INTERNATIONAL JOURNAL of ENGINEERING SCIENCE AND APPLICATION
H. Shibly and A. Shibly, Vol.5, No.2, July 2021

# Array Technique to Calculate the Breakpoints on Root Locus Graph and Related Gains

Hassan Shibly *, Aalaa Shibly **‡.

* Robotics and Mechatronics Engineering Technology, School of Engineering, Science and Technology Program, CCSU University, New Britain, CT, USA.

** Graphics , Comanox Corp, VA, USA.

‡Hassan Shibly; hshibly@ccsu.edu, Tel: +860 832 1882,

*Received: 06.01.2021 Accepted:27.06.2021*

**Abstract-** The root locus technique is a powerful and efficient mean to examine stability, and to analyze single input single output linear time invariant system. In addition, the gain range for any response type of the control system be determined. Some of the important points on a root locus graph of control system are the Breakaway, and Break-in points. In this article those points are called Break points, and the polynomial that some of its roots are Break points, is called Break polynomial. After leaving a Break point on root locus graph, the type of some roots of the system characteristic equation changes. The change is from real to a complex at Breakaway point, and from complex to real at break-in point. The type change of roots causes a type change of the system response. The response type of a system is a crucial matter for industrial machines applications. The development of a new method called the Array method is presented. The Array method is a technique to obtain the Break polynomial where several of its roots are the Break points. This technique is based on constructing an array. Then the array is filled by the polynomials' coefficients of the open loop transfer function's denominator and numerator. The mathematical proof of the method bases, and correctness is presented. It shows that the obtained Break polynomial by the proposed method is the same derived polynomial by the most used methods. The proposed method is compared with other methods in solution of examples of control systems to demonstrate its simplicity for the user and its correctness for any order of a single input single output linear invariant control system.
.

**Keywords** Array method for Break points calculation; Breakaway polynomial; Breakaway points' calculation; Break-in point calculation; Gain at Breakaway point.
.

## 1. Introduction

system. The theoretical range of the parameter of interest is The root locus technique is a common method used in the analysis and design of linear time invariant single input and single output of feedback control systems. This semi graphical technique was first introduced by Evans in years 1948 [1] and 1950 [2]. This technique is used to find the closed loop poles which are the roots of the closed loop characteristic equation. The paths of the roots' movement on the complex s-plane when varying a parameter of interest is known as the root loci graphs. Usually, the parameter of interest is the static gain of the open loop between zero and infinity.

A block diagram of a single input single output of a linear invariant control systems is shown in Figure 1. The open loop transfer function of this control system is $G(s) = G_c(s)G_p(s)$, and its feedback transfer function is $H(s)$. Then the general form of its closed loop transfer function T(s) with a negative feedback is

$$T(s) = \frac{G(s)}{1+G(s)H(s)}. \quad (1)$$

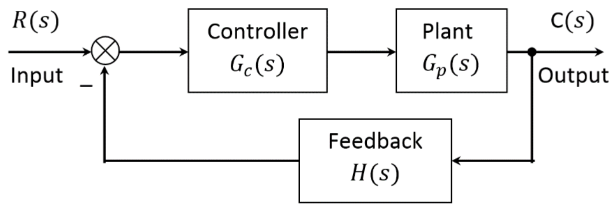



Figure 1. Feedback Control System Block Diagram.

Setting the denominator of the closed loop transfer function *T*(s), Equation (1), to zero gives the Characteristic Equation (C.E.), $\Delta(s)$, of the control system such as

$$\Delta(s) = 1 + G(s)H(s) = 0. \qquad (2)$$

The loci of the roots of the characteristic equation Δ(s), Equation (2), of the closed loop control system is the Root Locus graph. Typically, the function $G(s)H(s)$ has a fractional form such as

$$G(s)H(s) = K\frac{N(s)}{D(s)}. \qquad (3)$$

The degree of the numerator *N*(s) is *m,* and its roots are called zeros. While the degree of the denominator *D*(s) is *n*, and its roots are called poles, in general, $m \leq n$.

Substitute Equation (3) into Equation (2) to obtain the characteristic equation as

$$\Delta(s) = 1 + K\frac{N(s)}{D(s)} = 0. \qquad (4)$$

Rewriting Equation (4) as one fraction leads that the numerator of the fraction is equal to zero as

$$D(s) + KN(s) = 0. \qquad (5)$$

The range of the parameter K in Equation (5) is from zero to infinity. Solving the characteristic equation for each value of K gives n new roots. The graph which connects those roots is called the Root Locus plot.

At the two limits of the gain *K* the characteristic equation becomes

$$For \quad K = 0 \quad \Delta(s) = D(s) = 0, \ and$$
$$for \quad K \rightarrow \infty \quad \Delta(s) = N(s) = 0. \qquad (6)$$

As it can be seen from Equation (6) that the roots of the characteristic equation for $K = 0$ are the poles of the open loop transfer function, and for $K \rightarrow \infty$ the roots are the zeros of the open loop transfer function. As a result, the roots of the characteristic equation migrate from the poles to the zeros of the open loop transfer function. So, the number of root loci is equal to the number of poles.

The basics of the existence of any point on the root locus graph is that the associated open loop transfer function must satisfies two conditions: the magnitude condition, and the angle condition. The real roots give the root locus segments on the real axis. Satisfaction of the angle condition implies that the root locus segments on the real axis must be located to the left of an odd number of poles and zeros. While the conjugate pair roots give the symmetrical segments about the real axis.

From Laplace transform the time domain response of a real root is an exponential, while the response of a complex conjugate pair is oscillatory, [3-5]. The gain K in root locus technique defines the location of the roots on the s-plane, and consequently the response component's type. The gain at the Break points is a border value for some roots, and any increase above this value those roots change their type.

At the root locus method, the root loci begin at poles and move toward zeros for varying the value of the static gain K. If two roots' loci leave two poles on the real axis of the s-plane and move in opposite directions, as shown in Figure 2, they will collide at a point. This point is called a Breakaway point. If a pair of complex conjugate roots move in opposite directions toward the real axis they will collide at a point on the real axis. This point is called Break-in point. At the Break points the characteristic equation has double roots for a certain value of K. Geometrically, the Break points are on the segments of the root locus, which is between two poles, or between two zeros. A larger value of K than its value at the Breakaway point gives at least one pair of complex conjugate roots of the characteristic equation, and consequently the root loci split and leave the real axis in opposite directions and continue to move in a symmetrical way. For larger values of K than its value at the Break-in points the characteristic equation has two real roots. The associated root loci continue to move in opposite directions on the real axis. The process of the Break-in point is the opposite process of the Breakaway point. In this article Breakaway point and Break-in point are called just Break point.

There are several methods to find the Break points, and the corresponding gains, [3], [6], and [7]. The common method is based on rewriting the characteristic equation so, that K is equal to a fraction of two polynomials. Then finding a local extremum for K. The extremum is found via differentiation of the fraction of K with respect to the variable s, and then setting the derivate to zero. The derivation result is algebraic polynomial. Some of this polynomial's roots are the Break points, therefore, there is a need to search for the proper roots that are along the root loci segments on the real axis. Via differentiation method the complexity of the mathematical differentiation increases with a higher degree of the numerator's polynomial.

A second method was presented by Remec, [8], which it is a tabulation method that resemblance to the construction of Routh-Hurwitz array, and it is a long process. A third method was presented by Franklin, [9], is called the Transition method. The base of the method is that the natural logarithm has a zero derivative at the same point of the parameter of interest expression derivative is zero. In this method the there is no differentiation to be done by the user, but there are several multiplications of a polynomials' factors to obtain the final expression of the Break polynomial. A fourth method was developed by Krishnan, [9]. This method is based on a successive differentiation of numerator and denominator to construct arrays to obtain the algorithm, and it is a long process. A fifth method named the Formulated method developed by Shibly [6]. This method is based on obtaining the Break polynomial in a formulated method.

This article is divided into two parts. The first part presents a mathematical proof of the correctness of the basics of the proposed new Array method for calculating the Break points and the corresponding gain. The second part presents the development of the Array method, and a comparison with a commonly used methods in the solution of two examples.

## 2. Mathematical Proof of the Formulization Method

The proposed Array method is based on the Break polynomial $P_f(s)$ which is obtained by the Formulization method [6]. Therefore, to show the correctness of the Array method is to prove the correctness of its basis. The proof of the method basis is done by comparing this polynomial $P_f(s)$ with the polynomial $P_{vd}(s)$ that obtained from the commonly used Via Differentiation method. If both polynomials are the same, it means that the polynomial obtained from the Formulization method is correct.

The comparison of the two polynomials is show that the two polynomials mathematically are the same. To do the comparison the polynomial $P_f(s)$ is written at left side of equation and the polynomial $P_{vd}(s)$ is written at the right side of same equation, and then examining the equivalence of both sides of the equation as follows:

$$P_f(s) \overset{?}{=} P_{vd}(s) \quad (7)$$

### 2.2 Formulization Method

The Break polynomial obtained by the Formulization method, Shibly [6], as function of the Break points value sigma is

$$P_f(\sigma) = P_1(\sigma)P_4(\sigma) - P_2(\sigma)P_3(\sigma). \quad (8)$$

The four polynomials $P_i$ in Equation (8) are constructed from the denominator polynomial coefficients $a_i$, and the numerator polynomial coefficients $b_j$, of the open loop transfer function and they are

$$P_1(\sigma) = \sum_{q=1}^{n}(-1)^q q a_q \sigma^{q-1}. \quad (9)$$

$$P_2(\sigma) = \sum_{q=1}^{m}(-1)^q q b_q \sigma^{q-1}. \quad (10)$$

$$P_3(\sigma) = a_0 + \sum_{q=2}^{n}(-1)^{q-1}(q-1) a_q \sigma^{q}. \quad (11)$$

$$P_4(\sigma) = b_0 + \sum_{q=2}^{m}(-1)^{q-1}(q-1) b_q \sigma^{q}. \quad (12)$$

The values of the sigma variable which are on the real axis and satisfy the angle condition of root locus are the Break points, breakaway, and break-in points. To satisfy the angle condition along the real axis the root locus segments should be to the left of an odd number of poles and zeros of the system open loop transfer function.

Replace the variable sigma with negative s, $(\sigma = -s)$, into polynomial $P_1$, Equations (9) to have

$$P_1(s) = \sum_{q=1}^{n}(-1)^q q a_q (-s)^{q-1} =,$$
$$= \sum_{q=1}^{n}(-1)^q(-1)^{q-1} q a_q (s)^{q-1} \quad (13)$$

The products $(-1)^q(-1)^{q-1} = (-1)^{q-1}(-1)^q = -1$, is equal negative one, and as a result the polynomial $P_1$, Equation (13), becomes

$$P_1(s) = -\sum_{q=1}^{n} q a_q s^{q-1}. \quad (14)$$

Repeat the same substitution for the other three polynomials as in Equations (10-12) to have them as function of s.

$$P_2(s) = -\sum_{q=1}^{n} q b_q s^{q-1}. \quad (15)$$

$$P_3(s) = a_0 - \sum_{q=1}^{n}(q-1) a_q s^{q}. \quad (16)$$

$$P_4(s) = b_0 - \sum_{q=1}^{n}(q-1) b_q s^{q}. \quad (17)$$

Substitute Equations (14-17) into the Break polynomial, Equation (8), to have

$$P_f(s) = -\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 - \sum_{q=1}^{n}(q-1) b_q s^{q}\right) +,$$
$$+\left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(a_0 - \sum_{q=1}^{n}(q-1) a_q s^{q}\right). \quad 18)$$

### 2.2 Via Differentiation Method

The most used method for determining the Break points is the Via Differentiation method. In this method, K is solved from the characteristic equation, Equation (5). The result is that K is equal to a negative fraction that its numerator is the polynomial $D(s)$, and its denominator is the polynomial $N(s)$. Then the fractional expression of K is differentiated with respect to the complex variable s to find an extremum. The result of K is

$$K = -\frac{D(s)}{N(s)}. \quad (19)$$

Both polynomials may written as a summation such as

$$D(s) = a_0 + \sum_{q=1}^{n} a_q s^{q}. \quad (20)$$

$$N(s) = b_0 + \sum_{q=1}^{m} b_q s^{q}. \quad (21)$$

In Via Differentiation method, $K$ in Equation (19) is differentiated with respect to $s$ to obtain the local extremums of $K$. The derivative is set to zero to get

$$\frac{dK}{ds} = -\frac{D'N - N'D}{N^2} = 0. \quad (22)$$

The result of Equation (22) is that

$$N'D - D'N = 0. \quad (23)$$

Differentiate both polynomials D(s), Equation (20), and N(s), Equation (21), to have their derivatives as

$$D'(s) = \sum_{q=1}^{n} q a_q s^{q-1}. \quad (24)$$

$$N'(s) = \sum_{q=1}^{m} q b_q s^{q-1}. \quad (25)$$

Substitute the polynomials' derivatives, Equations (24-25), into Equation (23) to have the Break polynomial $P_{vd}$ of this method. The degree of this polynomial is (n+m-1).

$$P_{vd}(s) = \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(a_0 + \sum_{q=1}^{n} a_q s^{q}\right) -$$
$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 + \sum_{q=1}^{m} b_q s^{q}\right).$$

$$P_{vd}(s) = \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(a_0 + \sum_{q=1}^{n} a_q s^{q}\right) -,$$
$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 + \sum_{q=1}^{m} b_q s^{q}\right). \quad (26)$$

Then this polynomial, Equation (26), is solved to obtain the Break points.

### 2.3 Proof of Array Method Correctness

Substitute both polynomials, Equation (18) and Equation (26), into Equation (7) to have

$$-\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 - \sum_{q=1}^{n}(q-1) b_q s^q\right) +,$$

$$\left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(a_0 - \sum_{q=1}^{n}(q-1) a_q s^q\right) \stackrel{?}{=},$$

$$= \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(a_0 + \sum_{q=1}^{n} a_q s^q\right),$$

$$-\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 + \sum_{q=1}^{m} b_q s^q\right). \quad (27)$$

Algebraic simplifications of both sides of Equation (27) gives

$$-b_0\left(\sum_{q=1}^{n} q a_q s^{q-1}\right) + \left(\sum_{q=1}^{n} q a_q s^{q-1}\right),$$

$$\left(\sum_{q=1}^{n}(q-1) b_q s^q\right) + a_0\left(\sum_{q=1}^{n} q b_q s^{q-1}\right) -,$$

$$\left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) \stackrel{?}{=},$$

$$= a_0\left(\sum_{q=1}^{m} q b_q s^{q-1}\right) + \left(\sum_{q=1}^{m} q b_q s^{q-1}\right),$$

$$\left(\sum_{q=1}^{n} a_q s^q\right) - b_0\left(\sum_{q=1}^{n} q a_q s^{q-1}\right) -,$$

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right). \quad (28)$$

The removing of similar terms from both sides of Equation (28) gives

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) b_q s^q\right) -,$$

$$\left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) \stackrel{?}{=},$$

$$= \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) -,$$

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right). \quad (29)$$

The following two summations from Equation (29) may expanded to two summations each one to have

$$\left(\sum_{q=1}^{m}(q-1) b_q s^q\right) = \left(\sum_{q=1}^{m} q b_q s^q\right) - \left(\sum_{q=1}^{m} b_q s^q\right). \quad (30)$$

$$\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) = \left(\sum_{q=1}^{n} q a_q s^q\right) -,$$

$$\left(\sum_{q=1}^{n} a_q s^q\right). \quad (31)$$

Substitution of the results of Equations (30-31) into Equation (29) to have

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^q\right) -,$$

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right) -$$

$$\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^q\right) +,$$

$$+\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) \stackrel{?}{=}$$

$$= \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) -,$$

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right). \quad (32)$$

By removing equal terms on both sides of Equation (32) gives

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^q\right) -,$$

$$\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^q\right) \stackrel{?}{=} 0. \quad (33)$$

Having *s* in Equation (33) a factor gives

$$s\left[\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^{q-1}\right) -,\right.$$

$$\left.\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\right] \stackrel{?}{=} 0. \quad (34)$$

The two expressions of the two products within the brackets in Equation (34) are the same, and there subtraction gives zero value so that

$$s(0) = 0. \quad (35)$$

The result of Equation (35) is a mathematical proof which shows that the basis of the Array method gives the same Break polynomial as the obtained one by the popular Via Differentiation method. Therefore, the basis of the proposed Array method is correct and accurate.

## 3. Derivation of the Array Method

To simplify the understanding of the Array method derivation, an example of general four-order system is presented first. Then the four polynomials, $P_i(s)$, of this system for (n=4) are

$$P_1(s) = -\sum_{q=1}^{n} q a_q s^{q-1} =,$$

$$-(a_1 + 2a_2 s + 3a_3 s^2 + 4a_4 s^3). \quad (36)$$

$$P_2(s) = -\sum_{q=1}^{n} q b_q s^{q-1} =,$$

$$-(b_1 + 2b_2 s + 3b_3 s^2 + 4b_4 s^3). \quad (37)$$

$$P_3(s) = a_0 - \sum_{q=1}^{n}(q-1) a_q s^q =,$$

$$a_0 - (a_2 s^2 + 2a_3 s^3 + 3a_4 s^4). \quad (38)$$

$$P_4(s) = b_0 - \sum_{q=1}^{n}(q-1) b_q s^q =,$$

$$b_0 - (b_2 s^2 + 2b_3 s^3 + 3b_4 s^4). \quad (39)$$

Substitution of the four polynomials, Equations (36-39), into the Break polynomial, Equation (8), and then performing the multiplication to obtain the new form of the Break polynomial $P(s)$ such as

$$P(s) = (a_4 b_3 - a_3 b_4)s^6 + 2(a_4 b_2 - a_2 b_4)s^5 +,$$

$$[3(a_4 b_1 - a_1 b_4) + (a_3 b_2 - a_2 b_3)]s^4 +,$$

$$+[4(a_4 b_0 - a_0 b_4) + 2(a_3 b_1 - a_1 b_3)]s^3 +,$$

$$[3(a_3 b_0 - a_0 b_3) + (a_2 b_1 - a_1 b_2)]s^2 +,$$

$$+2(a_2 b_0 - a_0 b_2)s + (a_1 b_0 - a_0 b_1) = 0. \quad (40)$$

The Break polynomial, Equation (40), may divided into four parts as follows:

1st:

$$(a_4 b_3 - a_3 b_4)(1s^6) + (a_4 b_2 - a_2 b_4)(2s^5) +,$$

$$+(a_4 b_1 - a_1 b_4)(3s^4) + (a_4 b_0 - a_0 b_4)(4s^3).$$

2nd:

$$(a_3 b_2 - a_2 b_3)(1s^4) + (a_3 b_1 - a_1 b_3)(2s^3) +,$$

$$+(a_3 b_0 - a_0 b_3)(3s^2).$$

3rd

$$(a_2 b_1 - a_1 b_2)(1s^2) + (a_2 b_0 - a_0 b_2)(2s).$$

4th:

$$(a_1 b_0 - a_0 b_1)(1s^0). \quad (41)$$

The coefficients in each parts of Equation (41) may be written in determinant form as follows:

1st: $$\begin{vmatrix} a_4 & a_3 \\ b_4 & b_3 \end{vmatrix}(1s^6) + \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix}(2s^5) + \begin{vmatrix} a_4 & a_1 \\ b_4 & b_1 \end{vmatrix}(3s^4) + \begin{vmatrix} a_4 & a_0 \\ b_4 & b_0 \end{vmatrix}(4s^3),$$

2nd: $$\begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}(1s^4) + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}(2s^3) + \begin{vmatrix} a_3 & a_0 \\ b_3 & b_0 \end{vmatrix}(3s^2),$$

3rd: $$\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}(1s^2) + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix}(2s^1),$$

4th: $$\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}(1s^0).$$

(42)

There is a pattern of the terms given in Equation (42). This pattern can be realized by setting up an array.

### 3.1. Array Formation

The array may have horizontal orientation or vertical orientation. The horizontal orientation of an array contains four rows and $n+1$ columns for a system of order $n$. To flip horizontal orientation to a vertical orientation is by replacing the rows by the columns, and verse versa. The setting up of the array has the following steps.

i. The left column is added to the array to describe the content of each row. Later and after several practices this column may be omitted. This column does not consider in columns numbering.

ii. The top row is filled by the coefficients of the denominator's polynomial of $G(s)H(s)$. The filling is one coefficient per cell in a successive manner, starting from left column with the leading coefficient $a_n$.

iii. The second row from top is empty. It is optional, and it is added to obtain a reasonable space between rows.

iv. The third row from top is filled by the coefficients of the numerator's polynomial of $G(s)H(s)$. The filling is one coefficient per cell in a successive manner starting from left column with the leading coefficient $b_m$.

v. The fourth row contains the *Term Expression*. The filling starts at the second column. The exponent of the first term is $2(n-1)$, and exponents of the successive terms is reduced gradually by one.

vi. The numerical coefficients of the *Terms' Expression* are $1, 2, \cdots, n$ respectively.

vii. The terms of the Break polynomial are obtained by multiplying the *Term Expression* by the value of the determinant above it.

Following these steps, the array for a system of order $n$ can be set up as it is shown in Figure 2.

| $D(s)$ Coefficients | $a_n$ | $a_{n-1}$ | $a_{n-2}$ | $\cdots$ | $a_1$ | $a_0$ |
|---|---|---|---|---|---|---|
| | | | | | | |
| $N(s)$ Coefficients | $b_n$ | $b_{n-1}$ | $b_{n-2}$ | $\cdots$ | $b_1$ | $b_0$ |
| Term Expression | | $1s^{2n-l-1}$ | $2s^{2n-l-2}$ | $\cdots$ | $(n-1)s^1$ | $ns^0$ |

**Figure 2**. General array for n degrees system.

### 3.2. The Steps for obtaining the Break Polynomial

As an example, the array of a fourth order system, n=4, is shown in Figure 3. The successive determinants of the first part of the Break polynomial are: $[D_{4,j} = a_4 b_j - b_4 a_j, \;\; for \;\; j = 3,2,1,0]$. It can be noticed that there is pattern that is easy to remember. Which is that the first column of all determinants includes the two leading coefficients $a_n$ *and* $b_n$, and this column is the pivoted column for all the successive determinants. While the determinants' second column is the successive columns of the array. The elements multiplications of each one of the successive determinants are shown by the arrows of the same color as shown the right part of Figure 3.

| $D(s)$ Coefficients | $a_4$ | $a_3$ | $a_2$ | $a_1$ | $a_0$ |
|---|---|---|---|---|---|
| | | | | | |
| $N(s)$ Coefficients | $b_4$ | $b_3$ | $b_2$ | $b_1$ | $b_0$ |
| Term Expression | | $1s^6$ | $2s^5$ | $3s^4$ | $4s^3$ |

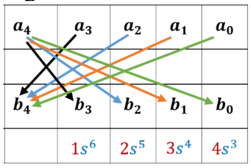



**Figure 3.** Array of first part of the Break polynomial in general and the arrows are Added on the right side to show the determinants calculations.

The products sum of all determinants and their Terms' Expression gives the first part of the Break Polynomial, the result is

$$(a_4b_3 - b_4a_3)(1s^6) + (a_4b_2 - b_4a_2)(2s^5) +,$$
$$+(a_4b_1 - b_4a_1)(3s^4) + (a_4b_0 - b_4a_0)(4s^3). \quad (43)$$

Equation (43) is the same as the first part of Equation (41), which is the first part of the Break polynomial.

The array of the second part of the Break polynomial is obtained by deleting the previous pivoted column, which is the left column that contains the leading coefficients $a_n$ and $b_n$. As a result, the new left column which contains the coefficients $a_{n-1}$ and $b_{n-1}$ becomes the pivoted column. Then reducing the *Term Expression's* exponent by two, and then shift the Term Expressions row one cell to the right. This process is repeated for all parts. The new arrays are shown in Figure 4.

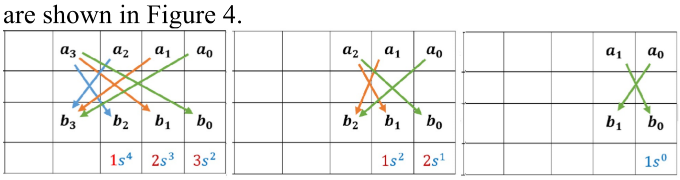



**Figure 4**. Arrays of the second part, third part, and fourth part of the Break polynomial. The arrows are added to show the sequence of determinants calculation.

By repeating the calculations steps of the first part of the Break polynomial, and after algebraic simplification gives

$$(a_3b_2 - b_3a_2)(1s^4) + (a_3b_1 -,$$
$$b_3a_1)(2s^3) + (a_3b_0 - b_3a_0)(3s^2). \quad (44)$$

Equation (44) is the same as the second part of Equation (41) which is the second part of the Break polynomial. Repeating the same steps as in part two to get the array of the third part of the Break polynomial. The result of the calculation after simplification is

$$(a_2b_1 - b_2a_1)(1s^2) + (a_2b_0 - b_2a_0)(2s^1). \quad (45)$$

Equation (45) is the same as the third part of Equation (41) which is the third part of the Break polynomial. Repeating the previous steps gives the array of the fourth part of the Break polynomial. The result of the calculation is

$$(a_1b_0 - b_1a_0)(1s^0). \quad (46)$$

Equation (46) is the same as the fourth part of Equation (41). Adding the results of the four parts of the Break polynomial and then collect equal terms to have

$$P(s) = (a_4b_3 - b_4a_3)s^6 + 2(a_4b_2 - b_4a_2)s^5 +,$$
$$+[3(a_4b_1 - b_4a_1) + (a_3b_2 - b_3a_2)]s^4 +,$$
$$+[4(a_4b_0 - b_4a_0) + 2(a_3b_1 - b_3a_1)]s^3 +,$$
$$+[3(a_3b_0 - b_3a_0) + (a_2b_1 - b_2a_1)]s^2 +,$$
$$+2(a_2b_0 - b_2a_0)s - (a_1b_0 - b_1a_0) = 0. \quad (47)$$

Equation (47) is the Break polynomial for a system of order four.

The first glimpse on the Break polynomial of a fourth order system, Equation (47), gives a wrong impression about the difficulty of obtaining this polynomial by using the Array method. In fact, this method in simple, and to show that, it is used in the solution of several examples, but in this article only two examples, Example 1, and Example 2 are presented.

**Example 1**. The transfer function $G(s)H(s)$ is

$$G(s)H(s) = K\frac{s^2+8s+15}{s^3+6s^2+8s}. \quad (48)$$

Each polynomial is expressed by $n+1$ coefficients. So, in this example the denominator polynomial coefficients $a_i$ are [1 6 8 0] and the numerator polynomial coefficients $b_i$ are [0 1 8 15]. Constructing the arrays as explained in section 2.1 to get the array of the three parts of the Break polynomial that are shown in Figure 5.

| | | | |
|---|---|---|---|
| 1 | 6 | 8 | 0 |
| | | | |
| 0 | 1 | 8 | 15 |
| | $1s^4$ | $2s^3$ | $3s^2$ |

| | | |
|---|---|---|
| 6 | 8 | 0 |
| | | |
| 1 | 8 | 15 |
| | $1s^2$ | $2s^1$ |

| | |
|---|---|
| 8 | 0 |
| | |
| 8 | 15 |
| | $1s^0$ |

**Figure 5.** Array of the first part of the Break polynomial of Example 1.

The detailed calculation of the first part is

$$\begin{vmatrix}1 & 6\\0 & 1\end{vmatrix}(1s^4) + \begin{vmatrix}1 & 8\\0 & 8\end{vmatrix}(2s^3) + \begin{vmatrix}1 & 0\\0 & 15\end{vmatrix}(3s^2) =,$$
$$= s^4 + 16s^3 + 45s^2. \quad (49)$$

As the first column of the three determinants is a pivot column.

The next two arrays, second and third, are obtained by repeating the steps explained before. The second array of this example is the middle array in Figure 5. Repeating the calculation steps gives the second part of the Break polynomial as

$$\begin{vmatrix}6 & 8\\1 & 8\end{vmatrix}(1s^2) + \begin{vmatrix}6 & 0\\1 & 15\end{vmatrix}(2s^1) = 40s^2 + 180s. \quad (50)$$

The third and final array is the third array of Figure 5. Then the third part of the Break polynomial is

$$\begin{vmatrix}8 & 0\\8 & 15\end{vmatrix}(1s^0) = 120. \quad (51)$$

Adding the three parts to get the Break polynomial as

$$P(s) = s^4 + 16s^3 + 85s^2 + 180s + 120. \quad (52)$$

After practice, all parts of the Break polynomial can be calculated by using only the first array. The other arrays are obtained virtually by neglecting the previous pivot column and the first *Term Expression*. Then reducing the exponent and coefficient of each *Term Expression* by one. The parts of the Break polynomial are calculated by repeating the previous steps.

**Example 2**. In this example a system was chosen so that its root locus graph has the two types of Break points: Break-away points and Break-in point. Therefore, the chosen transfer function, $G(s)H(s)$, has four poles and two zeros, and its order is four.

$$G(s)H(s) = K\frac{(s+3)(s+4)}{s(s+2)(s+5)(s+8)} = \frac{s^2+7s+12}{s^4+15s^3+66s^2+80s}. \quad (53)$$

The one array for this example, Equation (53), is shown in Figure 6 where the last three rows are added only to show the virtual steps.

| | | | | |
|---|---|---|---|---|
| 1 | 15 | 66 | 80 | 0 |
| | | | | |
| 0 | 0 | 1 | 7 | 12 |
| | $1s^6$ | $2s^5$ | $3s^4$ | $4s^3$ |
| | | $1s^4$ | $2s^3$ | $3s^2$ |
| | | | $1s^2$ | $2s^1$ |
| | | | | $1s^0$ |

**Figure 6**. Break polynomial's array of Example 2.

By using only the first array, and repeating the calculation steps for each of the four parts of the Break polynomial gives

$$P(\sigma) = [1(0)s^6 + 2(1)s^5 + 3(7)s^4 + 4(12)s^3\,],$$
$$+[1(15)s^4 + 2(105)s^3 + 3(180)s^2\,],$$
$$+[1(382)s^2 + 2(792)s^1\,],$$
$$+[1(960)s^0\,]. \quad (54)$$

Each row of Equation (54) is a part of the Break polynomial. Collect equal terms and adding the four parts to get the final form of the Break polynomial as

$$P(s) = 2s^5 + 36s^4 + 258s^3 + 922s^2 +, \quad +1584s + 960. \qquad (55)$$

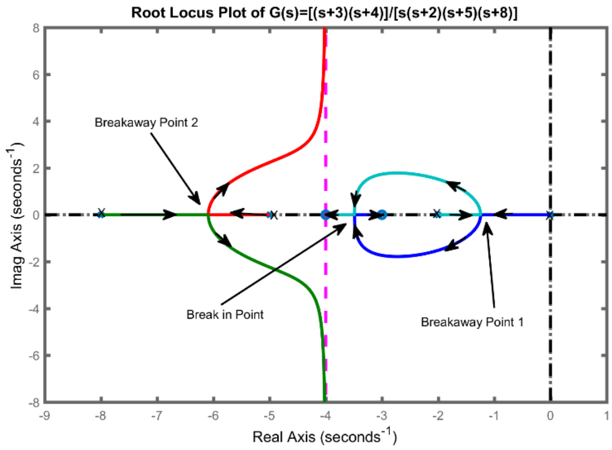
The roots of this Break polynomial, Equation (55), are

$$s_i = -1.2436, -3.4909, -3.5875 + j2.2987, \quad -3.5875 - j2.2987, -6.0906. \qquad (56)$$

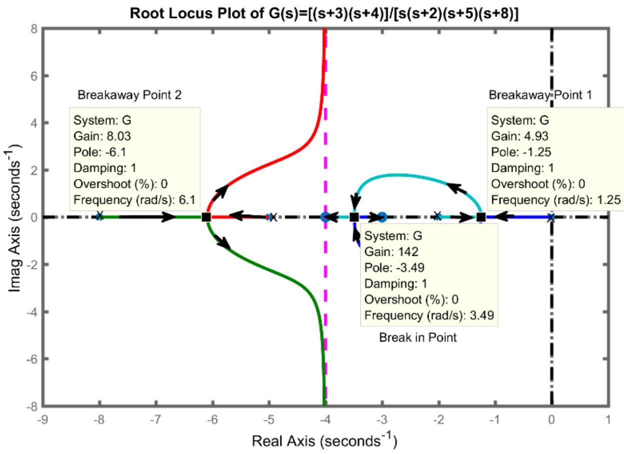
The proper roots are

$$[s_{1,2,3} = -1.2436, -3.4909, \ -6.0906].$$

There are two Break-away points are $[s_{1,2} = -1.2436, \ -6.0906]$, and one Break-in point is $[s_3 = -3.4909]$. By knowing the proper Breaks points $s_i$ the corresponding gains are calculated using Equation (19). The gains at the three points are $[K = 4.9312, \ 8.0296, \ 141.7091]$. The validation of the results is achieved by using MATLAB software for root locus plots. The MATLAB plots are shown in Figure 7a and 7b.




(a)




(b)

**Figure 7**. Root locus plots of Example 2; a) showing the Break points, and b) showing the location of the Break points and the corresponding gains.

## 4. Comparison with other Methods

To examine the Array method, it is compared with other methods for obtaining the Break polynomial for various control systems, but in this article one example is presented, Example 2. The complexity and the number of mathematical operations required in each method are factors of comparison. Therefore, the solutions are shown in detail.

### 4.1. Comparison with the Root Locus plot

The root locus plot, Figure 12, shows that the values of the Break points are too close to the result obtained from the Break polynomial. The small differences are a result of the root locus method being a semi-graphical method.

### 4.2. Comparison with Via Differentiation method

This method is based on finding the local extremums of the static gain *K*. These extremums are obtained by solving Equation (23) for K and then derivative with respect to the variable s is set to zero.

The denominator and numerator of the transfer function in this example are

$$D(s) = s^4 + 15s^3 + 66s^2 + 80s \ and, \quad N(s) = s^2 + 7s + 12. \qquad (57)$$

Their derivatives are

$$D'(s) = 4s^3 + 45s^2 + 132s + 80, \ \ and, \quad N'(s) = 2s + 7. \qquad (58)$$

Substitute into Equation (23) to get

$$(4s^3 + 45s^2 + 132s + 80)(s^2 + 7s + 12) -, \quad (s^4 + 15s^3 + 66s^2 + 80s)(2s + 7) = 0. \qquad (59)$$

Multiply and collect equal terms to have

$$2s^5 + 36s^4 + 258s^3 + 922s^2, \quad +1584s + 960 = 0. \qquad (60)$$

The roots of this polynomial are

$$s_i = -1.2436, -3.4909, -3.5875, +j\,2.2987, \quad -3.5875 - j\,2.2987, -6.0906. \qquad (61)$$

Then the Break points are:

$$[s_{1,2,3} = -1.2436, -3.4909, -6.0906].$$

This method requires polynomials differentiation and polynomials multiplication.

### 4.3. Comparison with Franklin Via Transition method:

The formula used in this method is

$$\sum_{i=1}^{n} \frac{1}{s+p_i} = \sum_{i=1}^{m} \frac{1}{s+z_i}. \qquad (62)$$

The application of this formula to this example gives

$$\frac{1}{s} + \frac{1}{s+2} + \frac{1}{s+5} + \frac{1}{s+8} = \frac{1}{s+3} + \frac{1}{s+4}. \qquad (63)$$

Algebraic simplification gives

$$\frac{(s+2)(s+5)(s+8)+s(s+2)(s+8)+s(s+2)(s+5)}{s(s+2)(s+5)(s+8)} =, \quad = \frac{(s+4)+(s+3)}{(s+3)(s+4)}. \qquad (64)$$

Cross multiplication gives

$$[(s + 2)(s + 5)(s + 8) + s(s + 2)(s + 8) +, \quad s(s + 2)(s + 8) + s(s + 2)(s + 5)], \quad [(s + 3)(s + 4)] - [(s + 4) + (s + 3)], \quad [s(s + 2)(s + 5)(s + 8)] = 0. \qquad (65)$$

This equation can be obtained by differentiating the factored form of the denominator and numerator. This result is the same as via differentiation method as shown in Equation (59).

## 5. Conclusion

The mathematical proof showed the correctness of the basis of the Array method. The comparison was done in the solution of one example only to minimize the article size. Other examples gave a similar result. In this example a system was chosen so that its root locus graph has the two types of Break points: Break-away points and Break-in point. The proposed Array method gives the same Break polynomial as in the two methods: the Via Differentiation method, and the Transition method. While Root Locus method gives very close numerical results because it is a semi graphical method. The Via Differentiation method, and the Transition method. Are similar in principle. The differentiation part is done by the user during problem solution In the Via Differentiation method, while it was done in the derivation of the final formula in the Transition method. In addition both methods their Break polynomial can be found by the following formula

$$[All\ \text{combinations of the product of, } (n-1) \text{ factors of } D(s)] \cdot \boldsymbol{N(s)} - [All\ \text{combinations of the product of, } (m-1) \text{ factors of } N(s)] \cdot \boldsymbol{D(s)} = \mathbf{0}. \quad (66)$$

Equation (66) is obtained by applying Equation (23) on the factored forms of the numerator N(s) and the denominator D(s). In the Via Differentiation method, the differentiation of a fraction of two polynomials may lead to calculation errors by users, while in the Transition method the use of the formula requires twelve multiplications of factors, and four multiplications of polynomials. This number of mathematical operations makes it tedious thing to do it by hand.

It can be concluded that the Array method is a simple technique, and a systematic method for finding all Break points, breakaway and break in points, and the associated gains. In the Array method there is no mathematical differentiation, and for a constant numerator all the coefficients are zeros which simplifies drastically the determinants calculation. After several use of this method the technique becomes more and easier to use.

## References

[1] Evans, W.R. “Graphical Analysis of Control System”, *AIEE Transactions*, vol. 67, pp. 547-551. 1948.
[2] Evans, W.R., “Control System Synthesis and Root Locus Method”, *AIEE Transactions*, vol. 69, pp. 66-69, 1950.
[3] Benjamin C. Kuo, “Automatic Control Systems”, Prentice Hall, 3rd Edition, 1975.
[4] Richard C. Dorf, Robert H. Bishop, “Modern Control Systems”, Pearson, 13th Edition, 2016.
[5] Katsuhiko Ogata, “Modern Control Engineering”, Pearson, 5th Edition, 2010.
[6] Hassan Shibly, Orwah H. Shibly, “Formulation Method for Root Locus Calculation of Breakaway and Break-in Points and the Corresponding Gain”, Advances in Systems Science and Applications, ASSA journal, Vol 21 No 1, 2021-03-31.
[7] Norman Nise, “Control Systems Engineering”, 7th Edition, 2015.