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Research Article

A new difference method for the singularly perturbed Volterra-Fredholm integro-differential equations on a Shishkin mesh

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Abstract

In this research, the finite difference method is used to solve the initial value problem of linear first order Volterra-Fredholm integro-differential equations with singularity. By using implicit difference rules and composite numerical quadrature rules, the difference scheme is established on a Shishkin mesh. The stability and convergence of the proposed scheme are analyzed and two examples are solved to display the advantages of the presented technique.

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1. Introduction

Volterra-Fredholm integro-differential equations (VFIDEs) have arisen in different areas of science and engineering. Their implementations can be found in electrostatics [14], biological models [37], atomic physics [20], astronomy [16], potential theory [17], fluid dynamics [2] and electromagnetic theory [32] (see, also references therein).

There are many researchs about VFIDEs in the literature. The existence, uniqueness and stabiliy of them were debated in [6, 25, 27, 28]. Furthermore, different numerical and semi-analytical methods have been proposed by many scholars. Adomian decomposition method and its modified versions were used in [12, 17, 18, 24, 29]. Variational iteration method was applied in [41]. Bernstein polynomials method was suggested for neutral type VFIDEs in [20]. Legendre collocation matrix method was developed for high-order VFIDEs in [39]. Homotopy perturbation method was used in [15, 34]. By using trapezoidal quadrature rules, the finite difference method was considered in [34]. Orthonormal Bernstein and Block Pulse functions method were presented in [35]. Shifted Jacobi spectral collocation method was performed to multi-dimensional VFIDEs in [10]. Nyström discretization approach was introduced in [13]. Galerkin method was carried out by using Legendre basis functions in [14, 23]. Nonlinear type VFIDEs were examined with the help of Chebyshev cardinal functions in [21]. Tau method was given for one and two-dimensional VFIDEs in [36, 37]. Collocation method with Boubaker wavelet functions was

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proposed for high-order VFIDEs in [32]. Integral collocation approximation was employed for high-order VFIDEs in [1]. The other techniques have been described in a series of papers [2, 7, 8, 16, 19, 31, 38]. The above-mentioned studies were only concerned with the regular cases (i.e. as the lacking of the singularity).

In this paper, we consider the following initial value problem of singularly perturbed Volterra-Fredholm integro-differential equation (SPVFIDE):

$$Lu := L_1 u + \int_0^x K_1(x,t)u(t)dt + \lambda \int_0^l K_2(x,t)u(t)dt = f(x),\ x \in I = [0,l], \tag{1.1}$$

$$u(0) = A \tag{1.2}$$

where

$$L_1 u = \varepsilon u'(x) + a(x)u(x),$$

$0 < \varepsilon \ll 1$ is a perturbation parameter, λ is a given parameter, $a(x) \geq \alpha > 0$ and $f(x)$ $(x \in I)$, $K_1(x,t)$ and $K_2(x,t)$ $((x,t) \in I \times I)$ are sufficiently smooth functions.

In recent times, there has been an increased interest in numerical solutions of singularly perturbed integro-differential equations and various difference schemes have been presented for both singularly perturbed Volterra integro-differential equations (SPVIDEs) and singularly perturbed Fredholm integro-differential equations (SPFIDEs). In [30], by using Simpson quadrature rule and Richardson extrapolation, the order of convergence of the numerical scheme has been enhanced for SPVIDEs. Exponentially fitted difference scheme has been constructed on a Shishkin mesh for second order SPFIDEs in [11]. The stability and convergence of the difference scheme have been analyzed for SPVIDEs in [22]. Boundary value problems of SPFIDEs have been investigated in [9]. Authors in [40] have presented second-order discretization on a piecewise uniform mesh for SPVIDEs. The finite difference scheme with exponential coefficient has been established on a uniform mesh for SPFIDEs in [3]. In [33], for SPVIDEs, a fitted mesh finite difference technique with Richardson extrapolation has been applied on piecewise-uniform Shishkin mesh. SPVIDEs with delay arguments have been discretized in [5].

Our motivation in this article is to present reliable and robust numerical approach for solving SPVFIDEs on a Shishkin mesh.

This work is arranged as follows: Asymptotic estimations of the exact solution are introduced in Section 2. In Section 3, the finite difference scheme is constructed on a Shishkin mesh. Error approximations and convergence analysis are presented in Section 4. Experimental results with some examples are given in Section 5. The paper ends with "Discussion and conclusion" section.

2. Properties of the exact solution

Conveinent asymptotic estimations of the exact solution and its derivatives are given in this section.

Lemma 2.1 ([26]). *Take into account the following initial-value problem*

$$\varepsilon v'(x) + a(x)v(x) = F(x),\ 0 < x < l, \tag{2.1}$$

$$v(0) = A. \tag{2.2}$$

Let $a(x) \geq \alpha > 0,\ F(x) \in C\left(\bar{I}\right),\ |F(x)| \leq \mathcal{F}(x)$ *and the function* $\mathcal{F}(x)$ *is nondecreasing. Then, the solution of the problem (2.1)-(2.2) satisfies that*

$$|v(x)| \leq |A| + \alpha^{-1}\mathcal{F}(x),\ 0 < x < l.$$

Lemma 2.2. *We assume that*

$$a, f \in C^1[0,l], \quad \frac{\partial}{\partial x}K_1(x,t) \in [0,l]^2, \ \frac{\partial}{\partial x}K_2(x,t) \in [0,l]^2 \tag{2.3}$$

and

$$\gamma = e^{\alpha^{-1}\bar{K}_1 l}\alpha^{-1}|\lambda| \max_{0\le x\le l}\int_0^l |K_2(x,t)|\,dt < 1.$$

Then, the solution $u(x)$ of the problem (1.1)-(1.2) holds

$$\|u\|_\infty \le C_0 \tag{2.4}$$

and

$$|u'(x)| \le C\left\{1 + \frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right\}, \quad x \in [0,l]. \tag{2.5}$$

where

$$C_0 = (1-\gamma)^{-1}\left(|A| + \alpha^{-1}\|f\|_\infty\right)e^{\alpha^{-1}\bar{K}_1 l}$$

and

$$\bar{K}_1 = \max_{x\in[0,l]}|K_1(x,t)|.$$

Proof. Firstly, we show the proof of (2.4). The equation (1.1) can be rewritten as follows

$$\varepsilon u'(x) + a(x)u(x) = F(x), \ 0 < x < l \tag{2.6}$$

where

$$F(x) = f(x) - \int_0^x K_1(x,t)\,u(t)dt - \lambda\int_0^l K_2(x,t)\,u(t)dt. \tag{2.7}$$

Then, we estimate (2.7) as the form

$$|F(x)| \le \|f\|_\infty + \bar{K}_1\int_0^x |u(t)|\,dt + |\lambda|\int_0^l |K_2(x,t)|\,|u(t)|\,dt.$$

Considering Lemma 2.1 for the equation (2.6), we obtain

$$|u(x)| \le \delta + \alpha^{-1}\bar{K}_1\int_0^x |u(t)|\,dt. \tag{2.8}$$

where

$$\delta = |A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}|\lambda|\int_0^l |K_2(x,t)|\,|u(t)|\,dt.$$

Applying the Gronwall's inequality to the inequality (2.8), we have

$$|u(x)| \le \delta\exp(\alpha^{-1}\bar{K}_1 x). \tag{2.9}$$

We can write the inequality (2.9) clearly that

$$\begin{aligned} |u(x)| \le{}& \left(|A| + \alpha^{-1}\|f\|_\infty\right)\exp(\alpha^{-1}\bar{K}_1 x) \\ &+ \alpha^{-1}|\lambda|\int_0^l |K_2(x,t)|\,|u(t)|\,dt\exp(\alpha^{-1}\bar{K}_1 x). \end{aligned} \tag{2.10}$$

Modifying the relation (2.10), it is found that

$$\|u\|_\infty\left(1 - \alpha^{-1}|\lambda|\max_{0\le x\le l}\int_0^l |K_2(x,t)|\,dt\exp(\alpha^{-1}\bar{K}_1 l)\right)$$

$$\leq \left(|A| + \alpha^{-1}\left\|f\right\|_{\infty}\right)\exp(\alpha^{-1}\bar{K}_1 l)$$

which validates the estimation (2.4). Now, we prove (2.5). Then, we estimate $u'(0)$. From (1.1), we have

$$|u'(0)| \leq \frac{1}{\varepsilon}\left(|f(0)| - |a(0)|\,|A| - |\lambda|\int_0^l |K_2(0,t)|\,|u(t)|\,dt\right).$$

Since $|K_2(x,t)| \leq \bar{K}_2$ and $|u| \leq C_0$, following inequality is written:

$$|u'(0)| \leq \frac{C}{\varepsilon}. \tag{2.11}$$

By differentiating (2.6), we get

$$\varepsilon v' + a(x)v = F(x) \tag{2.12}$$

with

$$v(x) = u'(x).$$

Here

$$F(x) = f'(x) - a'(x)u(x) - \int_0^x \frac{\partial}{\partial x}K_1(x,t)u(t)dt - K_1(x,x)u(x) - \lambda\int_0^l \frac{\partial}{\partial x}K_2(x,t)u(t)dt.$$

Considering (2.3) and (2.4), we can write

$$|F(x)| \leq C. \tag{2.13}$$

From (2.12), we obtain

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^x F(\xi)\,e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi.$$

Consequently, owing to (2.11) and (2.13), the following expression is found:

$$|u'(x)| \leq \frac{C}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}} + \alpha^{-1}\left\|F\right\|_{\infty}\left(1 - e^{\frac{-\alpha x}{\varepsilon}}\right)$$

which implies (2.5). Therefore, the lemma is proven. □

3. Description of the difference scheme

We denote by ω_N the non-uniform mesh on $[0,l]$

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < l,\ h_i = x_i - x_{i-1}\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{x = 0, x = l\}.$$

We use some notation for the mesh functions. For any mesh function we defined on $\bar{\omega}_N$, we use

$$v_i = v(x_i),\ v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i},\ \left\|v\right\|_{\infty} = \left\|v\right\|_{\infty,\bar{\omega}_N} = \max_{0\leq i\leq N}|v_i|.$$

We construct the difference scheme on Shishkin-type mesh for solving the problem (1.1)-(1.2). For an even number N, we divide each of the subintervals $[0,\sigma]$ and $[\sigma,l]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is determined as

$$\sigma = \min\{\frac{l}{2}, \alpha^{-1}\varepsilon\ln N\}.$$

We use the notation $h^{(1)}$ for the mesh width in $[0,\sigma]$ and the notation $h^{(2)}$ for the width in $[\sigma,l]$. Hence, the mesh stepsizes hold

$$h^{(1)}=\frac{2\sigma}{N},\quad h^{(2)}=\frac{2\,(l-\sigma)}{N},$$

$$h^{(1)}\le lN^{-1},\quad lN^{-1}\le h^{(2)}\le lN^{-1},\quad h^{(1)}+h^{(2)}=2lN^{-1}.$$

x_i node points are specified as

$$\bar{\omega}_N=\begin{cases} x_i=ih^{(1)}, & i=0,1,...,\frac{N}{2},\ x_i\in[0,\sigma]\,; \\ x_i=\sigma+\left(i-\frac{N}{2}\right)h^{(2)}, & i=\frac{N}{2}+1,...,N,\ x_i\in[\sigma,l]\,. \end{cases}$$

We start with the following integral identity for the equation (1.1):

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\varepsilon u'(x)\varphi_i dx+\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}a(x)u(x)\varphi_i dx+\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[\int\limits_0^x K_1(x,t)u(t)dt\right]\varphi_i dx$$

$$+\chi_i^{-1}h_i^{-1}\lambda\int\limits_{x_{i-1}}^{x_i}\left[\int\limits_0^l K_2(x,t)u(t)dt\right]\varphi_i dx=\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}f(x)\varphi_i dx, \tag{3.1}$$

where the basis function

$$\varphi_i(x)=e^{-\frac{a_i(x_i-x)}{\varepsilon}},\quad i=1,2,...,N.$$

$\varphi_i(x)$ is the solution of the following problem:

$$-\varepsilon\varphi_i'(x)+a_i\varphi_i(x)=0,\quad x_{i-1}\le x\le x_i$$

$$\varphi_i(x)=1.$$

For the first two term of (3.1), following relation is obtained:

$$h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[\varepsilon u'(x)+a(x)u(x)\right]\varphi_i(x)dx=h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[\varepsilon u'(x)+a(x_i)u(x)\right]\varphi_i(x)dx$$

$$+h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[a(x)-a\left(x_i\right)\right]u(x)\varphi_i(x)dx$$

$$=\varepsilon\vartheta_i u_{\bar{x},i}+a_i u_i+R_i^{(1)} \tag{3.2}$$

where

$$\vartheta_i=\frac{a_i\varrho_i}{1-e^{-a_i\varrho_i}}e^{-a_i\varrho_i},\quad \varrho_i=\frac{h_i}{\varepsilon}, \tag{3.3}$$

$$R_i^{(1)}=h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[a(x)-a\left(x_i\right)\right]u(x)\varphi_i(x)dx,$$

and the χ_i coefficient

$$\chi_i=h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\varphi_i(x)dx.$$

For the right-side integral term of (3.1), we have

$$h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}f(x)\varphi_i(x)dx=f_i+R_i^{(2)} \tag{3.4}$$

where

$$R_i^{(2)} = h_i^{-1}\chi_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[f(x) - f\left(x_i\right)\right]\varphi_i(x)dx.$$

For the third term in left side of (3.1), using interpolating quadrature rules in [4], we find

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}dx\varphi_i(x)\int\limits_0^x K_1(x,t)u(t)dt = \int\limits_0^x K_1(x_i,t)u(t)dt + R_i^{(3)} \tag{3.5}$$

where

$$R_i^{(3)} = -\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}dx\varphi_i(x)\int\limits_{x_{i-1}}^{x_i}\left(\int\limits_0^x \frac{\partial}{\partial x}K_1\left(x,t\right)u\left(t\right)dt\right)dx. \tag{3.6}$$

Using the right side rectangle rule to the right side of (3.5), we get

$$\int\limits_0^x K_1(x_i,t)u(t)dt + R_i^{(3)} = \sum_{j=1}^{i}h_jK_{1,ij}u_j + R_i^{(3)} + R_i^{(4)} \tag{3.7}$$

where

$$R_i^{(4)} = -\sum_{j=1}^{i}\int\limits_{x_{j-1}}^{x_j}\left(\xi - x_{j-1}\right)\frac{\partial}{\partial\xi}\left(\int\limits_0^x K_1\left(\xi,t\right)u\left(t\right)dt\right)d\xi. \tag{3.8}$$

Eventually, for the fourth term in left side of (3.1), applying the interpolating quadrature rules in [4], it is found

$$\chi_i^{-1}h_i^{-1}\lambda\int\limits_{x_{i-1}}^{x_i}dx\varphi_i(x)\int\limits_0^l K_2(x,t)u(t)dt = \lambda\int\limits_0^l K_2(x_i,t)u(t)dt + R_i^{(5)} \tag{3.9}$$

where

$$R_i^{(5)} = -\chi_i^{-1}h_i^{-1}\lambda\int\limits_{x_{i-1}}^{x_i}dx\varphi_i(x)\int\limits_{x_{i-1}}^{x_i}\left(\int\limits_0^l \frac{\partial}{\partial x}K_2\left(x,t\right)u\left(t\right)dt\right)dx.$$

After, applying right side rectangle rule to the right side of (3.9), we have

$$\lambda\int\limits_0^l K_2(x_i,t)u(t)dt + R_i^{(5)} = \lambda\sum_{j=1}^{N}h_jK_{2,ij}u_j + R_i^{(5)} + R_i^{(6)} \tag{3.10}$$

where

$$R_i^{(6)} = -\lambda\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(\xi - x_{j-1}\right)\frac{\partial}{\partial\xi}\left(\int\limits_0^x K_2\left(\xi,t\right)u\left(t\right)dt\right)d\xi.$$

Combining (3.2), (3.4), (3.7) and (3.10), we can write the following difference scheme:

$$\varepsilon\vartheta_i u_{\bar{x},i} + a_iu_i + \sum_{j=1}^{i}h_jK_{1,ij}u_j + \lambda\sum_{j=1}^{N}h_jK_{2,ij}u_j + R_i = f_i, \quad i = 1,2,...,N, \tag{3.11}$$

with remainder term

$$R_i = \sum_{k=1}^{6}R_i^{(k)}. \tag{3.12}$$

By omitting the error term in (3.11), the following difference scheme is presented for the approximate solution:

$$\varepsilon\vartheta_i y_{\bar{x},i} + a_iy_i + \sum_{j=1}^{i}h_jK_{1,ij}y_j + \lambda\sum_{j=1}^{N}h_jK_{2,ij}y_j = f_i, \quad i = 1,2,...,N, \tag{3.13}$$

$$y_0 = A, \tag{3.14}$$

where ϑ_i is stated by (3.3).

4. Error analysis

In this section, the convergence of the proposed method is examined. The error function $z_i = y_i - u_i$, $i = 0, 1, 2, ..., N$ is the solution of the following problem:

$$lz_i = R_i, \ i = 0, 1, 2, ..., N,$$

$$z_0 = 0.$$

Lemma 4.1 ([26])**.** *Consider the following difference problem*

$$\varepsilon \vartheta_i v_{\bar{x},i} + a_i v_i = F_i, \ i = 0, 1, 2, ..., N, \tag{4.1}$$

$$v_0 = A. \tag{4.2}$$

Let $|F_i| \leq \mathcal{F}_i$ and the function $\mathcal{F}_i$ be nondecreasing. Then, the solution of (4.1)-(4.2) holds

$$|v_i| \leq |A| + \alpha^{-1}\mathcal{F}_i, \ i = 0, 1, 2, ..., N.$$

Lemma 4.2. *Let y_i be the solution of (3.13)-(3.14). If*

$$\bar{\gamma} = \alpha^{-1} |\lambda| e^{\alpha^{-1}\bar{K}_1 x_i} \max_{1 \leq i \leq N} \sum_{j=1}^{N} h_j |K_{2,ij}| < 1$$

then, for the solution of the difference problem (3.13)-(3.14), the following estimate is satisfied:

$$\|y\|_\infty \leq (1 - \bar{\gamma})^{-1} \left(|A| + \alpha^{-1} \|f\|_\infty \right) e^{\alpha^{-1}\bar{K}_1 x_i}.$$

Proof. The difference scheme (3.13) can be rewritten in the form

$$\varepsilon \vartheta_i y_{\bar{t},i} + a_i y_i = F_i, \ i = 1, 2, ..., N-1$$

where

$$F_i = f_i - \sum_{j=1}^{i} h_j K_{1,ij} y_j - \lambda \sum_{j=1}^{N} h_j K_{2,ij} y_j \tag{4.3}$$

From (4.3), we get

$$|F_i| \leq \|f\|_\infty + \sum_{j=1}^{i} h_j |K_{1,ij}| |y_j| + |\lambda| \sum_{j=1}^{N} h_j |K_{2,ij}| |y_j|. \tag{4.4}$$

Moreover, applying Lemma 4.1. to (4.4), we have

$$|y_i| \leq |A| + \alpha^{-1} \|f\|_\infty + \alpha^{-1}\bar{K}_1 \sum_{j=1}^{i} h_j |y_j| + \alpha^{-1} |\lambda| \sum_{j=1}^{N} h_j |K_{2,ij}| |y_j|$$

$$\leq \bar{\delta} + \alpha^{-1}\bar{K}_1 \sum_{j=1}^{i} h_j |y_j| \tag{4.5}$$

where

$$\bar{\delta} = |A| + \alpha^{-1} \|f\|_\infty + \alpha^{-1} |\lambda| \sum_{j=1}^{N} h_j |K_{2,ij}| |y_j|.$$

Applying the difference analogue of Gronwall's inequality to the relation (4.5), we obtain

$$|y_i| \leq \bar{\delta} e^{\alpha^{-1}\bar{K}_1 x_i}$$

Then, it can be written obviously that

$$\|y\|_\infty \leq \left(|A| + \alpha^{-1}\|f\|_\infty\right) e^{\alpha^{-1}\bar{K}_1 x_i} + \alpha^{-1}|\lambda| e^{\alpha^{-1}\bar{K}_1 x_i} \max_{1\leq i\leq N} \sum_{j=1}^{N} h_j |K_{2,ij}| \|y\|_\infty . \quad (4.6)$$

To estimate the error function, by rewriting $A = 0$, $f = R$ and $y = z$ in (4.6), it is found that

$$\|z\|_\infty \leq (1-\bar{\gamma})^{-1}\alpha^{-1}\|R\|_\infty e^{\alpha^{-1}\bar{K}_1 x_i}.$$

Thus, the proof of the lemma is fulfilled. □

Lemma 4.3. *Under the conditions of Lemma 2.2 and* $\frac{\partial K_1(x,t)}{\partial t} \in [0,l]^2$, $\frac{\partial K_2(x,t)}{\partial t} \in [0,l]^2$, *the error term* R_i *holds*

$$\|R\|_\infty \leq CN^{-1}\ln N.$$

Proof. Using the mean value theorem, we have

$$|a(x) - a(x_i)| = \left|a'(\eta_i)\right| |x - x_i|, \quad \eta_i \in (x_i, x)$$
$$\leq Ch_i$$

Therefore, we find

$$\left|R_i^{(1)}\right| \leq \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} Ch_i\varphi_i(x)dx \leq Ch_i\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}\varphi_i(x)dx \leq Ch_i.$$

By the same way, it can be shown that $\left|R_i^{(2)}\right| \leq Ch_i$. For the remainder term $R_i^{(3)}$, applying Leibnitz rule to the integral term in (3.6), we have

$$R_i^{(3)} = -\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} dx\varphi_i(x)\int_{x_{i-1}}^{x_i}\left(K_1(x,x)u(x) + \frac{d}{dx}\int_0^x K_1(x,t)u(t)dt\right)dx$$

Thus, the following relation can be written:

$$\left|R_i^{(3)}\right| \leq \int_{x_{i-1}}^{x_i}\left(|K_1(x,x)||u(x)| + \left|\int_0^x \frac{\partial}{\partial x}K_1(x,t)u(t)dt\right|\right)dx$$

$$\left|R_i^{(3)}\right| \leq Ch_i.$$

In a similar way, $\left|R_i^{(5)}\right| \leq Ch_i$ is found. For the error term $R_i^{(4)}$, using Leibnitz rule for integral term in (3.8), we obtain

$$\left|R_i^{(4)}\right| \leq \sum_{j=1}^{i}\int_{x_{i-1}}^{x_i}(\xi - x_{j-1})\left[|K_1(\xi,x)u(x)| + \int_0^x\left|\frac{\partial}{\partial\xi}K_1(\xi,t)u(t)dt\right|\right]d\xi$$
$$\leq \int_0^l(\xi - x_{j-1})\left[|K_1(\xi,x)u(x)| + \int_0^x\left|\frac{\partial}{\partial\xi}K_1(\xi,t)u(t)dt\right|\right]d\xi$$
$$\leq C\left\{h_i + \int_{x_{i-1}}^{x_i}|u'(x)|dx\right\}.$$

Similarly,

$$\left|R_i^{(6)}\right| \leq C\left\{h_i + \int_{x_{i-1}}^{x_i}|u'(x)|dx\right\}.$$

According to the node points of Shishkin mesh, we take the following estimations. Initially, considering the first case $\sigma = \frac{l}{2}$ and $\frac{l}{2} < \alpha^{-1}\varepsilon \ln N$, we find $h^{(1)} = h^{(2)} = h = lN^{-1}$. Hence, we evaluate $R_i^{(k)}$ for $k = 1, 2, 3, 5$. Now, we estimate the remainder term $R_i^{(4)}$.

$$\left|R_i^{(4)}\right| \le C\left\{h + \int\limits_{x_{i-1}}^{x_i} \left|u'(x)\right| dx\right\} \le C\left\{h + \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} dx\right\}$$

$$\le C\left\{h + \varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i} e^{-\frac{\alpha x}{\varepsilon}} dx\right\} = 2\alpha^{-1}N^{-1}\ln N.$$

Likewise, we obtain $R_i^{(6)} = 2\alpha^{-1}N^{-1}\ln N$. In the second case, for the interval $[\sigma, l]$, we have the inequality

$$\left|R_i^{(4)}\right| \le C\left\{h + \int\limits_{x_{i-1}}^{x_i} \left|u'(x)\right| dx\right\} \le C\left\{h + \varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i} e^{-\frac{\alpha x}{\varepsilon}} dx\right\}$$

$$\le C\left\{h + \alpha^{-1}\left(e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}}\right)\right\}.$$

Since $x_i = \alpha^{-1}\varepsilon \ln N + \left(i - \frac{N}{2}\right) h$, we can write

$$e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}} = \frac{1}{N} e^{\frac{-\alpha\left(i-1-\frac{N}{2}\right)H}{\varepsilon}}\left(1 - e^{-\frac{\alpha H}{\varepsilon}}\right) < N^{-1}$$

$$\left|R_i^{(4)}\right| \le CN^{-1}.$$

For the interval $[0, \sigma]$, if we take $\sigma = \alpha^{-1}\varepsilon \ln N$, we get

$$\left|R_i^{(4)}\right| \le C\left(1 + \varepsilon^{-1}\right)\frac{\alpha^{-1}\varepsilon \ln N}{N/2},$$

$$\left|R_i^{(4)}\right| \le CN^{-1}\ln N.$$

Similarly, we find

$$\left|R_i^{(6)}\right| \le CN^{-1}\ln N.$$

Thus, substituting the estimations of all remainder terms in (3.12), we obtain

$$|R_i| \le CN^{-1}\ln N.$$

□

Theorem 4.4. *Let u be the solution of (1.1)-(1.2) and y be the solution of (3.13)-(3.14). Then, the following estimate is satisfied:*

$$\|y - u\|_{\infty,\bar{\omega}_N} \le CN^{-1}\ln N.$$

Proof. This follows immediately by combining of previous two lemmas. □

5. Illustrative examples

In this section, theoretical results are tested on two samples. In this context, we use the following iteration for solving discretization (3.13)-(3.14):

$$y_i^{(n)} = \frac{\varepsilon\vartheta_i y_{i-1}^{(n)} + h_i f_i - h_i\left(\sum\limits_{j=1}^{i} h_j K_{1,ij} y_j^{(n-1)} + \lambda\sum\limits_{j=1}^{N} h_j K_{2,ij} y_j^{(n-1)}\right)}{\varepsilon\vartheta_i + h_i a_i},$$

$$y_0^{(n)} = A.$$

Example 1: We take into account the following singularly perturbed Volterra-Fredholm equation:

$$\varepsilon u' + u + \int\limits_0^x xu(t)dt + \int\limits_0^1 u(t)dt = e^{\frac{-x}{\varepsilon}}\left(-\varepsilon^2 + \varepsilon + 1\right) + \varepsilon x - \varepsilon e^{\frac{-1}{\varepsilon}} + \varepsilon$$

subject to initial condition

$$u(0) = 1.$$

The exact solution of this problem is $u(x) = e^{\frac{-x}{\varepsilon}}$. Error approximations are computed as

$$e^N = |y_i - u_i|$$

where u_i is the exact solution and y_i is approximate solution. Besides, the order of convergence is defined as follows

$$p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}.$$

Experimental datas are displayed in Table 1.

Table 1. Maximum pointwise errors e^N and order of convergence p^N on $\bar{\omega}_N$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.00817305	0.00399709	0.00197592	0.00098227	0.00048971
	1.031	1.016	1.008	1.004	1.002
2^{-4}	0.00918949	0.00426125	0.00204326	0.00099927	0.00049398
	1.108	1.060	1.032	1.016	1.023
2^{-6}	0.01477887	0.00738994	0.00369472	0.00184726	0.00092348
	0.999	1.001	1.001	1.002	1.005
2^{-8}	0.01531296	0.00767148	0.00383674	0.00191787	0.00095693
	0.997	0.999	1.004	1.003	1.008
2^{-10}	0.01554171	0.00770116	0.00384749	0.00192375	0.00096087
	1.013	1.001	1.000	1.003	1.001
e^N	0.01554171	0.00770116	0.00384749	0.00192375	0.00096087
p^N	0.997	0.999	1.000	1.002	1.001

Example 2: Consider the another problem

$$\varepsilon u' + (3x^2 + 1)u + \int\limits_0^x tu(t)dt + \frac{1}{2}\int\limits_0^1 (1 - xt)u(t)dt = e^{\frac{-x}{\varepsilon}}(x + \varepsilon),$$

$$u(0) = 1.$$

The exact solution of this problem is unknown. Since the exact solution is unknown, we apply the double-mesh technique. The maximum pointwise errors are remarked by

$$e^N = \left|y_i^N - y_i^{2N}\right|$$

and the convergence rates are specified as follows

$$p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}.$$

The computational results are tabulated in Table 2.

Table 2. Maximum pointwise errors e^N and order of convergence p^N on $\bar{\omega}_N$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.00811413	0.00398210	0.00197214	0.00098132	0.00048947
	1.026	1.014	1.007	1.003	1.019
2^{-4}	0.00897773	0.00420438	0.00202853	0.00099552	0.00049304
	1.094	1.051	1.026	1.013	1.030
2^{-6}	0.01054824	0.00529412	0.00263706	0.00131833	0.00065826
	0.994	1.005	1.000	1.002	1.006
2^{-8}	0.01383275	0.00696637	0.00348319	0.00174159	0.00087080
	0.989	1.000	1.000	1.001	1.002
2^{-10}	0.01509617	0.00754809	0.00376404	0.00187702	0.00093510
	1.000	1.003	1.003	1.005	1.011
e^N	0.01509617	0.00754809	0.00376404	0.00187702	0.00093510
p^N	0.989	1.000	1.000	1.001	1.002

From Tables 1-2, it can be observed that almost first-order convergence is acquired for different values of the perturbation parameter and mesh stepsize. This shows that the numerical applications accordance with the theory.

6. Discussion and conclusion

A new difference scheme was introduced by using exponential basis functions and interpolating quadrature rules to get the numerical solution of SPVIDEs. The difference scheme was constructed on a Shishkin mesh. Error analysis of the method was completed and two test problems were solved. The obtained outcomes were shown in Tables 1-2 and the order of uniform convergence was found as $O(N^{-1}\ln N)$. The computed results show that the proposed method is stable and very effective for solving these problems. It can also be applied to partial and fractional types of integro-differential equations for future investigations.

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