{\rho_2 - \rho_1}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left[\left(|f'(\rho_1)|^q + 3 |f'(\rho_2)|^q\right)^{\frac{1}{q}} + \left(3 |f'(\rho_1)|^q + |f'(\rho_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

***Proof.*** For $\omega = 1$, we use (2.7) to calculate

$$G_p(\rho_1, \rho_2) = \int_{\rho_1}^{\rho_*} (x - \rho_1)^p \,\mathrm{d}x = \frac{(\rho_* - \rho_1)^{p+1}}{p+1} = \frac{(\rho_2 - \rho_1)^{p+1}}{(p+1)2^{p+1}} = G_p(\rho_2, \rho_1).$$

Noting $1/p + 1/q = 1$ and simplifying

$$\frac{\left(\frac{\rho_2 - \rho_1}{8}\right)^{\frac{1}{q}} \left(\frac{(\rho_2 - \rho_1)^{p+1}}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}}{\rho_2 - \rho_1} = \frac{(\rho_2 - \rho_1)\, 4^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}} 2^{\frac{1}{q}} 4^{\frac{1}{q}} 2^{1+\frac{1}{p}} 4^{\frac{1}{p}}} = \frac{\rho_2 - \rho_1}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$

and using Theorem 3.8 completes the proof. □

**Remark 3.10.** Note that Corollary 3.9 was proved first in [13, Theorem 2.3]. Hence, our Theorem 3.8 is a generalization of [13, Theorem 2.3].

**Table 4.1.** $F(z_1, z_2)$

| $\omega$ | $\frac{8}{(z_2-z_1)^2}F(z_1,z_2)$ |
|---|---|
| 1 | 1 |
| 2 | $\frac{1}{4}(5z_1+3z_2)$ |
| 3 | $\frac{3}{4}(2z_1^2+z_1z_2+z_2^2)$ |
| 4 | $\frac{1}{24}(41z_1^3+21z_1^2z_2+15z_1z_2^2+19z_2^3)$ |
| 5 | $\frac{5}{24}(9z_1^4+5z_1^3z_2+3z_1^2z_2^2+3z_1z_2^3+4z_2^4)$ |
| 6 | $\frac{1}{128}(257z_1^5+155z_1^4z_2+90z_1^3z_2^2+70z_1^2z_2+85z_1z_2^4+111z_2^5)$ |
| 7 | $\frac{7}{384}(116z_1^6+75z_1^5z_2+45z_1^4z_2^2+30z_1^3z_2z_2^3+30z_1^2z_2^4+39z_1z_2^5+49z_2^6)$ |

**Table 4.2.** $F_1(z_1, z_2)$

| $\omega$ | $\frac{24}{(z_2-z_1)^2}F_1(z_1,z_2)$ |
|---|---|
| 1 | 1 |
| 2 | $\frac{1}{8}(13z_1+3z_2)$ |
| 3 | $\frac{3}{40}(27z_1^2+11z_1z_2+2z_2^2)$ |
| 4 | $\frac{1}{80}(183z_1^3+99z_1^2z_2+33z_1z_2^2+5z_2^3)$ |
| 5 | $\frac{1}{112}(276z_1^4+177z_1^3z_2+81z_1^2z_2^2+23z_1z_2^3+3z_2^4)$ |
| 6 | $\frac{3}{1792}(1545z_1^5+1107z_1^4z_2+618z_1^3z_2^2+246z_1^2z_2+61z_1z_2^4+7z_2^5)$ |
| 7 | $\frac{1}{768}(2053z_1^6+1587z_1^5z_2+1014z_1^4z_2^2+502z_1^3z_2z_2^3+177z_1^2z_2^4+39z_1z_2^5+4z_2^6)$ |

## 4. Integer values of $\omega$

Although the conformable integral mentioned in Section 1 is defined only for $\omega \in (0,1]$, the results presented in this paper hold for any $\omega > 0$. By plugging the special values $\omega \in \mathbb{N}$, we can get some special cases of our presented inequalities. In Tables 4.1, 4.2, 4.3, and 4.4, we give the functions $F$, $F_1$, $F_2$, and $G$ for the values $\omega \in \{1,2,3,4,5,6,7\}$.

## 5. Examples and applications

In this final section, we give three distinct applications of our presented inequalities: First, inequalities involving the modified Bessel function of the first kind (for related inequalities, see [16, Section 4.3.1]). Second, inequalities involving the arithmetic mean and the $(\omega,r)$th generalized logarithmic mean, and third, inequalities involving the midpoint formula (for related inequalities, see [12, Section 3]).

**Table 4.3.** $F_2(z_1, z_2)$

| $\omega$ | $\frac{12}{(z_2-z_1)^2}F_2(z_1,z_2)$ |
|---|---|
| 1 | $1$ |
| 2 | $\frac{1}{16}\left(27z_1+5z_2\right)$ |
| 3 | $\frac{3}{80}\left(58z_1^2+19z_1z_2+3z_2^2\right)$ |
| 4 | $\frac{1}{160}\left(405z_1^3+177z_1^2z_2+51z_1z_2^2+7z_2^3\right)$ |
| 5 | $\frac{1}{224}\left(627z_1^4+327z_1^3z_2+129z_1^2z_2^2+33z_1z_2^3+4z_2^4\right)$ |
| 6 | $\frac{3}{3584}\left(3591z_1^5+2109z_1^4z_2+1014z_1^3z_2^2+362z_1^2z_2+83z_1z_2^4+9z_2^5\right)$ |
| 7 | $\frac{1}{1536}\left(4868z_1^6+3111z_1^5z_2+1713z_1^4z_2^2+758z_1^3z_2z_2^3+246z_1^2z_2^4+51z_1z_2^5+5z_2^6\right)$ |

**Table 4.4.** $G(z_1, z_2)$

| $\omega$ | $\frac{8}{(z_2-z_1)^2}G(z_1,z_2)$ |
|---|---|
| 1 | $1$ |
| 2 | $\frac{1}{3}\left(5z_1+z_2\right)$ |
| 3 | $\frac{1}{8}\left(17z_1^2+6z_1z_2+z_2^2\right)$ |
| 4 | $\frac{1}{20}\left(49z_1^3+23z_1^2z_2+7z_1z_2^2+z_2^3\right)$ |
| 5 | $\frac{1}{48}\left(129z_1^4+72z_1^3z_2+30z_1^2z_2^2+8z_1z_2^3+z_2^4\right)$ |
| 6 | $\frac{1}{112}\left(321z_1^5+201z_1^4z_2+102z_1^3z_2^2+38z_1^2z_2+9z_1z_2^4+z_2^5\right)$ |
| 7 | $\frac{1}{256}\left(769z_1^6+522z_1^5z_2+303z_1^4z_2^2+140z_1^3z_2z_2^3+47z_1^2z_2^4+10z_1z_2^5+z_2^6\right)$ |

### 5.1. Bessel functions

Consider the function $\mathcal{B}_\varrho : (0,\infty) \to [1,\infty)$ defined by

$$\mathcal{B}_\varrho(x) = 2^\varrho \Gamma(\varrho+1)x^{-\varrho}\mathcal{B}_\varrho^1(x),$$

where $\mathcal{B}_\varrho^1$ is the *modified Bessel function of the first kind* defined by (see [27, (2) on page 77])

$$\mathcal{B}_\varrho^1(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\varrho+2n}}{n!\Gamma(\varrho+1+n)}, \quad x \in \mathbb{R}.$$

In [27, (6) on page 79], the first derivative of $\mathcal{B}_\varrho$ is given by

$$\mathcal{B}_\varrho'(x) = \frac{x\mathcal{B}_{\varrho+1}(x)}{2(\varrho+1)}, \tag{5.1}$$

and the second derivative can easily be calculated from (5.1) to be

$$\mathcal{B}''_\varrho(x) = \frac{x^2\mathcal{B}_{\varrho+2}(x)}{4(\varrho+1)(\varrho+2)} + \frac{\mathcal{B}_{\varrho+1}(x)}{2(\varrho+1)}. \tag{5.2}$$

**Example 5.1.** Let $0 < \rho_1 < \rho_2$ and $\varrho > -1$. Then, by applying Corollary 3.2 with $f = \mathcal{B}'_\varrho$ (note that all assumptions are satisfied) and the identities (5.1) and (5.2), we have

$$\begin{aligned}&\left|\frac{\mathcal{B}_\varrho(\rho_2) - \mathcal{B}_\varrho(\rho_1)}{\rho_2 - \rho_1} - \frac{\rho_1+\rho_2}{4(\varrho+1)}\mathcal{B}_{\varrho+1}\left(\frac{\rho_1+\rho_2}{2}\right)\right| \\ &\qquad \le \frac{\rho_2-\rho_1}{16}\left[\frac{\rho_1^2\mathcal{B}_{\varrho+2}(\rho_1) + \rho_2^2\mathcal{B}_{\varrho+2}(\rho_2)}{2(\varrho+1)(\varrho+2)} + \frac{\mathcal{B}_{\varrho+1}(\rho_1) + \mathcal{B}_{\varrho+1}(\rho_2)}{\varrho+1}\right].\end{aligned} \tag{5.3}$$

**Example 5.2.** Let $0 < \rho_1 < \rho_2$, $\varrho > -1$, and $q \ge 1$. Then, by applying Corollary 3.6 with $f = \mathcal{B}'_\varrho$ (note that all assumptions are satisfied) and the identities (5.1) and (5.2), we get

$$\begin{aligned}&\left|\frac{\mathcal{B}_\varrho(\rho_2) - \mathcal{B}_\varrho(\rho_1)}{\rho_2 - \rho_1} - \frac{\rho_1+\rho_2}{4(\varrho+1)}\mathcal{B}_{\varrho+1}\left(\frac{\rho_1+\rho_2}{2}\right)\right| \\ &\le \frac{\rho_2-\rho_1}{16(\varrho+1)3^{\frac{1}{q}}}\left\{\left[2\left(\frac{\rho_1^2\mathcal{B}_{\varrho+2}(\rho_1)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_1)\right)^q + \left(\frac{\rho_2^2\mathcal{B}_{\varrho+2}(\rho_2)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_2)\right)^q\right]^{\frac{1}{q}}\right. \\ &\qquad \left. + \left[\left(\frac{\rho_1^2\mathcal{B}_{\varrho+2}(\rho_1)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_1)\right)^q + 2\left(\frac{\rho_2^2\mathcal{B}_{\varrho+2}(\rho_2)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_2)\right)^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

**Example 5.3.** Let $0 < \rho_1 < \rho_2$, $\varrho > -1$, $q > 1$, and $p = q/(q-1)$. Then, by applying Corollary 3.9 with $f = \mathcal{B}'_\varrho$ (note that all assumptions are satisfied) and the identities (5.1) and (5.2), we obtain

$$\begin{aligned}&\left|\frac{\mathcal{B}_\varrho(\rho_2) - \mathcal{B}_\varrho(\rho_1)}{\rho_2 - \rho_1} - \frac{\rho_1+\rho_2}{4(\varrho+1)}\mathcal{B}_{\varrho+1}\left(\frac{\rho_1+\rho_2}{2}\right)\right| \\ &\le \frac{\rho_2-\rho_1}{32(\varrho+1)}\sqrt[p]{\frac{4}{\varrho+1}}\left\{\left[\left(\frac{\rho_1^2\mathcal{B}_{\varrho+2}(\rho_1)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_1)\right)^q + 3\left(\frac{\rho_2^2\mathcal{B}_{\varrho+2}(\rho_2)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_2)\right)^q\right]^{\frac{1}{q}}\right. \\ &\qquad \left. + \left[3\left(\frac{\rho_1^2\mathcal{B}_{\varrho+2}(\rho_1)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_1)\right)^q + \left(\frac{\rho_2^2\mathcal{B}_{\varrho+2}(\rho_2)}{2(\varrho+2)} + \mathcal{B}_{\varrho+1}(\rho_2)\right)^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

## 5.2. Special means

Let $r \in \mathbb{R} \setminus \{-\omega, 0\}$, $\omega \in (0,1]$, and $\rho_1, \rho_2 \in \mathbb{R}$ with $0 < \rho_1 < \rho_2$. Then we consider the *arithmetic mean* and the $(\omega, r)$*th generalized logarithmic mean* defined by

$$A(\rho_1,\rho_2) = \frac{\rho_1+\rho_2}{2} \quad \text{and} \quad L_{(\omega,r)}(\rho_1,\rho_2) = \left[\frac{\omega\left(\rho_2^{r+\omega} - \rho_1^{r+\omega}\right)}{(r+\omega)\left(\rho_2^\omega - \rho_1^\omega\right)}\right]^{\frac{1}{r}}. \tag{5.4}$$

**Example 5.4.** Let $r > 1$ and $f(x) = x^r$ for all $x > 0$. It is easy to see that $|f'|$ is convex on $[\rho_1, \rho_2]$. Then, by applying Theorem 3.1, the $r$th powers of the two means in (5.4) differ from each other at most by

$$\frac{r}{\rho_2^\omega - \rho_1^\omega}\left(\rho_1^{r-1}F(\rho_1,\rho_2) + \rho_2^{r-1}F(\rho_2,\rho_1)\right),$$

where $F$ is given in Theorem 3.1.

**Example 5.5.** Let $q \geq 1$, $r > 1$, and $f(x) = x^r$ for all $x > 0$. Then, by applying Theorem 3.4, the $r$th powers of the two means in (5.4) differ from each other at most by

$$\frac{r}{\rho_2^\omega - \rho_1^\omega}\left\{(G(\rho_1,\rho_2))^{1-\frac{1}{q}}\left(\rho_1^{q(r-1)}F_2(\rho_1,\rho_2)+\rho_2^{q(r-1)}F_1(\rho_1,\rho_2)\right)^{\frac{1}{q}}\right.$$
$$\left.+(G(\rho_2,\rho_1))^{1-\frac{1}{q}}\left(\rho_1^{q(r-1)}F_1(\rho_2,\rho_1)+\rho_2^{q(r-1)}F_2(\rho_2,\rho_1)\right)^{\frac{1}{q}}\right\},$$

where $F_1$, $F_2$, and $G$ are given in Lemma 2.1.

**Example 5.6.** Let $q > 1$, $r > 1$, and $f(x) = x^r$ for all $x > 0$. Then, by applying Theorem 3.8, the $r$th powers of the two means in (5.4) differ from each other at most by

$$\frac{r}{\rho_2^\omega - \rho_1^\omega}\sqrt[p]{\frac{\rho_2-\rho_1}{8}}\left\{(G_p(\rho_1,\rho_2))^{\frac{1}{p}}\left(\rho_1^{q(r-1)}+3\rho_2^{q(r-1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+(G_p(\rho_2,\rho_1))^{\frac{1}{p}}\left(3\rho_1^{q(r-1)}+\rho_2^{q(r-1)}\right)^{\frac{1}{q}}\right\},$$

where $p = q/(q-1)$ and $G_p$ is given in Theorem 3.8.

## 5.3. Midpoint approximations

Let $\mathcal{P}$ be the partition $\rho_1 = x_0 < x_1 < \ldots < x_n = \rho_2$ of the interval $[\rho_1, \rho_2]$. We consider the quadrature formula

$$\int_{\rho_1}^{\rho_2} f(x)\mathrm{d}_\omega x = M_\omega(f,\mathcal{P}) + E_\omega(f,\mathcal{P}),$$

where

$$M_\omega(f,\mathcal{P}) = \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)\frac{x_{i+1}^\omega - x_i^\omega}{\omega}$$

is the midpoint approximation and $E_\omega(f,\mathcal{P})$ denotes the corresponding approximation error.

**Example 5.7.** If $(\mathrm{H}_\omega)$ holds and $|f'|$ is convex, then, by applying Theorem 3.1 on the subintervals $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the partition $\mathcal{P}$, we have

$$\left|f\left(\frac{x_i+x_{i+1}}{2}\right) - \frac{\omega}{x_{i+1}^\omega - x_i^\omega}\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}_\omega x\right|$$
$$\leq \frac{F(x_i,x_{i+1})\,|f'(x_i)| + F(x_{i+1},x_i)\,|f'(x_{i+1})|}{x_{i+1}^\omega - x_i^\omega},$$

where $F$ is given in Theorem 3.1, and then by adding, we get

$$|E_\omega(f,\mathcal{P})| \leq \frac{1}{\omega}\sum_{i=0}^{n-1}\left[F(x_i,x_{i+1})\,|f'(x_i)| + F(x_{i+1},x_i)\,|f'(x_{i+1})|\right].$$

**Example 5.8.** If $(\mathrm{H}_\omega)$ holds and $|f'|^q$ is convex for some $q \geq 1$, then, by applying Theorem 3.4 on the subintervals $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the partition $\mathcal{P}$, and then adding, we have

$$|E_\omega(f,\mathcal{P})| \leq \frac{1}{\omega}\sum_{i=0}^{n-1}\left\{(G(x_i,x_{i+1}))^{1-\frac{1}{q}}\left(F_2(x_i,x_{i+1})\,|f'(x_i)|^q + F_1(x_i,x_{i+1})\,|f'(x_{i+1})|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(G(x_{i+1},x_i))^{1-\frac{1}{q}}\left[F_1(x_{i+1},x_i)\,|f'(x_i)|^q + F_2(x_{i+1},x_i)\,|f'(x_{i+1})|^q\right]^{\frac{1}{q}}\right\},$$

where $F_1$, $F_2$, and $G$ are given in Lemma 2.1.

**Example 5.9.** If ($H_\omega$) holds and $|f'|^q$ is convex for some $q > 1$, then, by applying Theorem 3.8 on the subintervals $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$, of the partition $\mathcal{P}$, and then adding, we obtain

$$|E_\omega(f, \mathcal{P})| \leq \frac{1}{\omega \sqrt[q]{8}} \sum_{i=0}^{n-1} \sqrt[q]{x_{i+1} - x_i} \Big\{ (G_p(x_i, x_{i+1}))^{\frac{1}{p}} \left[ |f'(x_i)|^q + 3 |f'(x_{i+1})|^q \right]^{\frac{1}{q}} + (G_p(x_{i+1}, x_i))^{\frac{1}{p}} \left[ 3 |f'(x_i)|^q + |f'(x_{i+1})|^q \right]^{\frac{1}{q}} \Big\},$$

where $p = q/(q-1)$ and $G_p$ is given in Theorem 3.8.

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