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Initial Coefficient Estimates for a Certain Subclasses of m-Fold Symmetric Bi-Univalent Functions Involving ϕ-Pseudo-Starlike Functions Defined by Mittag-Leffler Function

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Abstract

In the present investigation, we introduce and study a certain subclasses $\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)$ and $\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)$ of analytic and m-fold symmetric bi-univalent functions involving ϕ-pseudo-starlike functions associated with Mittag-Leffler Function. We establish upper bounds for the second and third Taylor-Maclaurin coefficients for functions in each of these subclasses. Furthermore, we indicate several certain special cases for our results.

Keywords: *Analytic function; m-fold symmetric bi-univalent function; Coefficient bounds; Mittag-Leffler function; pseudo-starlike function.*
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1. Introduction and Definitions

Let $\mathscr{A}=\{f:\mathbb{U}\to\mathbb{C}: f$ is analytic in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}, f(0)=0=f^{'}(0)-1\}$ be the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all functions f univalent in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.
The Koebe on-quarter Theorem (see [8]) state that the image of $\mathbb{U}$ under every function $f(z)\in\mathscr{S}$ contains a disk of radius $1/4$. Therefore, all function $f(z)\in\mathscr{S}$ has an inverse $f^{-1}(z)$ which satisfies $f^{-1}(f(z))=z$ and $f(f^{-1}(w))=w$ $(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4})$, where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f^{-1}(z)$ and $f(z)$ are univalent in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ (see for more details [2, 17, 7, 25, 10, 20, 44, 41, 43, 46, 28, 32, 33, 34, 48, 49, 50, 51, 35, 38, 23, 37, 16, 39, 21, 42]).
For each function $f\in\mathscr{S}$, the function $h(z)=(f(z^m))^{1/m}$, $(z\in\mathbb{U},\ m\in\mathbb{N})$ is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [15] and [22]) if it has the following normalized form:

$$f(z)=z+\sum_{k=1}^{\infty}a_{mk+1}z^{mk+1}\quad(z\in\mathbb{U},m\in\mathbb{N}). \tag{1.3}$$

We denote by $\mathscr{S}_m$ the class of m-fold symmetric univalent function in $\mathbb{U}$, which are normalized by the series expansion (1.3). Also, the functions in the class $\mathscr{S}$ are one-fold symmetric.
Analogous to the concept of m-fold symmetric univalent function, the concept of m-fold symmetric bi-univalent functions has be introduced. From (1.3), Srivastava et al. [24] obtained the series expansion for f^{-1} as follows:

$$g(w)=f^{-1}(w)=w-a_{m+1}w^{m+1}+\left[(m+1)a_{m-1}^2-a_{2m+1}\right]w^{2m+1}-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]w^{3m+1}+\cdots.$$

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(1.4)

We denote by Σ_m the class of m-fold symmetric bi-univalent function in $\mathbb{U}$. We can note that for $m=1$, the formula (1.4) coincides with the formula (1.2) of the class Σ.
Recently, different researches related to this field investigated bounds for various subclasses of m-fold bi-univalent function (see [3, 4, 9, 26, 27, 24, 40, 31, 45, 29, 14, 36]).
In 1903, Mittag-Leffler [18, 19] introduced the function called Mittag-Leffler function $\mathscr{E}_\lambda$ which can be defined as

$$\mathscr{E}_\lambda=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\lambda k+1)},\quad (\lambda\in\mathbb{C},\Re(\lambda)>0),$$

for more details see [11].
Also, in 2009, Srivastava and Tomovski [30] introduced the function

$$\mathscr{E}_{\lambda,\eta}^{\delta,\tau}(z)=\sum_{k=0}^{\infty}\frac{(\delta)_{k\tau}z^k}{\Gamma(\eta+\lambda k)k!},\quad (\lambda\in\mathbb{C},\ \Re(\lambda)>0),$$

where $\lambda,\eta,\delta\in\mathbb{C}$, $\Re(\lambda)>\max\{0,\Re(\tau)-1\}$, $\Re(\tau)>0$ and $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}=\begin{cases}1, & \text{for } k=0;\\ x(x+1)\cdots(x+n-1), & \text{for } k\in\mathscr{N}.\end{cases}\tag{1.5}$$

Recently, Attiya [5] introduced and investigated a linear operator $\mathscr{R}_{\delta,\tau}^{\lambda,\eta}:\mathscr{A}\longrightarrow\mathscr{A}$ by using the Hadamard product (or convolution) and defined as follows

$$\mathscr{R}_{\delta,\tau}^{\lambda,\eta}=\mathscr{Q}_{\delta,\tau}^{\lambda,\eta}*f(z),\quad (z\in\mathbb{U}),$$

where "$*$" indicate the Hadamard product of two series and

$$\mathscr{Q}_{\delta,\tau}^{\lambda,\eta}=\frac{\Gamma(\lambda+\eta)}{(\delta)_\tau}\left(\mathscr{E}_{\lambda,\eta}^{\delta,\tau}(z)-\frac{1}{\Gamma(\eta)}\right),$$

$\lambda,\eta,\delta\in\mathbb{C}$, $\Re(\lambda)>\max\{0,\Re(\tau)-1\}$, $\Re(\tau)>0$.
By some easy computations, we conclude that

$$\mathscr{R}_{\delta,\tau}^{\lambda,\eta}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(\delta+k\tau)\Gamma(\lambda+\eta)}{\Gamma(\delta+\tau)\Gamma(\lambda k+\eta)\Gamma(k+1)}a_kz^k.$$

It is easily verified that if $f\in\mathscr{S}_m$, then we have

$$\mathscr{R}_{\delta,\tau}^{\lambda,\eta}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(\delta+(mk+1)\tau)\Gamma(\lambda+\eta)}{\Gamma(\delta+\tau)\Gamma(\lambda(mk+1)+\eta)\Gamma(mk+2)}a_{mk+1}z^{mk+1}.$$

The class $\mathfrak{L}_\phi(\varphi)$ of ϕ-pseudo- starlike functions of order $\varphi(0\le\varphi<1)$ were studied by Babalola [6] whose geometric conditions satisfy

$$\Re\left(\frac{z(f'(z))^\phi}{f(z)}\right)>\varphi.$$

He discover that every pseudo-starlike functions are Bazilevič of type $\left(1-\frac{1}{\phi}\right)$ order $\varphi^{\frac{1}{\phi}}$ and univalent in $\mathbb{U}$.

Lemma 1.1. *[8] Suppose $l(z)\in\mathscr{P}$, the class of functions which are holomorphic in $\mathbb{U}$ with $\Re(l(z))>0$, $(z\in\mathbb{U})$ and have the form $l(z)=1+l_1z+l_2z^2+l_3z^3+\cdots$, $(z\in\mathbb{U})$; then $|l_n|\le 2$ for each $n\in\mathbb{N}$.*

2. Coefficient estimates for the function class $\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)$

Definition 2.1. *A function $f\in\Sigma_m$ given by (1.3) is said to be in the class $\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)$ if it satisfies the following conditions:*

$$\left|\arg\left(1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}_{\delta,\tau}^{\lambda,\eta}f(z))']^\phi+\gamma z(\mathscr{R}_{\delta,\tau}^{\lambda,\eta}f(z))''-2\gamma-1\right]\right)\right|<\frac{\alpha\pi}{2},\tag{2.1}$$

$$\left|\arg\left(1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}_{\delta,\tau}^{\lambda,\eta}g(w))']^\phi+\gamma w(\mathscr{R}_{\delta,\tau}^{\lambda,\eta}g(w))''-2\gamma-1\right]\right)\right|<\frac{\alpha\pi}{2},\tag{2.2}$$

where $0\le\gamma\le1$, $m\in\mathbb{N}$, $\phi\ge1$, $\upsilon\in\mathbb{C}\setminus\{0\}$ and the function $g=f^{-1}$ is given by (1.4).

We have the following remarks:

Remark 2.2. *In the case of $\phi=\upsilon=1$ in Definition 2.1, we can have*

$$\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)=\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,1,1;\alpha)=\mathscr{R}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau;\alpha)\tag{2.3}$$

Studied by Wanas and Tang [47].

Remark 2.3. *In the case of $\gamma=\lambda=0$, $\phi=\upsilon=1$ and $\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)=\mathscr{H}_{\Sigma_m}(1,0,0,1,1,1,1;\alpha)=\mathscr{H}^{\alpha}_{\Sigma_m} \tag{2.4}$$

Studied by Srivastava et al [24].

Remark 2.4. *In the case of $m=1$, $\gamma=\lambda=0$ and $\upsilon=\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)=\mathscr{H}_{\Sigma_1}(1,0,0,1,1,\phi,1;\alpha)=\mathscr{H}^{\alpha,\phi}_{\Sigma} \tag{2.5}$$

Studied by Girgaonkar et al. [12].

Remark 2.5. *In the case of $m=1$, $\gamma=\lambda=0$, $\phi=\upsilon=1$ and $\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)=\mathscr{H}_{\Sigma_1}(1,0,0,1,1,1,1;\alpha)=\mathscr{H}^{\alpha}_{\Sigma} \tag{2.6}$$

Studied by Srivastava et al [25].

We state and prove the following results.

Theorem 2.6. *Let $f(z)$ given by (1.3) be in the class $\mathscr{H}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)$ $(0<\alpha\le 1,\ \phi>0,\upsilon\in\mathbb{C}\setminus\{0\},m\in\mathbb{N},0\le\gamma\le 1)$. Then*

$$|a_{m+1}|\le\frac{2\alpha\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\sqrt{\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}}{\alpha\upsilon M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+\alpha\upsilon\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(2\gamma+1)\\ \phi(\phi-1)M_2^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)-(\alpha-1)\Gamma(\lambda(2m+1)+\eta)\\ \Gamma(2(m+1))M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)} \tag{2.7}$$

and

$$|a_{2m+1}|\le\frac{2\upsilon^2\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}{M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)}+\frac{2\alpha\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]} \tag{2.8}$$

where

$$\begin{aligned}M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=&(m+1)\Gamma(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\\&\times\Gamma^2(m+1)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\\&\times[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]\end{aligned} \tag{2.9}$$

$$M_2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)(m+1) \tag{2.10}$$

$$\begin{aligned}M_3(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=&\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\\&\times[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi].\end{aligned} \tag{2.11}$$

Proof. We can write the inequality in (2.1) and (2.2) as

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))']^{\phi}+\gamma z(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))''-2\gamma-1\right]=[s(z)]^{\alpha} \tag{2.12}$$

and

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))']^{\phi}+\gamma w(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))''-2\gamma-1\right]=[t(w)]^{\alpha} \tag{2.13}$$

respectively.
Where $g(w)=f^{-1}$ and $s(z),t(w)$ in $\mathscr{P}$ have the following series representation:

$$s(z)=1+s_mz^m+s_{2m}z^{2m}+s_{3m}z^{3m}+\cdots \tag{2.14}$$

and

$$t(w)=1+t_mw^m+t_{2m}w^{2m}+t_{3m}w^{3m}+\cdots \tag{2.15}$$

Clearly,

$$[s(z)]^{\alpha}=1+\alpha s_mz^m+\left(\alpha s_{2m}+\frac{\alpha(\alpha-1)}{2}s_m^2\right)z^{2m}+\cdots \tag{2.16}$$

and

$$[t(w)]^{\alpha}=1+\alpha t_mw^m+\left(\alpha t_{2m}+\frac{\alpha(\alpha-1)}{2}t_m^2\right)w^{2m}+\cdots \tag{2.17}$$

Also

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))']^{\phi}+\gamma z(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))''-2\gamma-1\right]$$
$$=1+\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}z^m$$
$$+\left(\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{2m+1}\right.$$
$$\left.+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a^2_{m+1}\right)z^{2m}+\cdots \tag{2.18}$$

and

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))']^{\phi}+\gamma w(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))''-2\gamma-1\right]=1-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}w^m$$
$$+\left(\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\left((m+1)a^2_{m+1}-a_{2m+1}\right)\right.$$
$$\left.+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a^2_{m+1}\right)z^{2m}+\cdots \tag{2.19}$$

Now equating the coefficient in (2.12) and (2.13) we get

$$\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}=\alpha s_m, \tag{2.20}$$

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{2m+1}$$
$$+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a^2_{m+1}=\alpha s_{2m}+\frac{\alpha(\alpha-1)}{2}s^2_m, \tag{2.21}$$

$$-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}=\alpha t_m, \tag{2.22}$$

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}$$
$$\left((m+1)a^2_{m+1}-a_{2m+1}\right)+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a^2_{m+1}=\alpha t_{2m}+\frac{\alpha(\alpha-1)}{2}t^2_m. \tag{2.23}$$

From equation (2.20) and (2.22), we find that

$$s_m=-t_m \tag{2.24}$$

and

$$\frac{2\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2}{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2}a^2_{m+1}=\alpha^2(s^2_m+t^2_m). \tag{2.25}$$

Also, from (2.21), (2.23) and (2.25), we have

$$\frac{(m+1)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a^2_{m+1}$$
$$+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a^2_{m+1}=\alpha(s_{2m}+t_{2m})$$
$$+\frac{\alpha(\alpha-1)}{2}(t^2_m+s^2_m)=\alpha(s_{2m}+t_{2m})+\frac{(\alpha-1)\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2}{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2}$$

Therefore, we get

$$a^2_{m+1}=\frac{\alpha^2\upsilon^2\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\Gamma^2(\lambda(2m+1)+\eta)\Gamma^2(2(m+1))(s_{2m}+t_{2m})}{\alpha\upsilon M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+\alpha\upsilon\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(2\gamma+1)\phi(\phi-1)M_2^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)-(\alpha-1)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)} \tag{2.26}$$

where $M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)$, $M_2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)$ and $M_3(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)$ are given by (2.9), (2.10) and (2.11), respectively. Now, taking the absolute value of (2.26) and using Lemma 1.1 for the coefficient s_{2m} and t_{2m}, yields

$$|a_{m+1}| \leq \frac{2\alpha\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\sqrt{\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}}{\begin{array}{r}\alpha\upsilon M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+\alpha\upsilon\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(2\gamma+1)\\ \phi(\phi-1)M_2^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)-(\alpha-1)\Gamma(\lambda(2m+1)+\eta)\\ \Gamma(2(m+1))M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)\end{array}}.$$

For us to get te bound on $|a_{2m+1}|$, we subtract (2.23) from (2.21) to have

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)=\alpha(s_{2m}-t_{2m})+\frac{\alpha(\alpha-1)}{2}(t_m^2-s_m^2). \tag{2.27}$$

It follows from (2.24), (2.25) and (3.25)

$$a_{2m+1}=\frac{\upsilon^2\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)(t_m^2+s_m^2)}{4M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)}+\frac{\alpha\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(s_{2m}-t_{2m})}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}. \tag{2.28}$$

Taking the absolute value of (2.28) and using Lemma 1.1 for the coefficient s_m, s_{2m}, t_m and t_{2m}, we have

$$|a_{2m+1}| \leq \frac{2\upsilon^2\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}{M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)}+\frac{2\alpha\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}$$

which completes the proof of Theorem 2.6. □

Remark 2.7. *In Theorem 2.6, if we choose*

1. $\gamma=\lambda=0$, $\phi=\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Srivastava et al. [24, Theorem 2].*
2. $\phi=\upsilon=1$, *then we have results determined by Wanas and Tang. [47, Theorem 1].*

When $m=1$ which is the one-fold symmetric bi-univalent functions, Theorem 2.6 gives the following corollary:

Corollary 2.8. *Let $f(z)$ given by (1.1) be in the class $\mathscr{H}_\Sigma(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\alpha)$ $(0<\alpha\leq 1,\ \phi>0, \upsilon\in\mathbb{C}\setminus\{0\})$. Then*

$$|a_{m+1}| \leq \frac{4\alpha\upsilon\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)\sqrt{6\Gamma(3\lambda+\eta)}}{\begin{array}{r}\alpha\upsilon M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+6\alpha\upsilon\Gamma(3\lambda+\eta)(2\gamma+1)\phi(\phi-1)\\ M_2^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+6(1-\alpha)\Gamma(3\lambda+\eta)M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)\end{array}} \tag{2.29}$$

and

$$|a_{2m+1}| \leq \frac{4\upsilon^2\alpha^2\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}{M_3^2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)}+\frac{4\alpha\upsilon\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)[2\gamma+2\gamma\phi+\phi]} \tag{2.30}$$

where

$$M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=24\Gamma(\delta+\tau)\Gamma^2(2\lambda+\eta)\Gamma(\lambda+\eta)\Gamma(3\tau+\delta)[2\gamma+2\gamma\phi+\phi]$$

$$M_2(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=2\Gamma(\delta+2\tau)\Gamma(\lambda+\eta) \tag{2.31}$$

$$M_3(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)[2\gamma+2\phi+4\gamma\phi]. \tag{2.32}$$

Corollary 2.9. *Let $f(z)$ given by (1.1) be in the class $\mathscr{H}_\Sigma(\eta,\gamma,\lambda,\delta,\tau,\upsilon;\alpha)$ $(0<\alpha\leq 1,\ \upsilon\in\mathbb{C}\setminus\{0\})$. Then*

$$|a_{m+1}| \leq \frac{4\alpha\upsilon\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)\sqrt{6\Gamma(3\lambda+\eta)}}{\sqrt{\left|\begin{array}{r}\alpha\upsilon M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)+24(1-\alpha)\Gamma(3\lambda+\eta)\Gamma^2(\delta+2\tau)\\ \Gamma^2(\lambda+\eta)[3\gamma+1]^2\end{array}\right|}} \tag{2.33}$$

and

$$|a_{2m+1}| \leq \frac{4\upsilon^2\alpha^2\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}{\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)[3\gamma+1]^2}+\frac{4\alpha\upsilon\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)[4\gamma+1]} \tag{2.34}$$

where

$$M_1(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon,m)=24\Gamma(\delta+\tau)\Gamma^2(2\lambda+\eta)\Gamma(\lambda+\eta)\Gamma(3\tau+\delta)[4\gamma+1].$$

Remark 2.10. *In Corollary 2.8, if we choose*

1. $\gamma=\lambda=0$, $\phi=\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Srivastava et al. [25, Theorem 1].*
2. $\gamma=\lambda=0$, $\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Girgaonkar et al. [12, Theorem 1].*

3. Coefficient estimates for the function class $\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)$

Definition 3.1. *A function $f \in \Sigma_m$ given by (1.3) is said to be in the class $\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)$ if it satisfies the following conditions:*

$$\Re\left\{1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}f(z))']^{\phi}+\gamma z(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}f(z))''-2\gamma-1\right]\right\}>\beta, \tag{3.1}$$

$$\Re\left\{1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}g(w))']^{\phi}+\gamma w(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}g(w))''-2\gamma-1\right]\right\}>\beta, \tag{3.2}$$

where $0\le\gamma\le 1$, $m\in\mathbb{N}$, $\phi\ge 1$, $\upsilon\in\mathbb{C}\setminus\{0\}$ and the function $g=f^{-1}$ is given by (1.4).

We have the following remarks:

Remark 3.2. *In the case of $\phi=\upsilon=1$ in Definition 3.1, we can have*

$$\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)=\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,1,1;\beta)=\mathscr{R}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau;\beta) \tag{3.3}$$

Studied by Wanas and Tang [47].

Remark 3.3. *In the case of $\gamma=\lambda=0$, $\phi=\upsilon=1$ and $\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)=\mathscr{H}^{*}_{\Sigma_m}(1,0,0,1,1,1,1;\beta)=\mathscr{H}_{\Sigma_m}(\beta) \tag{3.4}$$

Studied by Srivastava et al [24].

Remark 3.4. *In the case of $m=1$, $\gamma=\lambda=0$ and $\upsilon=\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)=\mathscr{H}^{*}_{\Sigma_1}(1,0,0,1,1,\phi,1;\beta)=\mathscr{H}^{\phi}_{\Sigma}(\beta) \tag{3.5}$$

Studied by Girgaonkar et al. [12].

Remark 3.5. *In the case of $m=1$, $\gamma=\lambda=0$, $\phi=\upsilon=1$ and $\eta=\delta=\tau=1$ in Definition 2.1, we can have*

$$\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)=\mathscr{H}^{*}_{\Sigma_1}(1,0,0,1,1,1,1;\beta)=\mathscr{H}_{\Sigma}(\beta) \tag{3.6}$$

Studied by Srivastava et al [25].

We state and prove the following results.

Theorem 3.6. *Let $f(z)$ given by (1.3) be in the class $\mathscr{H}^{*}_{\Sigma_m}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)$ $(0\le\beta<1,\ \phi>0, \upsilon\in\mathbb{C}\setminus\{0\}, m\in\mathbb{N}, 0\le\gamma\le 1)$. Then*

$$|a_{m+1}|\le 2\sqrt{\frac{\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma(2(m+1))\Gamma^2(m+2)\Gamma(\lambda(2m+1)+\eta)(1-\beta)}{\begin{array}{c}(m+1)\Gamma(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\\ [\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]\Gamma^2(\delta+(m+1)\tau)+\Gamma^2(\lambda+\eta)\\ \phi(\phi-1)(m+1)^2(2\gamma+1)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))\end{array}}} \tag{3.7}$$

and

$$\begin{aligned}|a_{2m+1}|\le{}&\frac{2(1-\beta)^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2}{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2}\\&+\frac{2\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)\upsilon}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}\end{aligned} \tag{3.8}$$

Proof. First of all, the argument inequality in (3.1) and (3.2) can be written in their equivalent forms as:

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}f(z))']^{\phi}+\gamma z(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}f(z))''-2\gamma-1\right]=\beta+(1-\beta)s(z) \tag{3.9}$$

and

$$1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}g(w))']^{\phi}+\gamma w(\mathscr{R}^{\lambda;\eta}_{\delta,\tau}g(w))''-2\gamma-1\right]=\beta+(1-\beta)t(w). \tag{3.10}$$

respectively. Where $s(z), t(w)\in\mathscr{P}$ and have the forms

$$s(z)=1+s_m z^m+s_{2m}z^{2m}+s_{3m}z^{3m}+\cdots \tag{3.11}$$

and

$$t(w)=1+t_m w^m+t_{2m}w^{2m}+t_{3m}w^{3m}+\cdots \tag{3.12}$$

Clearly,

$$\beta+(1-\beta)s(z)=1+(1-\beta)s_m z^m+(1-\beta)s_{2m}z^{2m}+\cdots \tag{3.13}$$

and

$$\beta+(1-\beta)t(w)=1+(1-\beta)t_m w^m+(1-\beta)t_{2m}w^{2m}+\cdots. \tag{3.14}$$

Also

$$\begin{aligned}
&1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))']^{\phi}+\gamma z(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}f(z))''-2\gamma-1\right]\\
&=1+\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}z^m\\
&+\Bigg(\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{2m+1}\\
&\qquad+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2\Bigg)z^{2m}+\cdots
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&1+\frac{1}{\upsilon}\left[(2\gamma+1)[(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))']^{\phi}+\gamma w(\mathscr{R}^{\lambda,\eta}_{\delta,\tau}g(w))''-2\gamma-1\right]\\
&=1-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}w^m\\
&+\Bigg(\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\\
&\qquad\left((m+1)a_{m+1}^2-a_{2m+1}\right)+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2\Bigg)z^{2m}+\cdots
\end{aligned} \tag{3.16}$$

Now equating the coefficient in (3.9) and (3.10), we get

$$\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}=(1-\beta)s_m, \tag{3.17}$$

$$\begin{aligned}
&\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{2m+1}\\
&\qquad+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2=(1-\beta)s_{2m},
\end{aligned} \tag{3.18}$$

$$-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\upsilon}a_{m+1}=(1-\beta)t_m, \tag{3.19}$$

$$\begin{aligned}
&\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\\
&\qquad\left((m+1)a_{m+1}^2-a_{2m+1}\right)+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{2\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2=(1-\beta)t_{2m}.
\end{aligned} \tag{3.20}$$

From (3.17) and (3.19), we get

$$s_m=-t_m \tag{3.21}$$

and

$$\frac{2\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2}{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2}a_{m+1}^2=(1-\beta)^2(s_m^2+t_m^2). \tag{3.22}$$

Also, adding (3.18) and (3.20), we have

$$\begin{aligned}
&\frac{(m+1)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{m+1}^2\\
&\qquad+\frac{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)(2\gamma+1)(m+1)^2\phi(\phi-1)}{\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2=(1-\beta)(s_{2m}+t_{2m})
\end{aligned}$$

Therefore, we get

$$a_{m+1}^2 \leq \frac{\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma(2(m+1))\Gamma^2(m+2)\Gamma(\lambda(2m+1)+\eta) \atop (1-\beta)(s_{2m}+t_{2m})}{(m+1)\Gamma(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta) \atop [\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]\Gamma^2(\delta+(m+1)\tau)+\Gamma^2(\lambda+\eta) \atop \phi(\phi-1)(m+1)^2(2\gamma+1)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))} \tag{3.23}$$

Applying Lemma 1.1 for the coefficient s_{2m} and t_{2m}, we obtain

$$|a_{m+1}| \leq 2\sqrt{\frac{\upsilon\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma(2(m+1))\Gamma^2(m+2)\Gamma(\lambda(2m+1)+\eta)(1-\beta)}{(m+1)\Gamma(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta) \atop [\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]\Gamma^2(\delta+(m+1)\tau)+\Gamma^2(\lambda+\eta) \atop \phi(\phi-1)(m+1)^2(2\gamma+1)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}}. \tag{3.24}$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.7). Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (3.20) from (3.18), we have

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]}{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)=(1-\beta)(t_{2m}-s_{2m}). \tag{3.25}$$

or equivalently

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2+\frac{\upsilon\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)(t_{2m}-s_{2m})}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]} \tag{3.26}$$

By substituting the value of a_{m+1}^2 from (3.22), we have

$$a_{2m+1}=\frac{(1-\beta)^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2(t_m^2+s_m^2)}{4\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2} + \frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)\upsilon(t_{2m}-s_{2m})}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]} \tag{3.27}$$

Applying Lemma 1.1 once again for coefficients s_m, s_{2m}, t_m and t_{2m}, we have

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\upsilon^2}{\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)[\gamma(m^2+2\phi m+2\phi)+m(\gamma+\phi)+\phi]^2} + \frac{2\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)\upsilon}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)[\gamma(4m(m+\phi)+2\phi)+2m(\gamma+\phi)+\phi]} \tag{3.28}$$

which completes the proof of Theorem 3.6. □

Remark 3.7. *In Theorem 3.6, if we choose*

1. $\gamma=\lambda=0$, $\phi=\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Srivastava et al. [24, Theorem 3].*
2. $\phi=\upsilon=1$, *then we have results determined by Wanas and Tang [47, Theorem 2].*

When $m=1$ which is the one-fold symmetric bi-univalent functions, Theorem 3.6 gives the following corollary:

Corollary 3.8. *Let $f(z)$ given by (1.1) be in the class $\mathscr{H}_{\Sigma}^{*}(\eta,\gamma,\lambda,\delta,\tau,\phi,\upsilon;\beta)$ $(0\leq\beta<1,\ \phi>0, \upsilon\in\mathbb{C}\setminus\{0\})$. Then*

$$|a_2| \leq 2\sqrt{\frac{24(1-\beta)\upsilon\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)\Gamma(3\lambda+\eta)}{24\Gamma(\delta+\tau)\Gamma^2(2\lambda+\eta)\Gamma(\lambda+\eta)\Gamma(3\tau+\delta)[2\gamma+2\gamma\phi+\phi] \atop +24\Gamma^2(2\tau+\delta)\Gamma^2(\lambda+\eta)\Gamma(3\lambda+\eta)(2\gamma+1)\phi(\phi-1)}} \tag{3.29}$$

and

$$|a_3| \leq \frac{4\upsilon^2(1-\beta)^2\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}{\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)[\gamma+\phi+2\gamma\phi]^2}+\frac{4(1-\beta)\upsilon\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)[2\gamma+2\gamma\phi+\phi]} \tag{3.30}$$

Corollary 3.9. *Let $f(z)$ given by (1.1) be in the class $\mathscr{H}_{\Sigma}^{*}(\eta,\gamma,\lambda,\delta,\tau,\upsilon;\beta)$ $(0\leq\beta<1,\ \upsilon\in\mathbb{C}\setminus\{0\})$. Then*

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)\upsilon\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\lambda+\eta)\Gamma(3\tau+\delta)[4\gamma+1]}} \tag{3.31}$$

and

$$|a_3| \leq \frac{4\upsilon^2(1-\beta)^2\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}{\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)[3\gamma+1]^2}+\frac{4(1-\beta)\upsilon\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)[4\gamma+1]} \tag{3.32}$$

Remark 3.10. *In Corollary 3.8, if we choose*

1. $\gamma=\lambda=0$, $\phi=\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Srivastava et al. [25, Theorem 2].*
2. $\gamma=\lambda=0$, $\upsilon=1$ *and* $\eta=\delta=\tau=1$, *then we have results determined by Girgaonkar et al. [12, Theorem 2].*

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