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Classification of Ruled Surfaces Family with Common Characteristic Curve in Euclidean 3-space

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Abstract. Classification of ruled surfaces that satisfying certain geometric conditions has been studied by many researchers in the past years. The purpose of this paper is to study and classify the family of ruled surfaces whose common directrix satisfies the requirements of characteristic curves in three-dimensional Euclidean space. The family of ruled surfaces is parameterized by its directrix curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. According to the type of characteristic directrix curve, the family of ruled surfaces is classified into three types, and one type when the family is developable.

1. Introduction

A ruled surface is constructed by continuous motion of a straight line called the ruling (or generator) through a given curve which is called the base (or directrix) curve. The ruled surfaces are classical subject in differential geometry and they have many recent applications in different areas of sciences including computer-aided geometric design (CAGD), computer graphic, architectural designing, mechanics, robotics, product design and manufacturing [4, 10].

Developable surfaces are a special class of ruled surfaces that have vanishing Gaussian curvature. The developable surface is locally isometric to a plane, this means that the developable surface can be developed (flattened) onto a plane without stretching and tearing. In manufacturing, a developable surface can be produced from paper or sheet metal with no distortion. Therefore, the developable surfaces are commonly used in industrial design and modeling [3, 12].

Geodesic, asymptotic, and line of curvature are characteristic curves that lie on the surface and have been used in surface analysis. The geodesic curve gives the shortest path between two given points on curved spaces. All straight lines in the plane are geodesics, as are the rulings of any ruled surface [9]. A curve is an asymptotic if its normal curvature is equal to zero. All straight lines on a surface are asymptotic lines. Finally, a line of curvature is a curve whose tangent always points along with a principal directions i.e. a direction in which the normal curvature is maximum or minimum. In shape analysis, the line of curvature is one of the most important characteristic curves indicates the directions in which a shape bends extremely.

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There are several articles for designing the surfaces family that possess the given curve as a characteristic curve. Wang et al. [13] and Li et al. [6] studied the parametric representation of a surface pencil with a common spatial geodesic and line of curvature respectively. Bayram et.al. [1] studied surface pencil with a common asymptotic curve.

In this article, a family of ruled surfaces with common characteristic curve is studied and classified in Euclidean 3- space. A ruled surfaces family is constructed by its directrix curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. According to the characteristic directrix curve, three types of ruled surfaces family are classified. The main theorem shows that the common directrix curve is a geodesic or an asymptotic if and only if the ruled surfaces family is a rectifying or an osculating. And the family of generalized ruled surfaces with a special condition has a directrix curve as a line of curvature. Finally, the developability condition of such families is studied.

The paper is organized as follows. In section 2, some basic concepts about space curves and ruled surfaces are given. The main results are studied in section 3, where the family of ruled surfaces based on its common directrix curve are classified into three categories (rectifying, osculating, generalized) whose common directrix curve is (geodesic, asymptotic, line of curvature) respectively. The classification classes are investigated under developability condition in section 4. Finally, the conclusion and future works are given in section 5.

2. Preliminaries

This section introduces some basic facts about the differential geometry of space curves and ruled surfaces in three-dimensional Euclidean space, as well as some basic definitions and notions that are required subsequently. More details can be found in such standard references as [2, 9].

2.1. *Curves in Euclidean 3-space*

A smooth space curve in 3-dimensional Euclidean space E^3 is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$ where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector n(s) of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following Frenet-Serret formula.

$$\begin{cases} t'(s) &= \kappa(s)n(s), \\ n'(s) &= -\kappa(s)t(s) + \tau b(s), \\ b'(s) &= -\tau(s)n(s). \end{cases} \tag{1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions.The planes spanned by $\{t(s), n(s)\}$, $\{t(s), b(s)\}$ and $\{n(s), b(s)\}$ are respectively called the osculating plane, the rectifying plane and the normal plane. When the point moves along the unit speed curve with non-vanishing curvature and torsion, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector

$$\omega = \tau t + \kappa b. \tag{2}$$

The direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. The unit Darboux vector field is defined by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b. \tag{3}$$

For a regular curve on a surface, there exists another frame which is called Darboux frame and denoted by $\{t(s), g(s), N(s)\}$. In this frame, $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. Derivative of the Darboux frame according to arc-length parameter is governed by the following relations

$$\begin{cases} t'(s) &= \kappa_g g(s) + \kappa_n N(s), \\ g'(s) &= -\kappa_g t(s) + \tau_g N(s), \\ N'(s) &= -\kappa_n t(s) - \tau_g g(s). \end{cases} \tag{4}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion at each point of the curve $\gamma(s)$ which are given by

$$\kappa_g =< \gamma''(s), g >, \qquad \kappa_n =< \gamma''(s), N > \text{ and } \qquad \tau_g =< N', g > . \tag{5}$$

The relation between Darboux frame and Serret-Frenet frame can be given by the following matrix representation

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{6}$$

Where

$$\begin{cases} g(s) &= \cos\phi(s) n(s) + \sin\phi(s) b(s), \\ N(s) &= -\sin\phi(s) n(s) + \cos\phi(s) b(s). \end{cases} \tag{7}$$

Differentiating (7), using (4) and (1), we get the relation between geodesic curvature, normal curvature, and geodesic torsion with curvature and torsion as follows

$$\kappa_g = \kappa\cos\phi, \qquad \kappa_n = \kappa\sin\phi, \qquad \text{and } \tau_g = \tau + \frac{d\phi}{ds}. \tag{8}$$

Definition 2.1. *A curve lying on a surface is*

1. *a geodesic if and only if its geodesic curvature vanishes ($\kappa_g = 0$). By (8) and (7), it is equivalent to*

$$N = \pm n. \tag{9}$$

2. *an asymptotic if and only if its normal curvature vanishes ($\kappa_n = 0$). By (8) and (7), it is equivalent to*

$$N = \pm b. \tag{10}$$

3. *a line of curvature if and only if its geodesic torsion vanishes ($\tau_g = 0$). By (8), it is equivalent to*

$$\tau + \frac{d\phi}{ds} = 0. \tag{11}$$

2.2. *Ruled surfaces*

A ruled surface is generated by the motion of a straight line on a given curve and parameterized by

$$X(s, v) = \gamma(s) + vD(s), 0 \le s \le \ell,\ v \in \mathbb{R}. \tag{12}$$

A unit regular curve $\gamma(s)$ is called a base curve (or directrix), and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling (or generator) of the ruled surface at $\gamma(s)$. $D(s)$ is a unit director vector field that gives the direction of the ruling. Different ruled surfaces are constructed based on different $\gamma(s)$ and $D(s)$. The unit normal vector field to the ruled surface is defined by

$$N(s, v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{(\gamma' \times D) + v(D' \times D)}{|(\gamma' \times D) + v(D' \times D)|}. \tag{13}$$

A point on a ruled surface that satisfies $X_s \times X_v = 0$ is called a singular point, where the surface normal is not defined, a point that is not singular is called a regular. In general, a ruled surface may have singular points, they are located (if exist) on the striction curve which parameterized by [2]

$$C(s) = \gamma(s) - \frac{\langle \gamma'(s), D'(s)\rangle}{\langle D'(s), D'(s)\rangle} D(s), \qquad D'(s) \neq 0. \tag{14}$$

A ruled surface where all rulings tangent the directrix i.e. $\gamma'(s) = D(s)$, has singularities (edge of regression) along the directrix curve. The Gaussian curvature is non-positive for a ruled surface, it vanishes identically for special classes called the developable surfaces. Equivalently, a ruled surface (12) is developable if and only if [2]

$$det\langle \gamma'(s), D(s), D'(s)\rangle = 0. \tag{15}$$

The vector field $D(s)$ lies in the space formed by moving frame $\{t, n, b\}$ of $\gamma(s)$ and using (6) is defined by

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \text{ where } g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s). \tag{16}$$

Therefore $D(s)$ can be given by[11]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \tag{17}$$

The functions $\phi(s)$ and $\theta(s)$ are two scalar functions that are called the first and second angular functions [5]. When $\phi(s)$ and $\theta(s)$ take special choices, the director vector $D(s)$ lies in a rectifying, an osculating, or a normal plane of the directrix curve.

Definition 2.2. *The surface defined by*

$$\begin{cases} X(s,v) &= \gamma(s) + vD(s),\ 0 \leq s \leq \ell,\ v \in \mathbb{R}, \quad \text{where}, \\ D(s) &= \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \end{cases} \tag{18}$$

is called the family of ruled surfaces with a common directrix curve.

Through this paper, the singular points on the constructed ruled surface are avoided, therefore the vectors $\gamma'(s)$ and $D(s)$ are not collinear, it requires that $\sin\theta(s) \neq 0$ which can be used as a regularity condition, i.e. $0 < \theta(s) < \pi$. The surface normal along the directrix using (6) is given by

$$N(s,0) = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \tag{19}$$

By choosing different values of $\phi(s)$ and $\theta(s)$ we obtain not only the members of ruled surfaces family with common directrix curve but also yields different families. For example, when $\cos\phi(s) = 0$, $\sin\phi(s) = 0$, or $\cos\theta(s) = 0$, the corresponding families of ruled surfaces are parameterized respectively by:

$$X_{rec}(s,v) = \gamma(s) + v(\cos\theta(s)t(s) + \sin\theta(s)b(s)), \tag{20}$$
$$X_{osc}(s,v) = \gamma(s) + v(\cos\theta(s)t(s) + \sin\theta(s)n(s)), \tag{21}$$
$$X_{nor}(s,v) = \gamma(s) + v(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \tag{22}$$

which are called the rectifying, osculating and normal ruled surfaces family respectively, and:

$$D_{rec}(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s), \tag{23}$$
$$D_{osc}(s) = \cos\theta(s)t(s) + \sin\theta(s)n(s), \tag{24}$$
$$D_{nor}(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s). \tag{25}$$

are called the rectifying, the osculating and the normal director vector respectively. This means that, according to type of director vector field, three different families of ruled surfaces are constructed. Especially, when the director vector field $D(s)$ has the same direction of the tangent vector field $t(s)$, the principal normal vector field $n(s)$, the binormal vector field $b(s)$, and the unit Darboux vector field $\omega(s)$, then the corresponding ruled surfaces are parameterized respectively by

$$X_t(s,v) = \gamma(s) + vt(s), \tag{26}$$
$$X_n(s,v) = \gamma(s) + vn(s), \tag{27}$$
$$X_b(s,v) = \gamma(s) + vb(s), \tag{28}$$
$$X_{Dar}(s,v) = \gamma(s) + v(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}b(s)). \tag{29}$$

From the above discussion, we can easily obtain the following lemma which constructs the ruled surfaces families by choosing different values of angular functions $\phi(s)$ and $\theta(s)$.

Lemma 2.3. *The ruled surfaces family (18) with common directrix curve is*

1. *A generalized ruled surfaces family $X_{gen}(s,v)$ (18), if and only if* $\cos\theta(s) \neq 0$, $\cos\phi(s) \neq 0$ *and* $\sin\phi(s) \neq 0$.
2. *A rectifying ruled surfaces family $X_{rec}(s,v)$ (20), if and only if* $\cos\phi(s) = 0$.
3. *An osculating ruled surfaces family $X_{osc}(s,v)$ (21), if and only if* $\sin\phi(s) = 0$.
4. *A normal ruled surfaces family $X_{nor}(s,v)$ (22), if and only if* $\cos\theta(s) = 0$.

Similarly, the following members are constructed by choosing different values for angular functions.

Lemma 2.4. *The ruled surfaces family (18) have the following members :*

1. *Principal normal ruled surface $X_{pn}(s,v)$ (27), if and only if* $\cos\theta(s) = 0$ *and* $\sin\phi(s) = 0$.
2. *Binormal ruled surface $X_b(s,v)$ (28), if and only if* $\cos\theta(s) = 0$ *and* $\cos\phi(s) = 0$.
3. *Darboux ruled surface $X_{Dar}(s,v)$ (29), if and only if* $\cos\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}$ *and* $\sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}$.

Excluding the tangent ruled surface (26) via the regularity condition ($\sin\theta(s) \neq 0$), the other ruled surfaces families $X_{gen}(s,v)$, $X_{rec}(s,v)$, $X_{osc}(s,v)$, and $X_{nor}(s,v)$, also the members $X_n(s,v)$, $X_b(s,v)$, and $X_{Dar}(s,v)$ will be studied in the next section and classified depending on the type of characteristic curve. It is worth noting that during writing this paper, the geometry of $X_{rec}(s,v)$ (20) is studied in [8].

3. Classification of ruled surfaces family with common characteristic curve

In this section, we classify the ruled surfaces family whose common directrix satisfies the requirements of characteristic curve in three-dimensional Euclidean space. We show that the ruled surfaces family parameterized by (18) whose directrix curve is characteristic can be classified into three types: (rectifying, osculating, or generalized) ruled surfaces family. This is the main result of this article and will be given in the following main theorem.

Theorem 3.1 (Main Theorem). *Let $X(s,v)$ be a ruled surfaces family parameterized by (18), and let $\gamma(s)$ be its common characteristic directrix curve with non-vanishing curvature and torsion, then $\gamma(s)$ is :*

1. *Geodesic directrix curve, if and only if $X(s,v)$ is a rectifying ruled surfaces family $X_{osc}(s,v)$.*
2. *Asymptotic directrix curve, if and only if $X(s,v)$ is an osculating ruled surfaces family $X_{rec}(s,v)$.*
3. *Line of curvature directrix curve, if $X(s,v)$ is a generalized ruled surfaces family $X_{gen}(s,v)$ satisfying $\tau + \frac{d\phi}{ds} = 0$.*

Proof.

1. From (19), $N = \pm n$, if and only if $\cos\phi(s) = 0$, by lemma(2.4) and (9) this is equivalent to $\gamma(s)$ is a geodesic, if and only if $X(s,v)$ is a rectifying ruled surfaces family.
2. From (19), it is clear that $N = \pm b$, if and only if $\sin\phi(s) = 0$, by lemma(2.4) and (10) this equivalent to $\gamma(s)$ is an asymptotic, if and only if $X(s,v)$ is an osculating ruled surfaces family.

3. Let $X(s,v)$ be the ruled surfaces family (18) where the common directrix curve $\gamma(s)$ is a line of curvature, hence $\gamma(s)$ satisfies the constraint $\tau + \frac{d\phi}{ds} = 0$ (11), it follows that

$$\phi(s) = -\int_0^s \tau ds. \tag{30}$$

By using (30) in (18), we obtain

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\int_0^s \tau ds\, n(s) - \sin\int_0^s \tau ds.\, b(s))]. \tag{31}$$

Therefore, the family (31) is generalized ruled surfaces family $X_{gen}(s,v)$.
□

Remark 3.2. *The proof of the main theorem showed how to construct the family of ruled surfaces whose directrix curve is a line of curvature,* $\cos\theta(s)$ *is the family parameter when varies the family members are constructed. The normal ruled surfaces family* $X_{nor}(s,v)$ *(22) degenerates to a member of this family via the line of curvature condition (30) and can be produced by substitution* $\cos\theta(s) = 0$ *as the following*

$$X_{nor}(s,v) = \gamma(s) + v[\cos\int_0^s \tau ds\, n(s) - \sin\int_0^s \tau ds\, b(s)]. \tag{32}$$

The main theorem can be decomposed into the following three theorems which study the constructed families separately and can be used as tools together with the main theorem to build the classification theorem.

Theorem 3.3. *Let* $X(s,v)$ *be a ruled surfaces family given by (18), where the common directrix curve* $\gamma(s)$ *is a line of curvature. Then, the following statements hold:*

1. $X(s,v)$ *is neither rectifying ruled surfaces family* $X_{rec}(s,v)$ *nor osculating ruled surfaces family* $X_{osc}(s,v)$.
2. $X(s,v)$ *is a generalized ruled surfaces family* $X_{gen}(s,v)$.
3. $X_{nor}(s,v)$ *(13) is a member of this family.*

Theorem 3.4. *Let* $X(s,v)$ *be a ruled surfaces family given by (18), where the common directrix curve* $\gamma(s)$ *is a geodesic. Then, the following statements hold:*

1. $X(s,v)$ *is neither generalized ruled surfaces family* $X_{gen}(s,v)$ *nor osculating ruled surfaces family* $X_{osc}(s,v)$.
2. $X(s,v)$ *is a rectifying ruled surfaces family* $X_{rec}(s,v)$.
3. $X_b(s,v)$ *(28) is a member of this family.*

Theorem 3.5. *Let* $X(s,v)$ *be a ruled surfaces family given by (18), where the common directrix curve* $\gamma(s)$ *is asymptotic. Then, the following statements hold:*

1. $X(s,v)$ *is neither generalized ruled surfaces family* $X_{gen}(s,v)$ *nor rectifying ruled surfaces family* $X_{rec}(s,v)$.
2. $X(s,v)$ *is an osculating ruled surfaces family* $X_{osc}(s,v)$.
3. $X_n(s,v)$ *(27) is a member of this family.*

Based on the main Theorem and Theorems (3.3), (3.4) and (3.5), we can get the following theorem of classification.

Theorem 3.6. *(Classification of ruled surfaces family) Let* $X(s,v)$ *be a ruled surfaces family parametrized by (18), where the common directrix is characteristic curve. Then* $X(s,v)$ *is either a rectifying ruled surfaces family* $X_{rec}(s,v)$*, an osculating ruled surfaces family* $X_{osc}(s,v)$ *or a generalized ruled surfaces family* $X_{gen}(s,v)$ *satisfying (30).*

The explicit classification can be obtained in following equivalent theorem.

Theorem 3.7. *Let $X(s,v)$ be a ruled surfaces family whose common directrix is a geodesic, an asymptotic or a line of curvature. Then $X(s,v)$ is a rectifying ruled surfaces family $X_{rec}(s,v)$, an osculating ruled surfaces family $X_{osc}(s,v)$ or a generalized ruled surfaces family $X_{gen}(s,v)$ satisfying (30) respectively.*

According to the above classification theorems, the ruled surfaces family whose common directrix is a characteristic curve can be classified into three different type $X_{rec}(s,v)$, $X_{osc}(s,v)$ or $X_{gen}(s,v)$ based on the type of the common directrix curve. Also, without any conditions, the families $X_{rec}(s,v)$ and $X_{osc}(s,v)$ have common geodesic and asymptotic directrix curve respectively, whereas the family $X_{gen}(s,v)$ has a common line of curvature when satisfying (30).

Corollary 3.8. *Let $X(s,v)$ be a ruled surfaces family of type $X_{rec}(s,v)$, $X_{osc}(s,v)$ or $X_{gen}(s,v)$ satisfying (30). Then, the common directrix curve is geodesic, asymptotic or line of curvature .*

The existence of such families $X_{rec}(s,v)$, $X_{osc}(s,v)$, or $X_{gen}(s,v)$ can be ensured via the proof of main Theorem, so we obtain the following corollary.

Corollary 3.9. *Given a unit speed regular curve $\gamma(s)$ with non vanishing curvature and torsion. Then, there exists a rectifying, an osculating or a generalized ruled surfaces family in which $\gamma(s)$ its a common geodesic, asymptotic or line of curvature directrix curve respectively.*

Remark 3.10. *According to the previous discussion, the ruled surfaces family parameterized by (18) whose common directrix is a characteristic curve is one of the following:*

1. *A rectifying ruled surfaces family $X_{rec}(s,v)$ (20) where the common directrix is a geodesic.*
2. *An osculating ruled surfaces family $X_{osc}(s,v)$ (21) where the common directrix is an asymptotic.*
3. *A generalized ruled surfaces family $X_{gen}(s,v)$ (18) where the common directrix is a line of curvature.*

Determining the type of a unit director vector field $D(s)$ is enough to design ruled surfaces family with a common geodesic or asymptotic directrix curve, this is the subject of the following proposition which its proof is straightforward based on the main Theorem (3.1).

Proposition 3.11. *Let $X(s,v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve and $D(s)$ is the unit director vector, then $\gamma(s)$ is a common*

1. *geodesic directrix curve on $X(s,v)$, if and only if $D(s)$ is a rectifying director vector.*
2. *asymptotic directrix curve on $X(s,v)$, if and only if $D(s)$ is an osculating director vector.*

Finally, this section ended with the following three theorems which clarify that there is an equivalent between the type of ruled surfaces family, directrix curve, angular functions, and the unit director vector field $D(s)$.

Theorem 3.12. *Let $X(s,v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$X(s,v)$ is a rectifying ruled surface family $X_{rec}(s,v)$,*
2. *$\gamma(s)$ is a common geodesic directrix curve,*
3. $\cos\phi(s) = 0$,
4. *$D(s)$ is a rectifying director vector.*

Theorem 3.13. *Let $X(s,v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$X(s,v)$ is an osculating ruled surface family $X_{osc}(s,v)$,*
2. *$\gamma(s)$ is a common asymptotic directrix curve,*
3. $\sin\phi(s) = 0$,
4. *$D(s)$ is an osculating director vector.*

Theorem 3.14. *Let $X(s,v)$ be a ruled surfaces family (18), and let $\gamma(s)$ be its common directrix curve with non-vanishing curvature and torsion. Then the followings are equivalent:*

1. *$\gamma(s)$ is a common line of curvature directrix curve,*
2. $\phi(s) = -\int_0^s \tau ds$,
3. *$X(s,v)$ is a generalized ruled surface family $X_{gen}(s,v)$.*

4. Classification of developable ruled surfaces family with common characteristic curve

In this section, we give a classification of the developable ruled surfaces families whose directrix is characteristic curve. In particular, we study under what conditions the three different families $X_{rec}(s,v)$, $X_{osc}(s,v)$ and $X_{gen}(s,v)$ satisfying (30) are to be developable. Firstly, the developability condition (15) can be explicitly written for a ruled surfaces family (18) as the following

Lemma 4.1. *A ruled surfaces family (18) is a developable, if and only if the following condition is satisfied,*

$$\sin\theta(s)(\frac{d\phi}{ds}+\tau(s))-\kappa(s)\sin\phi(s)\cos\theta(s)=0. \tag{33}$$

Proof. Using the developability condition (15), a ruled surfaces family (18) is developable, if and only if $det\langle\gamma', D, D'\rangle = 0$, since $\gamma'(s) = t$ and $D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))$, by taking the derivative of $D(s)$ and using the Frenet-Serret formula of $\gamma(s)$, we get $D'(s) = -\sin\theta(s)[\kappa(s)\cos\phi + \frac{d\theta}{ds}]t(s) + [\cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) - \sin\theta(s)\sin\phi(s)(\frac{d\phi}{ds}+\tau)]n + [\sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} + \sin\theta(s)\cos\phi(\frac{d\phi}{ds}+\tau)]b$. Then we obtain $det\langle\gamma', D, D'\rangle = \sin\theta(s)(\frac{d\phi}{ds}+\tau(s)) - \kappa(s)\sin\phi(s)\cos\theta(s)$, this completes the proof of the lemma. □

In [11], the author gave the above condition (33) by using a different technique. The following definition and lemma are needed to study the developability of rectifying ruled surfaces family $X_{rec}(s,v)$ (20).

Definition 4.2. *For a rectifying vector* $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ *defined along a unit speed regular curve* $\gamma(s)$*, we define a scalar function* $H(s) = \kappa(s)\cos\theta(s) - \tau(s)\sin\theta(s)$ *and we call it Darboux function of* $D(s)$ *along* $\gamma(s)$.

Lemma 4.3. *Suppose that* $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ *is a rectifying vector field defined along a unit speed regular curve* $\gamma(s)$*, then* $D(s)$ *is a unit Darboux vector, if and only if* $H(s)$ *vanishes.*

Proof. Let $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ be a unit Darboux vector. From (3), $\cos\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}$, $\sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}$. This implies that $H(s) = \kappa\cos\theta - \tau\sin\theta = 0$, and vice versa. □

The following theorem shows the developability condition for the $X_{rec}(s,v)$, $X_{osc}(s,v)$ and $X_{gen}(s,v)$.

Theorem 4.4. *Let* $X(s,v)$ *be the ruled surfaces family of type* $X_{rec}(s,v)$, $X_{osc}(s,v)$ *or* $X_{gen}(s,v)$ *satisfying (30). Then,*

1. *A rectifying ruled surfaces family* $X_{rec}(s,v)$ *is developable, if and only if the director vector is a unit Darboux vector.*
2. *An osculating ruled surfaces family* $X_{osc}(s,v)$ *is developable, if and only if the directrix curve is a plane curve .*
3. *A generalized ruled surfaces family* $X_{gen}(s,v)$ *satisfying (30) is developable, if and only if* $\cos\theta(s) = 0$.

Proof. Using Lemma (2.4),

1. A ruled surfaces family (18) with a common directrix curve is of type $X_{rec}(s,v)$ (20), if and only if $\cos\phi(s) = 0$. Then the developability condition (33) turns into

$$\tau(s)\sin\theta(s)-\kappa(s)\cos\theta(s)=0. \tag{34}$$

   Using lemma (4.3), $X_{rec}(s,v)$ is developable, if and only if the director vector is a unit Darboux vector.

2. A ruled surfaces family (18) with a common directrix is of type $X_{osc}(s, v)$ (21), if and only if $\sin\phi(s) = 0$, this implies that the developability condition (33) becomes

$$\sin\theta(s)\tau(s) = 0. \tag{35}$$

Since $\sin\theta(s) \neq 0$ (regularity condition), then $X_{osc}(s, v)$ is developable if and only if $\tau = 0$, that is the directrix curve is a plane curve.

3. Let $X(s, v)$ be a ruled surfaces family of type $X_{gen}(s, v)$ that satisfying (30). After substitution, the developabity condition (33) can be expressed as

$$\kappa(s)\sin\phi(s)\cos\theta(s) = 0. \tag{36}$$

Since $\sin\phi(s) \neq 0$, and $\kappa(s) \neq 0$ (regular curve), then $X_{gen}(s, v)$ (18) is developable, if and only if $\cos\theta(s) = 0$.

□

Based on the above theorem and its proof, we conclude the following equivalent theorem.

Theorem 4.5. *Let $X(s, v)$ be a ruled surfaces family of type $X_{rec}(s, v)$, $X_{osc}(s, v)$, or $X_{gen}(s, v)$ satisfying (30) . Then,*

1. *Darboux ruled surface $X_{Dar}(s, v)$ (29) is the only member of the family of rectifying ruled surfaces $X_{rec}(s, v)$ (20), that is developable.*
2. *The family of osculating ruled surfaces $X_{osc}(s, v)$ (21) that has plane directrix curve is developable. .*
3. *Normal ruled surface $X_{nor}(s, v)$ (22) is the only member of the family of generalized ruled surfaces $X_{gen}(s, v)$ satisfying (30), that is developable. .*

Corollary 4.6. *There is no family of developable ruled surfaces having parametrization (18) whose common directrix curve is geodesic or line of curvature.*

Remark 4.7. *It is important to note that this result is restricted to parameterization (18) that uses the angular functions as coefficients. When other different parameterizations are used, the family of developable ruled surfaces whose common directrix curve is geodesic [14] or line of curvature [7] can be constructed by using marching-scale functions instead of angular functions in (18).*

Theorem 4.8. *A family of developable surfaces whose common directrix is characteristic curve is an osculating ruled surfaces family $X_{osc}(s, v)$ (21) whose common directrix is asymptotic plane curve.*

Remark 4.9. *When the common asymptotic of the osculating ruled surfaces family $X_{osc}(s, v)$ (21) is a plane curve, then this family is the osculating planes of asymptotic plane curve.*

Theorem 4.10. *Let $X(s, v)$ be a developable ruled surfaces family whose common directrix $\gamma(s)$ is a characteristic curve. Then, the following statements hold:*

1. *$X(s, v)$ is neither generalized ruled surfaces family $X_{gen}(s, v)$ nor a rectifying ruled surfaces family $X_{rec}(s, v)$.*
2. *$X(s, v)$ is an osculating ruled surfaces family $X_{osc}(s, v)$.*
3. *The directrix curve $\gamma(s)$ is a plane curve .*
4. *The common directrix curve $\gamma(s)$ is asymptotic.*
5. *$X_{osc}(s, v)$ is the family of osculating planes of $\gamma(s)$.*

Finally, we review a known result which characterizes $X_{pn}(s, v)$ and $X_b(s, v)$ with developability property.

Corollary 4.11. *Let X be a ruled surface member of type $X_{pn}(s, v)$ (27) or $X_b(s, v)$ (28) then $X_{pn}(s, v)$ and $X_b(s, v)$ are developable if and only if the directrix curve is a plane curve.*

5. Conclusion and future works

In this study, using parameterization (18), the family of ruled surfaces has been classified based on its common characteristic directrix curve into three families: Rectifying ruled surfaces whose common directrix is a geodesic, osculating ruled surfaces whose common directrix is an asymptotic, or generalized ruled surfaces whose common directrix is a line of curvature. The three families converted into one family under the developability condition. For future works, we will investigate how to extend these results to other ambient spaces with different dimensions and using other frames.

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