



Asymptotic behavior of a laminated beam with nonlinear delay and nonlinear structural damping

Kassimu Mpungu^{*1} , Tijani A. Apalara² 

¹*Department of Mathematical of Sciences, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia*

²*Department of Mathematics, University of Hafr Al Batin, Hafr Al-Batin 31991, Saudi Arabia*

Abstract

Our concern in the present work is a Timoshenko laminated beam system with nonlinear delay and nonlinear structural damping acting in the equation describing the dynamics of slip. The aim is to establish an explicit and general energy decay rates of the solution under suitable assumptions on the weight of delay and speeds of wave propagation. To achieve our desired stability results, we exploit some properties of convex functions, coupled with the multiplier technique, which involves constructing an appropriate Lyapunov functional equivalent to the energy of the system.

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1. Introduction

In this paper, we consider a Timoshenko laminated beam system with nonlinear structural damping and an internal nonlinear delay feedback acting in the kinetics of slip equation

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s \\ \quad + \beta g_1(s_t(x, t)) + \mu g_2(s_t(x, t - \tau)) = 0, \end{cases} \quad (1.1)$$

where $(x, t) \in (0, 1) \times (0, \infty)$. $w = w(x, t)$ is the transverse displacement, $\psi = \psi(x, t)$ is the angle of rotation, $s = s(x, t)$ is proportional to the magnitude of slip at the interface and, $3s - \psi$ denotes the effective rotation angle. The positive coefficients $\rho, I_\rho, G, D, \gamma$ and β , represent the density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive stiffness and, adhesive damping weight respectively. $\tau > 0$ is the time delay and the

*Corresponding Author.

Email addresses: mpungu@graduate.utm.my (K. Mpungu), tijani@uhb.edu.sa (T. A. Apalara)

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positive coefficient μ is the delay weight. System (1.1) is subject to the following initial and boundary conditions

$$\begin{cases} w(x, 0) = w_0, s(x, 0) = s_0, \psi(x, 0) = \psi_0, & x \in (0, 1), \\ w_t(x, 0) = w_1, s_t(x, 0) = s_1, \psi_t(x, 0) = \psi_1, & x \in (0, 1), \\ w(0, t) = s_x(0, t) = \psi_x(0, t) = 0, & t \geq 0, \\ w_x(1, t) = s(1, t) = \psi(1, t) = 0, & t \geq 0, \\ s_t(x, t - \tau) = f_0(x, t - \tau), & (x, t) \in (0, 1) \times [0, \tau). \end{cases} \quad (1.2)$$

The initial data $(w_0, s_0, \psi_0, w_1, s_1, \psi_1, f_0)$ belongs to a suitable function space.

The interface connection property in composite layered structures possesses a significant impact on deformation, as well as stress in form of internal structural damping depending on the material of the connector [45]. This renders laminated beam structures more preferred to single beam structures in application. In structural engineering, adhesives are among the most used type of connectors of these layered beam structures. To this effect, Hansen and Spies [18] introduced a differential model describing vibrations in a structure set up by a pair of equal rods with uniform thickness, conjoined with help of an adhesive of negligible mass and thickness, in a manner that interfacial slip is possible while in continuous contact with each other. Precisely, the model comprises of three closely related differential equations. The first two equations are derived on the assumptions of Timoshenko beam theory. These are coupled with the third equation which describes the dynamics of slip. In absence of any external force, the system takes the form

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{cases} \quad (1.3)$$

where $(x, t) \in (0, 1) \times (0, \infty)$. Clearly, if s is identically zero, then the standard Timoshenko system is recovered. Furthermore, in the presence of structural damping ($\beta \neq 0$), the adhesion at the interface produces a restorative comparable force to counteract the interfacial slip. Otherwise, the third equation of (1.3) describes the slip dynamics of the coupled laminated beam system without structural damping.

Composite beam structures are highly applicable especially in structural engineering [36, 40], and like in any other control system, it is crucial that the designed systems are stable. Thus, by exploring different energy dissipating mechanisms introduced to the system, researchers have considerably investigated and presented fascinating energy decay results of (1.3), we summarize some of them below.

The asymptotic properties (1.3) with boundary feedback controls

$$\begin{cases} w(0, t) = \psi(0, t) = s(0, t), & (\psi - w_x)(1, t) = k_1 w_t(1, t) \\ (3s_x - \psi_x)(1, t) = -k_2(3s_t - \psi_t)(1, t), \end{cases} \quad (1.4)$$

were first investigated by Wang et al. [44]. The authors established an exponential stability of the system provided that $r_1 = \sqrt{\rho/G} \neq \sqrt{I_\rho/D} = r_2$, $k_i \neq r_i (i = 1, 2)$. Later, Tatar [43] and Mustafa [33] obtained improved the results in [44] by establishing decay results under weaker conditions on the system parameters. Raposo [38] introduced additional linear frictional damping terms in the first two equations of (1.3) in addition to structural damping, and proved exponential decay with no other restrictions. Similarly, Apalara et al. [9] and [6] respectively established exponential stability results on a single linear frictional damping in the effective rotation angle and structural damping in case of equal wave speeds. Alves and Monteiro [2] further investigated system (1.3) with boundary feedback controls acting through complementary displacements and prove that in presence of structural damping ($\beta \neq 0$), no further dissipation or restrictions on parameters are required for exponential stability, otherwise the equal wave speeds assumption must hold.

On stabilization through thermal effect, Apalara [5] studied a thermoelastic laminated beam without structural damping, and proved that the dissipation through thermoelasticity is sufficient for exponential decay in case of equal wave speeds. For more results on thermal effect, we refer the reader to [4,17] for second sound and, [22] for thermoelasticity of type III. In these works, authors established both exponential and polynomial decay results with some restrictions on the system parameters.

In a considerable number of mechanical systems, the mechanism of dissipating energy is nonlinear, for instance, automotive, rotor, bridges and marine applications [1]. For the Timoshenko laminated beam system (1.3), viscoelastic damping is among the well studied nonlinear form of energy dissipation. For instance, we mention the work of Lo and Tatar [24] in which viscoelastic damping acting the effective rotation angle

$$\int_0^t g(t-r)(3s_{xx} - \psi_{xx})(r)dr,$$

was considered. The authors proved that this coupled with structural damping is sufficient for uniform decay with several conditions on the relaxation function g and the system coefficients. Mustafa [34] further investigated the system introduced in [24]. With a simpler assumption of equal wave propagation speeds, the author improved the decay results by exploiting the minimal and general conditions on g and, he established the optimal anticipated decay rates in respect to the presented degree of generality. Other results on viscoelastic damping can be found [13,25] and [21] for laminated beam system with infinity memory. Liu and Zhao [23] on the other hand, investigated a thermoelastic laminated beam with past history, and proved that in presence of structural damping, the solution decays exponentially and polynomially without any restrictions on the parameters. The authors further established that, for a system without structural damping, exponential and polynomial decay of the solution are possible in case of equal wave speeds, otherwise, the system lacks exponential stability. Other results in which authors employ both viscoelastic and thermoelastic damping mechanisms can be found in [31,32]. The asymptotic behaviour a laminated beam system due to nonlinear structural damping was investigated by Apalara et al. [8] and they established an explicit and general decay results.

In systems where propagation and transmission of information or material are involved, time delays are intrinsic. Time delay may be exhibited in form of lags between the input and output processing, or lags in equipose attainment or stability restoration of a system after perturbations resulting from internal or external forces, among others. To comprehensively investigate the delay effect on the properties of instantaneous systems, it is preferred that they are explicitly represented by control differential models with delay. Even though there are isolated scenarios in which introduction of delay may aid energy decay [42], time lags are often diagnosed as a cause of instability to the extent that an arbitrarily small delay may drive an asymptotically stable system into chaos [15]. In some cases, additional feedback or conditions may help to retain stability. For instance, Nicaise and Pignotti [35] investigated a wave equation whose stability has long been established with linear damping in absence of delay, on inclusion of the delay term, the system becomes chaotic and stability is only attainable if the delay weight is less than the damping coefficient.

The amplitude of vibrations in a single or laminated Timoshenko beam vanishes due to feedback. Time delay amplifies the phase lag thus increasing the early time response, which is seen to cause frequency dispersion in displacements [26]. This necessitate stronger damping to match up the increase in energy decay time. For example, we mention the work of Benaissa and Bahlil [12], in which a nonlinear Timoshenko system with nonlinear

delay

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 g_1(\psi_t(x, t)) + \mu_2 g_2(\psi_t(x, t - \tau)) = 0, \end{cases}$$

in $]0, 1[\times]0, +\infty[$, was investigated. The authors established the global existence of the system's solutions by exploiting the Faedo-Galerkin strategy. Additionally, by employing the multiplier method and some convexity arguments, the authors proved the general decay rate estimates provided $\frac{\rho_1}{K} = \frac{\rho_2}{b}$. For more works regarding constant time delay effect on stability of a single Timoshenko beam, we refer the reader to [3, 7, 39], and references therein.

In a classical laminated beam setting (1.3), the dissipation through structural damping is sufficient for exponential stability in absence of delay on assumption of equal wave speeds [6, 8]. In application, structures are exposed to external factors such as ultra violet radiation, high temperatures, moisture, etc. This affects the adhesive stiffness, and being the dominant source of energy dissipation, the effect often translates into time lag in attaining system stability. Owing to the level of complexity in structural damping and delay feedback, we find a general (nonlinear setting) representation g_1 and g_2 respectively, imperative for a more inclusive analysis of this delay effect on the general energy decay.

Investigating delay effect on stability of Timoshenko laminated beam system is gaining interest especially among mathematicians. Below is a summary of what has been done so far. Feng [16] studied a laminated beam with three internal constant delay feedbacks

$$\begin{cases} \rho w_{tt} + G(3s - \xi - w_x)_x + a_1 w_t(x, t - \tau) = 0, \\ I_\rho \xi_{tt} - D\xi_{xx} - G(3s - \xi - w_x) + a_2 \xi_t(x, t - \tau) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(3s - \xi - w_x) + a_3 s_t(x, t - \tau) = 0, \end{cases} \quad (1.5)$$

where $\xi = 3s - \psi$ and $(x, t) \in (0, L) \times (0, \infty)$. With help of three boundary controls to create necessary damping, the author proved wellposedness and exponential decay of the solution provided $\frac{4GL^2}{\pi^2} \leq D$ and, delay weights satisfying $0 \leq a_i < a_i^0$, $i = 1, 2, 3$ where a_i^0 depends on the parameters of the system. Seghour et al. [41] investigated a thermoelastic laminated beam with neutral delay in dynamics of slip equation. In addition to the dissipation through thermal effect, the authors introduced a linear frictional damping term in the transverse displacement to establish exponential stability for $\rho = GI_\rho$ and, polynomial decay otherwise. In a similar development, Choucha et al. [14], considered a thermoelastic laminated Timoshenko beam with distributed delay term in the third equation

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| s_t(x, t - \varrho) d\varrho.$$

Using dissipation through structural and thermoelastic damping, the authors established exponential and polynomial decay results with some restriction of the parameters provided that $\beta > \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho$. Lastly, Nonato et al. [37] recently studied a thermoelastic laminated beam with nonlinear weights and time-varying delay. With help of the dissipation through thermal effect and nonlinear frictional damping, the authors established exponential decay with and without the structural damping with suitable relationship between friction damping and delay weight, provided the condition of equal wave speeds holds. For more recent results on constant, distributed, and neutral delay effect on decay of vibrations in a laminated beam subject to structural and/or linear frictional damping, we refer the reader to [27–30].

From the above results, it is evident that for Timoshenko laminated beam with delay so far, authors have exploited boundary controls or dissipation through thermal effect, coupled with either structural or linear frictional damping in addition to restrictions on delay weight and system parameters, to establish asymptotic behavior of the solution.

Taking into account all that, a natural question arises, is energy decay of (1.3) with nonlinear delay feedback possible with the intrinsic nonlinear structural damping as the only source of dissipation? To affirmatively answer this question, we consider the system (1.1)–(1.2) and establish general decay of the solution with appropriate assumptions on functions g_1 and g_2 , the delay weight, and equal wave speeds.

The rest of the manuscript is organized as follows. In section 2, we present some preliminaries which include a necessary transformation and the required hypotheses. Technical lemmas are given in section 3. Finally, in section 4, we state and prove our stability results.

2. Preliminaries

Like in [12], we proceed by assuming the following hypotheses:

(H1) $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing C^0 -function such that there exist positive constants c_1, c_2, ϵ , and a strictly increasing function $H \in C^1([0, +\infty))$, with $H(0) = 0$, and H is linear on $[0, \epsilon]$ or strictly convex C^2 -function on $(0, \epsilon]$ such that

$$\begin{cases} r^2 + g_1^2(r) \leq H^{-1}(rg_1(r)), & \text{for all } |r| \leq \epsilon, \\ c_1 |r| \leq |g_1(r)| \leq c_2 |r|, & \text{for all } |r| \geq \epsilon. \end{cases} \quad (2.1)$$

(H2) $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 r g_2(r) \leq \zeta(r) \leq \alpha_2 r g_1(r), \quad (2.2)$$

where

$$\zeta(r) = \int_0^r g_2(y) dy,$$

and

$$\alpha_2 \mu < \alpha_1 \beta. \quad (2.3)$$

Remark 2.1. Hypothesis (H1) implies that $rg_1(r) > 0$, for all $r \neq 0$. Furthermore Lasiecka and Tataru in [20] exploited the monotonicity and continuity properties of g_1 to prove the existence of H defined in (H1).

Remark 2.2. Using the mean value Theorem for integrals and the monotonicity of g_2 , we conclude that

$$\zeta(r) = \int_0^r g_2(y) dy \leq r g_2(r), \quad (2.4)$$

and in what follows

$$\alpha_1 \leq \alpha_2 \leq 1.$$

To cater for the nonlinear delay, we introduce a positive constant η satisfying

$$\frac{\mu(1 - \alpha_1)}{\alpha_1} < \eta < \frac{\beta - \alpha_2 \mu}{\alpha_2}. \quad (2.5)$$

Example 2.3. Here we give example of functions g_1 and g_2 to illustrate (2.1)–(2.5). Let the function $g_1(r) = r^\kappa$, $r \in (0, 1]$ ($\epsilon = 1$), and $\kappa \geq 1$. $g_1'(r) = \kappa r^{\kappa-1}$ which is strictly positive. In the neighborhood of 0, the function H is defined by

$$H(r) = c_\kappa r^{\frac{\kappa+1}{2}},$$

where $c_\kappa = (2\kappa)^{-\frac{\kappa+1}{2}}$. Clearly, for $\kappa = 1$, H is linear on $[0, 1]$, otherwise strictly convex on $(0, 1]$ ($H'(0) = 0$ and $H'' > 0$ on $(0, 1]$). Next, observe that

$$H^{-1}(r) = 2\kappa r^{\frac{2}{\kappa+1}},$$

and in what follows, with r near 0, (2.1) can be deduced from fact that $r^\kappa + r^{2\kappa} \leq 2\kappa r^2$. Next, suppose we set the non-decreasing odd function $g_2(r) = 3^{-\kappa} r^3$ ($g_2' \geq 0$), then $rg_2(r) \leq rg_1(r)$ on $(0, 1]$. Furthermore, (2.4) follows automatically, that is $\zeta(r) = \frac{3^{-\kappa}}{4} r^4 \leq$

$rg_2(r) = 3^{-\kappa}r^4$, moreover, from the fact that $\zeta(r) \leq rg_1(r)$, choosing $\alpha_1 \leq \frac{1}{4}$ and $\alpha_2 \geq \alpha_1$, (2.2) can be easily deduced. Once α_1, α_2 , are fixed, relations (2.3) and (2.5) are achievable provided μ is sufficiently small in comparison to β . Lastly, the physical interpretation is illustrated in Example 4.2, in terms of energy decay rates of the solution under assumptions (H1) and (H2).

Similar to [35], we introduce a new variable

$$z(x, \sigma, t) = s_t(x, t - \tau\sigma) \quad \text{in } (0, 1) \times (0, 1) \times (0, \infty). \tag{2.6}$$

It is easy to show that z satisfies

$$\tau z_t(x, \sigma, t) + z_\sigma(x, \sigma, t) = 0, \quad \text{in } (0, 1) \times (0, 1) \times (0, \infty). \tag{2.7}$$

As a result, the system (1.1) is equivalent to

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s \\ \quad + \beta g_1(s_t(x, t)) + \mu g_2(z(x, 1, t)) = 0, \\ \tau z_t(x, \sigma, t) + z_\sigma(x, \sigma, t) = 0, \end{cases} \tag{2.8}$$

where $(x, \sigma, t) \in (0, 1) \times (0, 1) \times [0, \infty)$, with the following initial and boundary data

$$\begin{cases} w(x, 0) = w_0, \quad s(x, 0) = s_0, \quad \psi(x, 0) = \psi_0, \quad x \in (0, 1), \\ w_t(x, 0) = w_1, \quad s_t(x, 0) = s_1, \quad \psi_t(x, 0) = \psi_1, \quad x \in (0, 1), \\ w(0, t) = s_x(0, t) = \psi_x(0, t) = 0, \quad t \geq 0, \\ w_x(1, t) = s(1, t) = \psi(1, t) = 0, \quad t \geq 0, \\ z(x, 0, t) = s_t(x, t), \quad x \in (0, 1), \quad t \geq 0, \\ z(x, \sigma, 0) = f_0(x, -\tau\sigma), \quad (x, \sigma) \in (0, 1) \times (0, 1). \end{cases} \tag{2.9}$$

Henceforth, we consider (2.8)–(2.9) instead of (1.1)–(1.2) and $z(\sigma)$ to represent $z(x, \sigma, t)$.

We define the energy of the solution of the system (2.8)–(2.9) as follows

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 \right] dx \\ &\quad + \frac{1}{2} \int_0^1 \left[4\gamma s^2 + G(\psi - w_x)^2 + \tau\eta \int_0^1 \zeta(z(\sigma))d\sigma \right] dx. \end{aligned} \tag{2.10}$$

The existence, uniqueness, and smoothness of solution of problem (2.8)–(2.9), can be established by continuing the arguments of the FaedoGalerkin method as in [10, 12] in which a single Timoshenko beam and nonlinear damped porous systems were investigated respectively.

3. Technical lemmas

This section concentrates on statement and proof of technical lemmas required in establishing our energy decay result.

Lemma 3.1. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the energy functional (2.10) satisfies*

$$E'(t) \leq -m_0 \int_0^1 s_t g_1(s_t) dx - m_1 \int_0^1 z(1) g_2(z(1)) dx, \quad \forall t \geq 0, \tag{3.1}$$

where m_0 and m_1 are positive constants given by

$$m_0 = \beta - \alpha_2(\eta + \mu) \text{ and } m_1 = \alpha_1\eta - \mu(1 - \alpha_1).$$

Proof. We begin by multiplying the first equations of (2.8), by w_t , $(3s_t - \psi_t)$ and, s_t respectively, followed by integrating by parts over $(0, 1)$ using the boundary conditions (2.9), to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left[\rho w_t^2 + I_\rho (3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3D s_x^2 \right] dx \\ & + \frac{1}{2} \frac{d}{dt} \int_0^1 \left[4\gamma s^2 + G(\psi - w_x)^2 \right] dx \\ & = -\beta \int_0^1 s_t g_1(s_t) dx - \mu \int_0^1 s_t g_2(z(1)) dx. \end{aligned} \quad (3.2)$$

Next, multiplying the last equation of (2.8), by $\eta g_2(z(\sigma))$, then integrating over $(0, 1) \times (0, 1)$ to get

$$\begin{aligned} \tau \eta \int_0^1 \int_0^1 g_2(z(\sigma)) z_t(\sigma) d\sigma dx & = -\eta \int_0^1 \int_0^1 \frac{\partial}{\partial \sigma} \zeta(z(\sigma)) d\sigma dx \\ & = -\eta \int_0^1 \zeta(z(1)) dx + \eta \int_0^1 \zeta(z(0)) dx. \end{aligned} \quad (3.3)$$

Using $z(0) = s_t$, we observe that

$$\tau \eta \frac{d}{dt} \int_0^1 \int_0^1 \zeta(z(\sigma)) d\sigma dx = -\eta \int_0^1 \zeta(z(1)) dx + \eta \int_0^1 \zeta(s_t) dx. \quad (3.4)$$

Combining (3.2) and (3.4), and using (2.2), we note that the energy derivative satisfies

$$E'(t) \leq -(\beta - \alpha_2 \eta) \int_0^1 s_t g_1(s_t) dx - \eta \int_0^1 \zeta(z(1)) dx - \mu \int_0^1 s_t g_2(z(1)) dx. \quad (3.5)$$

If ζ^* is the conjugate function of the convex function ζ , that is, $\zeta^*(r) = \sup_{v \in \mathbb{R}^+} (vr - \zeta(r))$, then ζ^* is the Legendre transform of ζ , defined by

$$\zeta^*(r) = r(\zeta')^{-1}(r) - \zeta \left[(\zeta')^{-1}(r) \right], \quad \forall r \geq 0, \quad (3.6)$$

and satisfies the relation

$$vr \leq \zeta^*(r) + \zeta(v), \quad \forall v, r \geq 0, \quad (3.7)$$

(see Arnol'd [11, pg. 61–62], Benaissa and Bahlil [12]). Going by the definition of ζ and (3.6), we deduce that

$$\zeta^*(r) = r g_2^{-1}(r) - \zeta \left(g_2^{-1}(r) \right). \quad (3.8)$$

Making use of (3.8) and (2.2), we note that

$$\begin{aligned} \zeta^*(g_2(z(1))) & = z(1) g_2(z(1)) - \zeta(z(1)) \\ & \leq (1 - \alpha_1) z(1) g_2(z(1)). \end{aligned} \quad (3.9)$$

Next, from (3.7), (3.9) and (2.2), we estimate the last term of (3.5) as follows

$$\begin{aligned} -\mu \int_0^1 s_t g_2(z(1)) dx & \leq \mu \int_0^1 \zeta(s_t) dx + \mu \int_0^1 \zeta^*(g_2(z(1))) dx \\ & \leq \mu \int_0^1 \zeta(s_t) dx + \mu(1 - \alpha_1) \int_0^1 z(1) g_2(z(1)) dx \\ & \leq \alpha_2 \mu \int_0^1 s_t g_1(s_t) dx + \mu(1 - \alpha_1) \int_0^1 z(1) g_2(z(1)) dx, \end{aligned} \quad (3.10)$$

and substituting (3.10) in (3.5), and exploiting (2.5) and (2.3) completes our proof. \square

Lemma 3.2. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_1 , defined by*

$$F_1(t) := -\rho \int_0^1 w w_t dx$$

satisfies, for any $\varepsilon_1, \varepsilon_2 > 0$, the estimate

$$\begin{aligned} \frac{d}{dt} F_1(t) &\leq -\rho \int_0^1 w_t^2 dx + \varepsilon_1 \int_0^1 s_x^2 dx + \varepsilon_2 \int_0^1 (3s_x - \psi_x)^2 dx \\ &\quad + \left(G + \frac{9G^2}{4\varepsilon_1} + \frac{G^2}{4\varepsilon_2} \right) \int_0^1 (\psi - w_x)^2 dx. \end{aligned} \quad (3.11)$$

Proof. Differentiate F_1 and make use the first equation in (2.8), to get

$$\begin{aligned} \frac{d}{dt} F_1(t) &= -\rho \int_0^1 w_t^2 dx + G \int_0^1 (\psi - w_x)^2 dx - 3G \int_0^1 (\psi - w_x) s dx \\ &\quad + G \int_0^1 (3s - \psi)(\psi - w_x) dx. \end{aligned} \quad (3.12)$$

The last two terms of (3.12) are estimated as

$$\begin{aligned} -3G \int_0^1 (\psi - w_x) s dx &\leq \varepsilon_1 \int_0^1 s^2 dx + \frac{9G^2}{4\varepsilon_1} \int_0^1 (\psi - w_x)^2 dx \\ &\leq \varepsilon_1 \int_0^1 s_x^2 dx + \frac{9G^2}{4\varepsilon_1} \int_0^1 (\psi - w_x)^2 dx \end{aligned}$$

and

$$\begin{aligned} G \int_0^1 (3s - \psi)(\psi - w_x) dx &\leq \varepsilon_2 \int_0^1 (3s - \psi)^2 dx + \frac{G^2}{4\varepsilon_2} \int_0^1 (\psi - w_x)^2 dx \\ &\leq \varepsilon_2 \int_0^1 (3s_x - \psi_x)^2 dx + \frac{G^2}{4\varepsilon_2} \int_0^1 (\psi - w_x)^2 dx \end{aligned}$$

for $\varepsilon_1, \varepsilon_2 > 0$, thanks Young's and Poincaré's inequalities. Thus, (3.11) follows by substituting the above two estimates in (3.12). \square

Lemma 3.3. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_2 , defined by*

$$F_2(t) := -I_\rho \int_0^1 (3s_t - \psi_t)(3s - \psi) dx$$

satisfies the estimate

$$\begin{aligned} \frac{d}{dt} F_2(t) &\leq -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + \frac{3D}{2} \int_0^1 (3s_x - \psi_x)^2 dx \\ &\quad + \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx. \end{aligned} \quad (3.13)$$

Proof. Direct computations and using the second equation of (2.8) gives

$$\begin{aligned} \frac{d}{dt} F_2(t) &= -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + D \int_0^1 (3s_x - \psi_x)^2 dx \\ &\quad - G \int_0^1 (\psi - w_x)(3s - \psi) dx. \end{aligned} \quad (3.14)$$

Exploiting Young's and Poincaré's inequalities, we have

$$\begin{aligned} -G \int_0^1 (\psi - w_x)(3s - \psi) dx &\leq \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2} \int_0^1 (3s - \psi)^2 dx \\ &\leq \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2} \int_0^1 (3s_x - \psi_x)^2 dx, \end{aligned}$$

and substituting the above in (3.14) completes the proof. \square

Establishing the next two lemmas necessitates the estimate

$$g_2^2(z(1)) \leq 2z(1)g_2(z(1)),$$

which is a consequence of (3.7), (3.9) and (2.4). Furthermore, Lemmas 3.5 and 3.6 require the assumption of equal wave speeds $\frac{G}{\rho} = \frac{D}{I_\rho}$.

Lemma 3.4. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_3 , defined by*

$$F_3(t) := 3I_\rho \int_0^1 s_t s dx + 3\rho \int_0^1 w_t \int_0^x s(y) dy dx$$

for any $\varepsilon_3 > 0$, satisfies the estimate

$$\begin{aligned} \frac{d}{dt} F_3(t) &\leq -3D \int_0^1 s_x^2 dx - 2\gamma \int_0^1 s^2 dx + \varepsilon_3 \int_0^1 w_t^2 dx + \frac{4\beta^2}{\gamma} \int_0^1 g_1^2(s_t) dx \\ &\quad + \frac{\mu^2}{2\gamma} \int_0^1 z(1)g_2(z(1)) dx + \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_3}\right) \int_0^1 s_t^2 dx. \end{aligned} \quad (3.15)$$

Proof. Differentiating F_3 , and using the first and third equations of (2.8) followed by integrating by parts that term with $(\psi - w_x)_x$, we obtain

$$\begin{aligned} \frac{d}{dt} F_3(t) &= 3I_\rho \int_0^1 s_t^2 dx - 3D \int_0^1 s_x^2 dx - 4\gamma \int_0^1 s^2 dx - 4\beta \int_0^1 s g_1(s_t) dx \\ &\quad - \mu \int_0^1 s g_2(z(1)) dx + 3\rho \int_0^1 w_t \int_0^x s_t(y) dy dx. \end{aligned} \quad (3.16)$$

Exploiting Young's and Poincaré's inequalities, we estimate the last three terms of (3.16) as follows

$$-4\beta \int_0^1 s g_1(s_t) dx \leq \gamma \int_0^1 s^2 dx + \frac{4\beta^2}{\gamma} \int_0^1 g_1^2(s_t) dx, \quad (3.17)$$

$$\begin{aligned} -\mu \int_0^1 s g_2(z(1)) dx &\leq \gamma \int_0^1 s^2 dx + \frac{\mu^2}{4\gamma} \int_0^1 g_2^2(z(1)) dx \\ &\leq \gamma \int_0^1 s^2 dx + \frac{\mu^2}{2\gamma} \int_0^1 z(1)g_2(z(1)) dx, \end{aligned} \quad (3.18)$$

and for any $\varepsilon_3 > 0$,

$$\begin{aligned} 3\rho \int_0^1 w_t \int_0^x s_t(y) dy dx &\leq \varepsilon_3 \int_0^1 w_t^2 dx + \frac{9\rho^2}{4\varepsilon_3} \int_0^1 \left(\int_0^x s_t(y) dy \right)^2 dx \\ &\leq \varepsilon_3 \int_0^1 w_t^2 dx + \frac{9\rho^2}{4\varepsilon_3} \int_0^1 s_t^2 dx. \end{aligned} \quad (3.19)$$

Substituting (3.17)–(3.19) into (3.16) completes the proof. \square

Lemma 3.5. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_4 , defined by*

$$F_4(t) := \int_0^1 (\psi - w_x) s_t dx - \int_0^1 w_t s_x dx$$

for any $\varepsilon_4 > 0$, satisfies the estimate

$$\begin{aligned} \frac{d}{dt} F_4(t) &\leq -\frac{G}{2I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{8\gamma^2}{3I_\rho} \int_0^1 s_x^2 dx + \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2 dx \\ &\quad + \frac{8\beta^2}{3I_\rho} \int_0^1 g_1^2(s_t) dx + \frac{\mu^2}{3I_\rho} \int_0^1 z(1)g_2(z(1)) dx \\ &\quad + \left(3 + \frac{1}{4\varepsilon_4}\right) \int_0^1 s_t^2 dx. \end{aligned} \quad (3.20)$$

Proof. Differentiate F_4 , integrating by parts that term with s_{xt} and using the transformation $\psi_t = 3s_t - (3s_t - \psi_t)$. Using the first and third equations of (2.8), we observe that

$$\begin{aligned} \frac{d}{dt}F_4(t) &= -\frac{G}{I_\rho} \int_0^1 (\psi - w_x)^2 dx - \frac{4\gamma}{3I_\rho} \int_0^1 s(\psi - w_x) dx + 3 \int_0^1 s_t^2 dx \\ &\quad - \int_0^1 (3s_t - \psi_t) s_t dx - \frac{4\beta}{3I_\rho} \int_0^1 g_1(s_t)(\psi - w_x) dx \\ &\quad - \frac{\mu}{3I_\rho} \int_0^1 (\psi - w_x) g_2(z(1)) dx. \end{aligned} \quad (3.21)$$

Exploiting Young's, Young's, Poincaré's and Cauchy-Schwarz inequalities, we estimate the non square terms of (3.21) as follows

$$\begin{aligned} -\frac{4\gamma}{3I_\rho} \int_0^1 s(\psi - w_x) dx &\leq \frac{G}{6I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{8\gamma^2}{3I_\rho} \int_0^1 s^2 dx \\ &\leq \frac{G}{6I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{8\gamma^2}{3I_\rho} \int_0^1 s_x^2 dx, \end{aligned} \quad (3.22)$$

$$-\frac{4\beta}{3I_\rho} \int_0^1 (\psi - w_x) s_t dx \leq \frac{G}{6I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{8\beta^2}{3I_\rho} \int_0^1 g_1^2(s_t) dx, \quad (3.23)$$

$$\begin{aligned} -\frac{\mu}{3I_\rho} \int_0^1 (\psi - w_x) g_2(z(1)) dx &\leq \frac{G}{6I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{\mu^2}{6I_\rho} \int_0^1 g_2^2(z(1)) dx \\ &\leq \frac{G}{6I_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{\mu^2}{3I_\rho} \int_0^1 z(1) g_2(z(1)) dx, \end{aligned} \quad (3.24)$$

and for any $\varepsilon_4 > 0$,

$$-\int_0^1 (3s_t - \psi_t) s_t dx \leq \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{1}{4\varepsilon_4} \int_0^1 s_t^2 dx. \quad (3.25)$$

Relation (3.20) follows by substitution of (3.22)–(3.25) into (3.21). \square

Lemma 3.6. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_5 , defined by*

$$F_5(t) := -\int_0^1 (3s_t - \psi_t) w_x dx - \int_0^1 (3s_x - \psi_x) w_t dx + 3 \int_0^1 (3s_t - \psi_t) s dx$$

for any $\varepsilon_5 > 0$, satisfies the estimate

$$\begin{aligned} \frac{d}{dt}F_5(t) &\leq -\frac{D}{2I_\rho} \int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_5 \int_0^1 (3s_t - \psi_t)^2 dx \\ &\quad + \frac{9}{\varepsilon_5} \int_0^1 s_t^2 dx + \left(\frac{G^2}{2DI_\rho} + \frac{G}{I_\rho} \right) \int_0^1 (\psi - w_x)^2 dx. \end{aligned} \quad (3.26)$$

Proof. Differentiating F_5 followed by integrating by parts the term containing w_{xt} . Similarly, using the first two equations of (2.8), and further integrating by parts term containing $3s_{xx} - \psi_{xx}$, we arrive at

$$\begin{aligned} \frac{d}{dt}F_5(t) &= -\frac{D}{I_\rho} \int_0^1 (3s_x - \psi_x)^2 dx + \frac{G}{I_\rho} \int_0^1 (\psi - w_x)^2 dx \\ &\quad + 3 \int_0^1 (3s_t - \psi_t) s_t dx + \frac{G}{I_\rho} \int_0^1 (3s - \psi)(\psi - w_x) dx. \end{aligned} \quad (3.27)$$

Exploiting Young's and Poincaré's inequalities, we have the following estimates

$$\begin{aligned} \frac{G}{I_\rho} \int_0^1 (\psi - w_x)(3s - \psi) dx &\leq \frac{G^2}{2DI_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2I_\rho} \int_0^1 (3s - \psi)^2 dx \\ &\leq \frac{G^2}{2DI_\rho} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2I_\rho} \int_0^1 (3s_x - \psi_x)^2 dx, \end{aligned} \quad (3.28)$$

and, for any $\varepsilon_5 > 0$

$$3 \int_0^1 (3s_t - \psi_t) s_t dx \leq \varepsilon_5 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_5} \int_0^1 s_t^2 dx. \quad (3.29)$$

The estimate (3.26) follows by substituting (3.28) and (3.29) in (3.27). \square

Lemma 3.7. *If (w, ψ, s, z) is a solution of (2.8)–(2.9), then the functional F_6 , defined by*

$$F_6(t) := \tau \int_0^1 \int_0^1 e^{-2\sigma\tau} \zeta(z(\sigma)) d\sigma dx$$

satisfies the estimate

$$\begin{aligned} \frac{d}{dt} F_6(t) &\leq -\alpha_1 e^{-\tau} \int_0^1 z(1) g_2(z(1)) dx + \alpha_2 \int_0^1 s_t g_1(s_t) dx \\ &\quad - \tau e^{-\tau} \int_0^1 \int_0^1 \zeta(z(\sigma)) d\sigma dx. \end{aligned} \quad (3.30)$$

Proof. We proceed by differentiating F_6 and using the fact that $z(0) = s_t$ as follows

$$\begin{aligned} \frac{d}{dt} F_6(t) &= \tau \int_0^1 \int_0^1 e^{-\tau\sigma} z_t(\sigma) g_2(z(\sigma)) d\sigma dx \\ &= - \int_0^1 \int_0^1 e^{-\tau\sigma} z_\sigma(\sigma) g_2(z(\sigma)) d\sigma dx \\ &= - \int_0^1 \int_0^1 e^{-\tau\sigma} \frac{\partial}{\partial\sigma} \zeta(z(\sigma)) d\sigma dx \\ &= - \int_0^1 \int_0^1 \frac{\partial}{\partial\sigma} [e^{-\tau\sigma} \zeta(z(\sigma))] d\sigma dx - \tau \int_0^1 \int_0^1 e^{-\tau\sigma} \zeta(z(\sigma)) d\sigma dx \\ &= -e^{-\tau} \int_0^1 \zeta(z(1)) dx + \int_0^1 \zeta(s_t) dx - \tau \int_0^1 \int_0^1 e^{-\tau\sigma} \zeta(z(\sigma)) d\sigma dx. \end{aligned}$$

The relation (3.30) follows by virtue of (2.2) and the fact that $e^{-\tau} \leq e^{-\sigma\tau} \leq 1$ for all $\sigma \in (0, 1)$. \square

Lemma 3.8. *Let $N, N_k, k = 1 \dots 6$, be positive constants. The functional defined by*

$$\mathcal{L}(t) := NE(t) + \sum_{k=1}^6 N_k F_k(t), \quad \forall t \geq 0, \quad (3.31)$$

for some positive constants b_1, b_2, α_3 and α_4 , satisfies the relations

$$b_1 E(t) \leq \mathcal{L}(t) \leq b_2 E(t), \quad \forall t \geq 0, \quad (3.32)$$

and

$$\mathcal{L}'(t) \leq -\alpha_3 E(t) + \alpha_4 \int_0^1 (s_t^2 + g_1^2(s_t)) dx \quad \forall t \geq 0. \quad (3.33)$$

Proof. Regarding relation (3.32), it easy to deduce that for some $b > 0$,

$$|\mathcal{L}(t) - NE(t)| \leq bE(t).$$

Consequently,

$$(N - b)E(t) \leq \mathcal{L}(t) \leq (N + b)E(t),$$

and choosing N large enough concludes the proof of (3.32).

To establish relation (3.33), we proceed by differentiating (3.31), and then substitute for the derivatives of F_1 to F_6 using estimates (3.11), (3.13), (3.15), (3.20), (3.26) and (3.26) respectively. Setting

$$N_1 = N_2 = N_6 = 1, \quad \varepsilon_3 = \frac{\rho}{2N_3}, \quad \varepsilon_1 = \varepsilon_2 = \frac{G}{4}, \quad \varepsilon_4 = \frac{I_\rho}{4N_4}, \quad \varepsilon_5 = \frac{I_\rho}{4N_5},$$

for $c_4 > 0$, we have

$$\begin{aligned} \mathcal{L}'(t) \leq & - [m_1N + e^{-\tau}\alpha_1 - c_4N_3 - c_4N_4] \int_0^1 z(1)g_2(z(1))dx - \frac{\rho}{2} \int_0^1 w_t^2 dx \\ & - [m_0N - \alpha_2] \int_0^1 s_t g_1(s_t) dx - \left[\frac{DN_5}{2I_\rho} - c_4 \right] \int_0^1 (3s_x - \psi_x)^2 dx \\ & - [3DN_3 - c_4N_4 - c_4] \int_0^1 s_x^2 dx + \left[\frac{4\beta^2 N_3}{\gamma} + c_4N_4 \right] \int_0^1 g_1^2(s_t) dx \\ & - \left[\frac{GN_4}{2I_\rho} - c_4N_5 - c_4 \right] \int_0^1 (\psi - w_x)^2 dx - \frac{I_\rho}{2} \int_0^1 (3s_t - \psi_t)^2 dx \\ & - 2\gamma N_3 \int_0^1 s^2 dx - \tau e^{-\tau} \int_0^1 \int_0^1 \zeta(z(\sigma)) d\sigma dx \\ & + [c_4N_4(1 + c_4N_4) + c_4N_5^2 + c_4N_3(1 + c_4N_3)] \int_0^1 s_t^2 dx. \end{aligned}$$

Next, choose N_5 large enough such that

$$\frac{DN_5}{2I_\rho} - c_4 > 0.$$

We then pick N_4 large enough so that

$$\frac{GN_4}{2I_\rho} - c_4N_5 - c_4 > 0.$$

Fixing N_4 allows us to choose N_3 sufficiently large such that

$$3DN_3 - cN_4 - c_4 > 0.$$

Finally, choose N large enough such

$$m_1N + e^{-\tau}\alpha_1 - c_4N_3 - c_4N_4 > 0 \text{ and } m_0N - \alpha_2 > 0,$$

while maintaining the validity of (3.32). To this end, the energy estimate (3.33) follows automatically for some $\alpha_3, \alpha_4 > 0$. □

4. Asymptotic behavior

In this section, we state and prove our stability result.

Theorem 4.1. *Let (w, ψ, s, z) be the solution of system (2.8) and assume (H1) and (H2) and $\frac{G}{\rho} = \frac{D}{I_\rho}$ hold. Then there exist positive constants k_0, k_1, k_2 , and ϵ_0 such that this solution satisfies*

$$E(t) \leq k_0 H_1^{-1}(k_1 t + k_2), \quad t \geq 0, \tag{4.1}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_0(r)} dr$$

and

$$H_0(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon], \\ tH'(\epsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0, \epsilon]. \end{cases}$$

Example 4.2. We illustrate some energy decay rates given by Theorem 4.1 corresponding to the functions defined in Example 2.3. Observe that, for $H(r) = c_\kappa r^{\frac{\kappa+1}{2}}$, we have

$$H'(r) = \frac{\kappa + 1}{2} c_\kappa r^{\frac{\kappa-1}{2}}$$

thus for $c = c_\kappa(\epsilon_0)^{\frac{\kappa-1}{2}}$,

$$H_0(r) = \frac{\kappa + 1}{2} cr^{\frac{\kappa+1}{2}},$$

and

$$H_1(t) = \frac{2}{c(\kappa + 1)} \int_t^1 r^{-\frac{\kappa+1}{2}} dr.$$

Therefore, in neighborhood of 0

$$H_1(t) = \begin{cases} -\frac{1}{c} \ln t; & \kappa = 1, \\ \frac{4}{c(\kappa^2 - 1)} \left(t^{-\frac{\kappa-1}{2}} - 1 \right); & \kappa > 1, \end{cases}$$

and as $t \rightarrow +\infty$

$$H_1^{-1}(t) = \begin{cases} e^{-ct}; & \kappa = 1, \\ \left(\frac{1}{4}c(\kappa^2 - 1)t + 1 \right)^{-\frac{2}{\kappa-1}}; & \kappa > 1. \end{cases}$$

To this end, the energy decay rates are given as

$$E(t) \leq \begin{cases} k_0 e^{-c(k_1 t + k_2)}; & \kappa = 1, \\ k_0 \left(\frac{1}{4}c(\kappa^2 - 1)(k_1 t + k_2) + 1 \right)^{-\frac{2}{\kappa-1}}; & \kappa > 1. \end{cases}$$

We now proceed to the proof of Theorem 4.1 as follows.

Proof. In consideration of (3.33), we consider the following two cases.

Case I: H is linear on $[0, \epsilon]$. Observe that from **(H1)**

$$c_1 r^2 \leq r g_1(r) \leq c_2 r^2 \quad \text{and} \quad c_1 r g_1(r) \leq g_1^2(r) \leq c_2 r g_1(r), \quad \forall r \in \mathbb{R},$$

consequently from (3.33), and for some $\tilde{\alpha}_4 > 0$, we have

$$\mathcal{L}'(t) \leq -\alpha_3 E(t) + \tilde{\alpha}_4 \int_0^1 s_t g_1(s_t) dx, \quad \forall t \geq 0. \tag{4.2}$$

Clearly, using (3.1) and (4.2), for some $\alpha_5 > 0$,

$$\mathcal{L}'(t) \leq -\alpha_3 E(t) - \alpha_5 E'(t), \quad \forall t \geq 0. \tag{4.3}$$

Next, if we define \mathfrak{L} as

$$\mathfrak{L}(t) := \mathcal{L}(t) + \alpha_5 E(t), \quad \forall t \geq 0, \tag{4.4}$$

then from (3.32), is easy to show that for some $\tilde{b}_1, \tilde{b}_2 > 0$

$$\tilde{b}_1 E(t) \leq \mathfrak{L}(t) \leq \tilde{b}_2 E(t), \quad \forall t \geq 0, \tag{4.5}$$

thus, using (4.4) with (4.5), for $k_1 = \frac{\alpha_3}{\tilde{b}_2}$, we note that

$$\mathfrak{L}'(t) \leq -k_1 \mathfrak{L}(t), \quad \forall t \geq 0. \tag{4.6}$$

A simple integration of (4.6) and using (4.5), for $k_0 = \frac{\tilde{b}_2 E(0)}{\tilde{b}_1}$, yields

$$E(t) \leq k_0 e^{-k_1 t}, \quad \forall t \geq 0. \tag{4.7}$$

Case II: For H nonlinear on $(0, \epsilon]$, as in [20], we choose $0 < \epsilon_1 \leq \epsilon$ such that

$$rg_1(r) \leq \min \{ \epsilon, H(\epsilon) \}, \quad \forall |r| \leq \epsilon_1.$$

Using (H1) and continuity of g_1 together with the fact that $|g_1(r)| > 0$, for $r \neq 0$, we deduce that

$$\begin{cases} r^2 + g_1^2(r) \leq H^{-1}(rg_1(r)), \text{ for all } |r| \leq \epsilon_1, \\ c_1 |r| \leq |g_1(r)| \leq c_2 |r|, \text{ for all } |r| \geq \epsilon_1. \end{cases} \tag{4.8}$$

We now concentrate on the last term of (3.33)

$$\int_0^1 (s_t^2 + g_1^2(t)) dx.$$

To estimate this integral, similar to [19], we introduce the following partitions

$$J_1 = \{x \in (0, 1) : |s_t| \leq \epsilon_1\}, \quad J_2 = \{x \in (0, 1) : |s_t| > \epsilon_1\}.$$

Hence, for J defined as

$$J(t) = \int_{J_1} s_t g_1(t) dx,$$

it follows by Jensen inequality and the concavity of H^{-1} that

$$H^{-1}(J(t)) \geq c_5 \int_{J_1} H^{-1}(s_t g_1(s_t)) dx, \tag{4.9}$$

for some $c_5 > 0$. By virtue of (4.8), (4.9) and (3.1), we note that

$$\begin{aligned} \int_0^1 (s_t^2 + g_1^2(t)) dx &= \int_{J_1} (s_t^2 + g_1^2(t)) dx + \int_{J_2} (s_t^2 + g_1^2(t)) dx \\ &\leq \int_{J_1} H^{-1}(s_t g_1(s_t)) dx + c_6 \int_{J_2} (s_t g_1(s_t)) dx \\ &\leq c_6 H^{-1}(J(t)) - c_6 E'(t), \end{aligned} \tag{4.10}$$

for some $c_6 > 0$. Next, for $c_7 > 0$, we define the functional \mathfrak{L}_0 as

$$\mathfrak{L}_0(t) := \mathcal{L}(t) + c_7 E(t), \quad \forall t \geq 0. \tag{4.11}$$

Using (3.32), it is easy to deduce that for some $\mathfrak{b}_1, \mathfrak{b}_2 > 0$, $\mathfrak{L}_0 \sim E$, that is

$$\mathfrak{b}_1 E(t) \leq \mathfrak{L}_0(t) \leq \mathfrak{b}_2 E(t), \quad \forall t \geq 0. \tag{4.12}$$

Similarly, substituting (4.10) in (3.33) and using (4.11) we observe that

$$\mathfrak{L}'_0(t) \leq -\alpha_3 E(t) + c_7 H^{-1}(J(t)), \quad \forall t \geq 0. \tag{4.13}$$

For $\epsilon_0 < \epsilon$ and $\delta_0 > 0$, using (4.12) and the following properties of E and H :

$$E' \leq 0, \quad H' > 0, \quad H'' > 0 \quad \text{on } (0, \epsilon],$$

the functional \mathfrak{L}_1 defined as

$$\mathfrak{L}_1(t) := H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) \mathfrak{L}_0(t) + \delta_0 E(t), \quad \forall t \geq 0, \tag{4.14}$$

is equivalent to E , that is, for some $\tilde{\mathfrak{b}}_1, \tilde{\mathfrak{b}}_2 > 0$,

$$\tilde{\mathfrak{b}}_1 E(t) \leq \mathfrak{L}_1(t) \leq \tilde{\mathfrak{b}}_2 E(t), \quad \forall t \geq 0. \tag{4.15}$$

Furthermore, using (4.13), we deduce that

$$\begin{aligned} \mathfrak{L}'_1(t) &= \epsilon_0 \frac{E'(t)}{E(0)} H'' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) \mathfrak{L}_0(t) + H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) \mathfrak{L}'_0(t) + \delta_0 E'(t) \\ &\leq -\alpha_3 H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) E(t) + \underbrace{c_7 H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) H^{-1}(J(t))}_{\mathbf{Y}} + \delta_0 E'(t). \end{aligned} \tag{4.16}$$

Like in (3.6), to estimate \mathbf{Y} we let H^* be the convex conjugate of H defined by

$$H^*(r) = r(H')^{-1}(r) - H \left[(H')^{-1}(r) \right] \leq r(H')^{-1}(r), \quad \text{for } r \in (0, H'(\epsilon)), \tag{4.17}$$

Furthermore, exploiting the general Youngs inequality we observe that

$$rv \leq H^*(r) + H(v), \quad \text{for } r \in (0, H'(\epsilon)), v \in (0, \epsilon] \tag{4.18}$$

Setting

$$r = H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) \text{ and } v = H^{-1}(J(t)),$$

and using (4.17), (4.18), coupled with fact that

$$J(t) = \int_{J_1} s_t g_1(t) dx \leq \int_0^1 s_t g_1(s_t) dx \leq -m_0^{-1} E'(t),$$

for some $c_8 > 0$, we deduce

$$c_7 H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) H^{-1}(J(t)) \leq c_7 \epsilon_0 \frac{E(t)}{E(0)} H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) - c_8 E'(t). \tag{4.19}$$

Substituting (4.19) in (4.16), we observe that

$$\mathfrak{L}'_1(t) \leq -[\alpha_3 E(0) - c_7 \epsilon_0] \frac{E(t)}{E(0)} H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) + (\delta_0 - c_8) E'(t). \tag{4.20}$$

At this point, we choose $\epsilon_0 = \frac{\alpha_3 E(0)}{2c_7}$ and $\delta_0 = 2c_8$ to obtain,

$$\mathfrak{L}'_1(t) \leq -\tilde{\alpha}_3 \frac{E(t)}{E(0)} H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) + c_8 E'(t); \quad \tilde{\alpha}_3 = \frac{\alpha_3 E(0)}{2}.$$

Moreover, using the fact that $E'(t) \leq 0$, we deduce that

$$\mathfrak{L}'_1(t) \leq -\tilde{\alpha}_3 \frac{E(t)}{E(0)} H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) = -\tilde{\alpha}_3 H_0 \left(\frac{E(t)}{E(0)} \right), \tag{4.21}$$

were $H_0(r) = rH'(\epsilon_0 r)$. Going by the fact that H is strictly convex on $(0, \epsilon]$, we observe that $H_0(r), H'_0(r) > 0$ on $(0, 1]$. Thus, if we let

$$\tilde{\mathfrak{L}}_1(t) := \frac{\tilde{\mathfrak{b}}_1 \mathfrak{L}_1(t)}{E(0)}, \tag{4.22}$$

then from (4.15), it is clear that

$$\bar{b}_1 E(t) \leq \tilde{\mathfrak{L}}_1(t) \leq \bar{b}_2 E(t), \quad \forall t \geq 0, \tag{4.23}$$

and, (4.21) implies that

$$\tilde{\mathfrak{L}}'_1(t) \leq -\frac{\tilde{\mathfrak{b}}_1 \tilde{\alpha}_3}{E(0)} H_0 \left(\frac{E(t)}{E(0)} \right)$$

Furthermore, by putting into consideration the fact that H_0 is increasing together with (4.23), for some $k_1 > 0$, we obtain

$$\tilde{\mathfrak{L}}'_1(t) \leq -k_1 H_0 \left(\tilde{\mathfrak{L}}_1(t) \right), \quad \forall t \geq 0. \tag{4.24}$$

The consequence of (4.24) is

$$\left[H_1 \left(\tilde{\mathcal{L}}_1(t) \right) \right]' \geq k_1, \quad \forall t \geq 0, \quad (4.25)$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_0(r)} dr.$$

Integrating (4.25) over $(0, t)$, we obtain

$$H_1 \left(\tilde{\mathcal{L}}_1(t) \right) \geq k_1 t + k_2, \quad k_2 = H_1 \left(\tilde{\mathcal{L}}_1(0) \right). \quad (4.26)$$

By virtue of H_1^{-1} being a decreasing function, it follows that

$$\tilde{\mathcal{L}}_1(t) \leq H_1^{-1} (k_1 t + k_2), \quad \forall t \geq 0. \quad (4.27)$$

Lastly, from the relation (4.23), we note that for $k_0 = \frac{1}{b_1}$,

$$E(t) \leq k_0 H_1^{-1} (k_1 t + k_2), \quad \forall t \geq 0. \quad (4.28)$$

which completes the proof of Theorem 4.1. \square

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