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Generalized Turán-type Inequalities for Polar Derivative of a Polynomial

Kshetrimayum Krishnadas*, Thangjam Birkramjit Singh and Barchand Chanam

Abstract
Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$. We obtain an improvement and a generalization of an inequality in polar derivative proved by Somsuwan and Nakprasit [1]. Further, we also extend a result proved by Chanam and Dewan [2] to its polar version. Besides, our results are also found to generalize and improve some known inequalities.

Keywords: *Turán-type inequality; polynomial; polar derivative; maximum modulus*

AMS Subject Classification (2020): *Primary: 30C15: 30C10 ; Secondary: 30A10.*

****Corresponding author***

1. Introduction and statement of results

The study of geometric relationship between the maximum moduli of a complex polynomial and its derivative on the same circle or different circles by taking into account the position of zeros of the polynomial inside or outside the same or a different circle has been drawing great interest among researchers for many decades. One of the pioneering works in this area is due to S. Bernstein.

If $P(z)$ is a polynomial of degree n, Bernstein [3] proved

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality is the famous Bernstein's Inequality. Equality holds in (1.1) if all zeros of $P(z)$ are found at the origin.

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If we restrict ourselves to the class of polynomials $P(z)$ of degree n having no zero in $|z| < 1$, then inequality (1.1) can be refined and substituted by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later proved by Lax [4]. The result is sharp and equality holds for the polynomial $P(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

On the other hand, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, Turán [5] proved

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.3), which is often referred to as Turán's Inequality, is best possible and equality occurs if $P(z)$ has all its zeros on $|z| = 1$.

It was asked by Professor R.P. Boas that if $P(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can $\left\{ \max\limits_{|z|=1} |P'(z)| / \max\limits_{|z|=1} |P(z)| \right\}$ be. A partial answer to this problem was given by Malik [6], who proved that if $P(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The result is sharp and equality is attained for $P(z) = (z + k)^n$. Whereas, for the polynomial $P(z)$ having all its zeros in $|z| \leq k$, $k \leq 1$, by applying the above inequality (1.4) to the polynomial $q(z)$, where $q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, Malik [6] further obtained a generalization of (1.3) as

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Inequality (1.5) is sharp and equality holds for $P(z) = (z + k)^n$.

In inequalities (1.2) and (1.3) the boundaries of zero-free regions and the circle on which the estimates of $P(z)$ and its derivative are compared is the unit circle, which is not the case in inequalities (1.4) and (1.5) where the two circles are not same. It is of interest to obtain generalization of the above inequalities by considering the maximum moduli of the polynomial and its derivative on different circles other than the unit circle. In this direction the following result was proved by Aziz and Zargar [7].

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for real numbers r and R such that $rR \geq k^2$ and $r \leq R$,*

$$\max_{|z|=R} |P'(z)| \geq \frac{n\,(R+k)^{n-1}}{(r+k)^n} \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.6}$$

Equality in (1.6) holds for the polynomial $P(z) = (z + k)^n$.

Chanam and Dewan [2] generalized and improved Theorem 1.1 by involving certain coefficients of $P(z)$. They proved

Theorem 1.2. *Let $P(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $a_0 \neq 0$ and $1 \leq \mu < n$, be a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, where $k > 0$, then for $rR \geq k^2$ and $r \leq R$,*

$$\begin{aligned} \max_{|z|=R} |P'(z)| \geq n &\left\{ \frac{n|a_n|R^{\mu}k^{\mu-1} + \mu|a_{n-\mu}|R^{\mu-1}}{n|a_n|R^{\mu+1}k^{\mu-1} + n|a_n|k^{2\mu} + \mu|a_{n-\mu}|\left(Rk^{\mu-1} + R^{\mu}\right)} \right\} \\ &\times \left(\frac{R+k}{r+k}\right)^n \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \end{aligned} \tag{1.7}$$

Let $P(z)$ be a polynomial of degree n and let α be any complex number. Then, the polar derivative of $P(z)$ with respect to α, denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z). \tag{1.8}$$

The polynomial $D_\alpha P(z)$ is of degree at most $(n-1)$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

The following result, proved by Aziz and Rather [8], generalizes and extends Turán's inequality (1.3) to its polar version.

Theorem 1.3. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n(|\alpha|-k)}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.9}$$

Further, Dewan and Upadhye [9] improved Theorem 1.3 by involving $\min\limits_{|z|=k} |P(z)|$. They proved

Theorem 1.4. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n(|\alpha|-k) \left\{ \frac{1}{1+k^n} \max_{|z|=1} |P(z)| + \frac{1}{2k^n} \left(\frac{k^n-1}{k^n+1} \right) \min_{|z|=k} |P(z)| \right\}. \tag{1.10}$$

Nakprasit and Somsuwan [1] generalized Theorem 1.4 by proving the following result.

Theorem 1.5. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$ and $1 \leq R \leq k$,*

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z)| \geq nR^{n-1}(|\alpha|-k) &\left[\frac{R^n}{R^n+k^n} \left(\frac{k^\mu+1}{k^\mu+R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=1} |P(z)| \right. \\ &\left. + \left\{ \frac{k^n}{R^n+k^n} \left(1 - \left(\frac{k^\mu+1}{k^\mu+R^\mu} \right)^{\frac{n}{\mu}} \right) + \frac{1}{2k^n} \left(\frac{k^n-R^n}{k^n+R^n} \right) \right\} \min_{|z|=k} |P(z)| \right]. \end{aligned} \tag{1.11}$$

In this paper, we first obtain an improvement and a generalization of Theorem 1.5. Theorem 1.5 is generalized in the sense that inequality (1.11) is extended to circles with smaller radii, viz., for $0 < r \leq 1$ when the estimate of $\max |D_\alpha P(z)|$ is considered. More precisely, we prove

Theorem 1.6. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$ and $0 < r \leq 1 \leq R \leq k$,*

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z)| \geq nR^{n-1}(|\alpha|-k) &\left[\frac{R^n}{k^n+R^n} B \max_{|z|=r} |P(z)| \right. \\ &\left. + \left\{ \frac{k^n}{k^n+R^n}(1-B) + \frac{1}{2k^n} \left(\frac{k^n-R^n}{k^n+R^n} \right) \right\} \min_{|z|=k} |P(z)| \right], \end{aligned} \tag{1.12}$$

where

$$B = \exp\left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} (k^{\mu+1}t^\mu + k^{2\mu} t)} dt \right\} \tag{1.13}$$

and $m = \min\limits_{|z|=k} |P(z)|$.

The following result is obtained by taking $r = 1$ in Theorem 1.6.

Corollary 1.1. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$ and $1 \leq R \leq k$,*

$$\begin{aligned}\max_{|z|=R} |D_\alpha P(z)| \geq nR^{n-1} (|\alpha| - k) \Bigg[&\frac{R^n}{k^n + R^n} B_1 \max_{|z|=1} |P(z)| \\ &+ \left\{ \frac{k^n}{k^n + R^n} (1 - B_1) + \frac{1}{2k^n} \left(\frac{k^n - R^n}{k^n + R^n} \right) \right\} \min_{|z|=k} |P(z)| \Bigg], \end{aligned} \tag{1.14}$$

where

$$B_1 = \left\{ -n \int_1^R \frac{\dfrac{\mu}{n} \dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \dfrac{\mu}{n} \dfrac{\frac{|a_{n-\mu}|}{k^{2\mu}}}{|a_n| - \frac{m}{k^n}} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \tag{1.15}$$

and $m = \min\limits_{|z|=k} |P(z)|$.

Remark 1.1. Corollary 1.1 is an improvement of Theorem 1.5. It is sufficient to show that the bound given by inequality (1.14) is bigger than the bound given by inequality (1.11) concerning the estimate of $\max |D_\alpha P(z)|$, i.e.,

$$\begin{aligned} &\left[\frac{R^n}{k^n + R^n} B_1 \max_{|z|=1} |P(z)| + \left\{ \frac{k^n}{k^n + R^n} (1 - B_1) + \frac{1}{2k^n} \left(\frac{k^n - R^n}{k^n + R^n} \right) \right\} \min_{|z|=k} |P(z)| \right] \\ &\geq \left[\frac{R^n}{R^n + k^n} \left(\frac{k^\mu + 1}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=1} |P(z)| + \left\{ \frac{k^n}{R^n + k^n} \left(1 - \left(\frac{k^\mu + 1}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \right) + \frac{1}{2k^n} \left(\frac{k^n - R^n}{k^n + R^n} \right) \right\} \min_{|z|=k} |P(z)| \right]. \end{aligned}$$

From (2.1) of Lemma 2.2, we have

$$\max_{|z|=1} |P(z)| \geq k^n \min_{|z|=k} |P(z)|.$$

Since $R \geq 1$, it follows that

$$R^n \max_{|z|=1} |P(z)| - k^n \min_{|z|=k} |P(z)| \geq 0. \tag{1.16}$$

Putting $r = 1$ in (2.9) of Lemma 2.5, we get

$$B_1 \geq \left(\frac{k^\mu + 1}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}}, \tag{1.17}$$

where B_1 is given by (1.15).
In view of inequality (1.17), it is sufficient to show that the function f such that

$$f(x) = R^n \max_{|z|=1} |P(z)| x + k^n (1 - x) \min_{|z|=k} |P(z)|$$

is a non-decreasing function of x. Now as

$$\begin{aligned} f'(x) &= R^n \max_{|z|=1} |P(z)| - k^n \min_{|z|=k} |P(z)| \\ &\geq 0 \quad (by \ \ (1.16)), \end{aligned}$$

f is a non-decreasing function of x, which proves our claim.

Further, for $r = R = 1$ in Theorem 1.6, we have the following result.

Corollary 1.2. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n(|\alpha| - k) \left\{ \frac{1}{k^n + 1} \max_{|z|=1} |P(z)| + \frac{1}{2k^n} \left(\frac{k^n - 1}{k^n + 1} \right) \min_{|z|=k} |P(z)| \right\}. \tag{1.18}$$

Remark 1.2. It is clear from Corollary 1.1 that Theorem 1.6 is a generalization of Theorem 1.4, as taking $\mu = 1$ along with $r = R = 1$, inequality (1.12) of Theorem 1.6 reduces to (1.10) of Theorem 1.4.

Dividing both sides of (1.12) by $|\alpha|$ and letting $|\alpha| \to \infty$ and putting $R = k$, we have the next result.

Corollary 1.3. *If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for $0 < r \le k$,*

$$\max_{|z|=k} |P'(z)| \ge \frac{nk^{n-1}}{2} \left\{ B \max_{|z|=r} |P(z)| + (1 - B) \min_{|z|=k} |P(z)| \right\}, \tag{1.19}$$

where B is given by (1.13).

Remark 1.3. In particular, if we let $r = k = 1$ and $\mu = 1$, (1.19) reduces to Turán's inequality (1.3).

Next, we extend Theorem 1.2 due to Chanam and Dewan [2] to its polar version in which the assumption $a_0 \neq 0$ in the constant term of the polynomial $P(z)$ is also dropped. The result also improves as well as generalizes other well known inequalities.

Theorem 1.7. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then for every real or complex number α with $\frac{|\alpha|}{R} \ge A_{\mu,n}$ and for $rR \ge k^2$ and $r \le R$,*

$$\max_{|z|=R} |D_\alpha P(z)| + mn \ge \frac{n}{1 + A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n} \right) \left(\frac{R + k}{r + k} \right)^n \left[\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right], \tag{1.20}$$

where $m = \min_{|z|=k} |P(z)|$ *and*

$$A_{\mu,n} = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|Rk^{\mu-1}}{\mu|a_{n-\mu}|R^\mu + n|a_n|R^{\mu+1}k^{\mu-1}}. \tag{1.21}$$

Dividing on both sides of (1.20) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following result.

Corollary 1.4. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then for $rR \ge k^2$ and $r \le R$,*

$$\max_{|z|=R} |P'(z)| \ge \frac{n}{R(1 + A_{\mu,n})} \left(\frac{R + k}{r + k} \right)^n \left[\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right], \tag{1.22}$$

where $A_{\mu,n}$ is given by (1.21).

Remark 1.4. In view of Corollary 1.4, Theorem 1.7 is the polar derivative version of Theorem 1.2 in a richer form for restrictions concerning the polynomial $P(z)$, namely $a_0 \neq 0$, $\mu \neq n$ and $n \neq 1$ in the hypotheses of Theorem 1.2 have all been dropped in Theorem 1.7 and hence consequently in Corollary 1.4. In other words, Corollary 1.4 is a better version of Theorem 1.2.

Further, taking $k = R = r = 1$ in Corollary 1.4, we have the following result.

Corollary 1.5. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le 1$, then*

$$\max_{|z|=R} |P'(z)| \ge \frac{n}{2} \left\{ \max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.23}$$

Inequality (1.23) verifies that Corollary 1.4 is a generalization as well as an improvement of inequality (1.3) due to Turán [5].

Remark 1.5. Corollary 1.4 is also an improvement and a generalization of Theorem 1.1 as explained by Chanam and Dewan [2, Remark 2].

2. Lemmas

We need the following lemmas to prove our theorems.

The following lemma was proved by Gardner et al.[10].

Lemma 2.1. *If $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, $P(z) \neq 0$ in $|z| < k$, $k > 0$, then $|P(z)| \geq m$ for $|z| \leq k$, where $m = \min\limits_{|z|=k} |P(z)|$.*

Lemma 2.2. *If $P(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P(z)| \geq k^n \min_{|z|=k} |P(z)|. \tag{2.1}$$

Proof. Let $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = \overline{a}_0 z^n + \sum\limits_{\nu=\mu}^{n} \overline{a}_\nu z^{n-\nu}$, $1 \leq \mu \leq n$. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, therefore $q(z)$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \leq 1$. Let $Q(z) = q\left(\frac{z}{k^2}\right) = \frac{\overline{a}_0}{k^{2n}} z^n + \sum\limits_{\nu=\mu}^{n} \frac{\overline{a}_\nu}{k^{2(n-\nu)}} z^{n-\nu} = \overline{a}_n + \sum\limits_{\nu=\mu}^{n} \frac{\overline{a}_{n-\nu}}{k^{2\nu}} z^\nu$, then $Q(z) \neq 0$ in $|z| < k$, $k \geq 1$.
Therefore, by applying Lemma 2.1 to $Q(z)$, we get for $|z| = k$

$$\begin{aligned}
|Q(z)| &\geq \min_{|z|=k} |Q(z)| \\
&= \min_{|z|=k} \left| q\left(\frac{z}{k^2}\right) \right| \\
&= \min_{|z|=k} \left| \left(\frac{z}{k^2}\right)^n \overline{P\left(\frac{1}{\overline{z}/k^2}\right)} \right| \\
&= \frac{1}{k^n} \min_{|z|=k} \left| P\left(\frac{k^2}{\overline{z}}\right) \right| \\
&= \frac{1}{k^n} \min_{|z|=k} |P(z)|.
\end{aligned} \tag{2.2}$$

Now as $1 \leq k$ and hence in particular inequality (2.2) gives for $|z| = 1$

$$|Q(z)| \geq \frac{1}{k^n} \min_{|z|=k} |P(z)|, \quad \text{from which it is implied that}$$

$$\max_{|z|=1} |Q(z)| \geq \frac{1}{k^n} \min_{|z|=k} |P(z)|, \quad \text{which is equivalent to}$$

$$\max_{|z|=1} \left| q\left(\frac{z}{k^2}\right) \right| \geq \frac{1}{k^n} \min_{|z|=k} |P(z)|,$$

which implies

$$\max_{|z|=1} \left| \left(\frac{z}{k^2}\right)^n \overline{P\left(\frac{1}{\overline{z}/k^2}\right)} \right| \geq \frac{1}{k^n} \min_{|z|=k} |P(z)|,$$

which is equivalent to

$$\frac{1}{k^{2n}} \max_{|z|=1} \left| P\left(\frac{k^2}{\overline{z}}\right) \right| \geq \frac{1}{k^n} \min_{|z|=k} |P(z)|. \tag{2.3}$$

Since $k \geq 1$, it is obvious that $k^2 \geq k \geq 1$ and hence by Maximum Modulus Principle [11]

$$\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|, \quad \text{which is equivalent to}$$
$$\frac{1}{(k^2)^n}\max_{|z|=k^2}\left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \frac{1}{(k^2)^n}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|, \quad \text{which simplifies to}$$
$$\frac{1}{(k^n)^2}\max_{|z|=1}|P(z)| \geq \frac{1}{k^{2n}}\max_{|z|=1}\left|P\left(\frac{k^2}{\overline{z}}\right)\right|. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$\frac{1}{(k^n)^2}\max_{|z|=1}|P(z)| \geq \frac{1}{k^n}\min_{|z|=k}|P(z)|. \tag{2.5}$$

Hence,

$$\max_{|z|=1}|P(z)| \geq k^n \min_{|z|=k}|P(z)|.$$

□

Lemma 2.3. *If $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\max_{|z|=r}|P(z)| \geq B' \max_{|z|=R}|P(z)| + (1 - B') \min_{|z|=k}|P(z)|, \tag{2.6}$$

where

$$B' = \exp\left\{-n\int_r^R \frac{\dfrac{\mu}{n}\dfrac{|a_\mu|}{|a_0| - m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \dfrac{\mu}{n}\dfrac{|a_\mu|}{|a_0| - m}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right)}dt\right\} \tag{2.7}$$

and $m = \min\limits_{|z|=k}|P(z)|$. Equality holds in (2.6) for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Lemma 2.4. *If $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\mu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, where $k > 0$, then for $0 < r \leq R \leq k$,*

$$B' \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \tag{2.8}$$

where B' is given by (2.7).

Lemma 2.3 and Lemma 2.4 are due to Chanam and Dewan [12].

Lemma 2.5. *Let $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$B \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \tag{2.9}$$

where B is given by (1.13).

Proof. Let $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$ and $Q(z) = q\left(\frac{z}{k^2}\right)$. Then, $Q(z) = \frac{\overline{a}_0}{k^{2n}} z^n + \sum\limits_{\nu=\mu}^{n} \frac{\overline{a}_\nu}{k^{2(n-\nu)}} z^{n-\nu} = \overline{a}_n + \sum\limits_{\nu=\mu}^{n} \frac{\overline{a}_{n-\nu}}{k^{2\nu}} z^\nu$, where $1 \leq \mu \leq n$. Since $P(z) \neq 0$ in $|z| < k$, $k > 0$, we have $Q(z) \neq 0$ in $|z| < k$, $k > 0$. Hence, applying Lemma 2.4 to $Q(z)$, we get

$$B \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}},$$

where B is given by (1.13). □

The next lemma is due to Qazi [13, Proof and Remark of Lemma 1].

Lemma 2.6. *If $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$|q'(z)| \geq k^{\mu+1} \frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}} |P'(z)| \quad on \quad |z| = 1 \tag{2.10}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \leq 1, \tag{2.11}$$

where $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$.

Lemma 2.7. *If $P(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for every real or complex number β with $|\beta| \geq A$,*

$$\max_{|z|=1} |D_\beta P(z)| \geq \frac{n}{1+A} (|\beta| - A) \max_{|z|=1} |P(z)|, \tag{2.12}$$

where

$$A = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}.$$

Inequality (2.12) is best possible for $\mu = 1$ and equality occurs for $P(z) = (z-k)^n$ with $|\beta| \geq A = k$.

Proof. Let $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$. Then it can be easily verified that

$$|q'(z)| = |nP(z) - zP'(z)|, \quad \text{for} \quad |z| = 1. \tag{2.13}$$

Since the polynomial $P(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, $q(z) = \overline{a}_n + \sum\limits_{\nu=\mu}^{n} \overline{a}_{n-\nu} z^\nu$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$, therefore, by applying Lemma 2.6 to $q(z)$, we have from (2.10)

$$\begin{aligned} |P'(z)| &\geq \frac{1}{k^{\mu+1}} \left(\frac{\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu-1}} + 1}{1 + \frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu+1}}} \right) |q'(z)| \\ &= \frac{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}} |q'(z)|. \end{aligned}$$

Therefore,

$$\begin{aligned}|q'(z)| &\le \frac{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}|+n|a_n|k^{\mu-1}}|P'(z)|\\ &= A|P'(z)|. \end{aligned} \tag{2.14}$$

From (2.14), we have

$$|P'(z)|+|q'(z)| \le (1+A)|q'(z)|. \tag{2.15}$$

Also, for $|z|=1$, with the help of (2.13), we have

$$\begin{aligned} n|P(z)| &= |nP(z)-zP'(z)+zP'(z)|\\ &\le |nP(z)-zP'(z)|+|P'(z)|\\ &= |q'(z)|+|P'(z)|. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we get

$$n|P(z)| \le (1+A)|P'(z)|$$

i.e.,

$$|P'(z)| \ge \frac{n}{1+A}|P(z)|, \quad \text{for} \quad |z|=1. \tag{2.17}$$

For every real or complex number β, by definition, we have

$$D_\beta P(z) = nP(z)+(\beta-z)P'(z),$$

from which for $|z|=1$, we have

$$\begin{aligned} |D_\beta P(z)| &\ge ||\beta||P'(z)|-|nP(z)-zP'(z)||\\ &= ||\beta||P'(z)|-|q'(z)|| \quad \text{(by (2.13))}. \end{aligned} \tag{2.18}$$

Further, by (2.14)

$$\begin{aligned} |\beta||P'(z)|-|q'(z)| &\ge |\beta||P'(z)|-A|P'(z)|\\ &= (|\beta|-A)|P'(z)|, \end{aligned} \tag{2.19}$$

which is non-negative, since $|\beta| \ge A$.
Combining (2.18) and (2.19), we get

$$|D_\beta P(z)| \ge (|\beta|-A)\,|P'(z)|,$$

which on using (2.17), gives

$$|D_\beta P(z)| \ge (|\beta|-A)\,\frac{n}{1+A}|P(z)|.$$

□

The following lemma is due to Aziz and Zargar [7].

Lemma 2.8. *If $P(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k>0$, then for $rR\ge k^2$ and $r\le R$, we have for $|z|=1$,*

$$|P(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n |P(rz)|. \tag{2.20}$$

Equality holds in (2.20) for $P(z)=(z+k)^n$.

3. Proof of the theorems

Proof of Theorem 1.6. Let $F(z) = P(Rz)$. Then $F(z)$ has all its zeros in the closed disk $|z| \leq \frac{k}{R}$, $\frac{k}{R} \geq 1$. Applying Theorem 1.5 to $F(z)$, we have

$$\max_{|z|=1} |D_{\alpha/R} F(z)| \geq n\left(\frac{|\alpha|-k}{R}\right)\left[\frac{1}{1+\frac{k^n}{R^n}} \max_{|z|=1} |F(z)| + \frac{1}{2\frac{k^n}{R^n}}\left(\frac{\frac{k^n}{R^n}-1}{\frac{k^n}{R^n}+1}\right) \min_{|z|=\frac{k}{R}} |F(z)|\right]$$
$$= n\left(|\alpha|-k\right) R^{n-1}\left[\frac{1}{k^n+R^n} \max_{|z|=1} |F(z)| + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right) \min_{|z|=\frac{k}{R}} |F(z)|\right]. \tag{3.1}$$

Using the relations,

$$\max_{|z|=1} |D_{\alpha/R} F(z)| = \max_{|z|=R} |D_\alpha P(z)|,$$
$$\max_{|z|=1} |F(z)| = \max_{|z|=R} |P(z)|$$
$$\text{and} \quad \min_{|z|=\frac{k}{R}} |F(z)| = \min_{|z|=k} |P(z)|$$

in inequality (3.1), we get

$$\max_{|z|=R} |D_\alpha P(z)| \geq nR^{n-1}\left(|\alpha|-k\right)\left[\frac{1}{k^n+R^n} \max_{|z|=R} |P(z)| + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right) \min_{|z|=k} |P(z)|\right]. \tag{3.2}$$

Let $q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$ and $Q(z) = q\left(\frac{z}{k^2}\right)$, then

$$Q(z) = \frac{z^n}{k^{2n}} \overline{P\left(\frac{k^2}{\overline{z}}\right)}. \tag{3.3}$$

Therefore, $q(z)$ has no zero in $|z| < \frac{1}{k}$ and $Q(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$. Thus, applying Lemma 2.3 to $Q(z)$, we have

$$\max_{|z|=r} |Q(z)| \geq B \max_{|z|=R} |Q(z)| + (1-B) \min_{|z|=k} |Q(z)|, \tag{3.4}$$

where B is given by (1.13).
From (3.3) we have for $r > 0$

$$\max_{|z|=r} |Q(z)| = \frac{r^n}{k^{2n}} \max_{|z|=r} \left|P\left(\frac{k^2}{\overline{z}}\right)\right|. \tag{3.5}$$

Since $0 < r \leq 1 \leq R \leq k$, we have by Maximum Modulus Principle [11],

$$\max_{|z|=k^2} \left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \max_{|z|=1} \left|P\left(\frac{k^2}{\overline{z}}\right)\right| \quad (\because\ k^2 \geq 1)$$
$$i.e. \quad \frac{1}{k^{2n}} \max_{|z|=k^2} \left|P\left(\frac{k^2}{\overline{z}}\right)\right| \geq \frac{1}{k^{2n}} \max_{|z|=1} \left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$i.e. \quad \frac{1}{k^{2n}} \max_{|z|=1} |P(z)| \geq \frac{1}{k^{2n}} \max_{|z|=1} \left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$\geq \frac{r^n}{k^{2n}} \max_{|z|=1} \left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$\geq \frac{r^n}{k^{2n}} \max_{|z|=r} \left|P\left(\frac{k^2}{\overline{z}}\right)\right|$$
$$= \max_{|z|=r} |Q(z)| \quad (by\ (3.5)).$$

Hence

$$\begin{aligned}\max_{|z|=r}|Q(z)| &\le \frac{1}{k^{2n}}\max_{|z|=1}|P(z)| \\ &\le \frac{1}{k^{2n}}\max_{|z|=R}|P(z)| \quad (\because R\ge 1).\end{aligned} \tag{3.6}$$

Again from (3.3) we have

$$\begin{aligned}\max_{|z|=R}|Q(z)| &= \max_{|z|=R}\left|\frac{z^n}{k^{2n}}\overline{P\left(\frac{k^2}{\overline{z}}\right)}\right| \\ &= \frac{R^n}{k^{2n}}\max_{|z|=R}\left|\overline{P\left(\frac{k^2}{\overline{z}}\right)}\right| \\ &= \frac{R^n}{k^{2n}}\max_{|z|=k^2/R}|P(z)| \\ &\ge \frac{R^n}{k^{2n}}\max_{|z|=r}|P(z)| \quad \left(\because \frac{k^2}{R}\ge r\right).\end{aligned} \tag{3.7}$$

Also, we know that

$$\min_{|z|=k}|Q(z)| = \frac{1}{k^n}\min_{|z|=k}|P(z)|. \tag{3.8}$$

Using (3.6), (3.7) and (3.8) in inequality (3.4), we get

$$\frac{1}{k^{2n}}\max_{|z|=R}|P(z)| \ge \frac{R^n}{k^{2n}}B\max_{|z|=r}|P(z)| + (1-B)\frac{1}{k^n}\min_{|z|=k}|P(z)|$$

i.e.,

$$\max_{|z|=R}|P(z)| \ge R^nB\max_{|z|=r}|P(z)| + k^n(1-B)\min_{|z|=k}|P(z)|. \tag{3.9}$$

Combining inequalities (3.2) and (3.9), we obtain

$$\begin{aligned}\max_{|z|=R}|D_\alpha P(z)| \ge nR^{n-1}(|\alpha|-k)\Bigg[&\frac{1}{k^n+R^n}\left(R^nB\max_{|z|=r}|P(z)| + k^n(1-B)\min_{|z|=k}|P(z)|\right) \\ &+ \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\min_{|z|=k}|P(z)|\Bigg],\end{aligned}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=R}|D_\alpha P(z)| \ge nR^{n-1}(|\alpha|-k)\Bigg[&\frac{R^n}{k^n+R^n}B\max_{|z|=r}|P(z)| \\ &+ \left\{\frac{k^n}{k^n+R^n}(1-B) + \frac{1}{2k^n}\left(\frac{k^n-R^n}{k^n+R^n}\right)\right\}\min_{|z|=k}|P(z)|\Bigg].\end{aligned}$$

This completes the proof of Theorem 1.6. □

Proof of Theorem 1.7. Let $m = \min_{|z|=k}|P(z)|$, then $m \le |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| \le k$, $k > 0$, therefore, for every complex number λ with $|\lambda| < 1$, it follows from Rouche's Theorem that for $m > 0$, the polynomial $G(z) = P(z) + \lambda m$ has all its zeros in $|z| \le k$, $k > 0$.
Let

$$\begin{aligned}H(z) &= G(Rz) \\ &= P(Rz) + \lambda m \\ &= a_nR^nz^n + a_{n-\mu}R^{n-\mu}z^{n-\mu} + a_{n-\mu-1}R^{n-\mu-1}z^{n-\mu-1} + \ldots + a_1Rz + a_0 + \lambda m.\end{aligned}$$

Therefore, $H(z)$ has all its zeros in $|z| \leq \frac{k}{R}, \frac{k}{R} \leq 1$. Hence applying Lemma 2.7 to $H(z)$, we get from (2.12)

$$\max_{|z|=1} \left|D_{\alpha/R}H(z)\right| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=1} |H(z)|, \tag{3.10}$$

where $A_{\mu,n}$ is given by (1.21). Therefore,

$$\max_{|z|=1} \left|D_{\alpha/R}G(Rz)\right| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=1} |G(Rz)|,$$

which is equivalent to

$$\max_{|z|=R} |D_\alpha G(z)| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \max_{|z|=R} |G(z)|. \tag{3.11}$$

Applying (2.20) of Lemma 2.8 to $G(z)$, we have

$$\max_{|z|=R} |G(z)| \geq \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |G(z)| \quad \text{for } r \leq R \text{ and } rR \geq k^2. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned} &\max_{|z|=R} |D_\alpha G(z)| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |G(z)| \\ i.e. \quad &\max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |P(z) + \lambda m| \\ &\qquad \text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned} \tag{3.13}$$

Let z_0 on the circle $|z| = r$ be such that $\max_{|z|=r} |P(z)| = |P(z_0)|$. Then, in particular,

$$\max_{|z|=r} |P(z) + \lambda m| \geq |P(z_0) + \lambda m|. \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n |P(z_0) + \lambda m| \quad \text{for } r \leq R \text{ and } rR \geq k^2.$$

Choosing the argument of λ on the right hand side of (3.14) such that $|P(z_0) + \lambda m| = |P(z_0)| + |\lambda| m$, we get

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z) + \lambda mn| \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \left\{\max_{|z|=r} |P(z)| + |\lambda| \min_{|z|=k} |P(z)|\right\} \\ \text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned} \tag{3.15}$$

Using the simple fact that

$$|D_\alpha P(z) + \lambda mn| \leq |D_\alpha P(z)| + |\lambda| mn$$

in (3.15) and letting $|\lambda| \to 1$, we get

$$\begin{aligned} \max_{|z|=R} |D_\alpha P(z)| + mn \geq \frac{n}{1+A_{\mu,n}} \left(\frac{|\alpha|}{R} - A_{\mu,n}\right) \left(\frac{R+k}{r+k}\right)^n \left\{\max_{|z|=r} |P(z)| + \min_{|z|=k} |P(z)|\right\} \\ \text{for } r \leq R \text{ and } rR \geq k^2. \end{aligned}$$

This completes the proof of Theorem 1.7. □

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References

[1] Somsuwan, J., Nakprasit, K. M.: *Some bounds for the polar derivative of a polynomial*. International Journal of Mathematics and Mathematical Sciences. **2018**, 5034607 (2018).

[2] Chanam, B., Dewan, K. K.: *Inequalities for a polynomial and its derivatives*. Journal of Interdisciplinary Mathematics. **11**(4), 469-478 (2008).

[3] Bernstein, S.: *Lecons sur les propriétés extrémales et la meilleure approximation desfonctions analytiques d'une variable réelle*. Gauthier Villars. Paris (1926).

[4] Lax, P. D.: *Proof of a conjecture of P. Erdös on the derivative of a polynomial*. Bull. Amer. Math. Soc. **50**, 509-513 (1944).

[5] Turán, P.: *Über die Ableitung von Polynomen*. Compositio Mathematica. **7**, 89-95 (1939).

[6] Malik, M. A.: *On the derivative of a polynomial*. Journal of the London Mathematical Society. **2**(1), 57-60 (1969).

[7] Aziz, A., Zargar, B. A.: *Inequalities for a polynomial and its derivative*. Mathematical Inequalities and Applications. **1**(4), 543-550 (1998).

[8] Aziz, A., Rather, N. A.: *A refinement of a theorem of Paul Turán concerning polynomials*. Mathematical Inequalities and Applications. **1**(2), 231-238 (1998).

[9] Dewan, K. K., Upadhye, C. M.: *Inequalities for the polar derivative of a polynomial*. Journal of Inequalities in Pure and Applied Mathematics. **9**(4), 1-9 (2008).

[10] Gardner, R. B., Govil, N. K., Musukala, S. R.: *Rate of growth of polynomials not vanishing inside a circle*. Journal of Inequalities in Pure and Applied Mathematics. **6**(2), 1-9 (2005).

[11] Pólya, G., Szegö, G.: Aufgaben and Lehratze ous der Analysis I (Problems and Theorems in Analysis I). Springer-Verlag. Berlin (1925).

[12] Chanam, B., Dewan, K. K.: *Inequalities for a polynomial and its derivatives*. J. Math. Anal. Appl. **336**, 171-179 (2007).

[13] Qazi, M. A.: *On the maximum modulus of polynomials*. Proc. Amer. Math. Soc. **115**, 337-343 (1992).

Affiliations

KSHETRIMAYUM KRISHNADAS
ADDRESS: Shaheed Bhagat Singh College (University of Delhi), Dept. of Mathematics, 110017, New Delhhi-India.
E-MAIL: kshetrimayum.krishnadas@sbs.du.ac.in
ORCID ID:0000-0001-5091-2874

THANGJAM BIRKRAMJIT SINGH
ADDRESS: National Institute of Technology Manipur, Dept. of Mathematics, 795004, Imphal-India.
E-MAIL: birkramth@gmail.com
ORCID ID:0000-0002-9061-2400

BARCHAND CHANAM
ADDRESS: National Institute of Technology Manipur, Dept. of Mathematics, 795004, Imphal-India.
E-MAIL: barchand_2004@yahoo.co.in
ORCID ID:0000-0001-6397-9066