http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 2, Pages 326–338 (2022)
DOI:10.31801/cfsuasmas.950707
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: June 10, 2021; Accepted: October 11, 2021

DIRECTION CURVES OF GENERALIZED BERTRAND CURVES AND INVOLUTE-EVOLUTE CURVES IN E^4

Mehmet ÖNDER
Delibekirli Village, Kırıkhan, 31440 Hatay, TURKEY

Abstract. In this study, we define (1,3)-Bertrand-direction curve and (1,3)-Bertrand-donor curve in the 4-dimensional Euclidean space E^4. We introduce necessary and sufficient conditions for a special Frenet curve to have a (1,3)-Bertrand-direction curve. We introduce the relations between Frenet vectors and curvatures of these direction curves. Furthermore, we investigate whether (1,3)-evolute-donor curves in E^4 exist and show that there is no (1,3)-evolute-donor curve in E^4.

1. Introduction

Associated curves are the most interesting subject of curve theory. Such curves have a special property between their Frenet apparatus. Bertrand curves are one of the most famous type of such curve pairs. These curves were first discovered by J. Bertrand in 1850 [1]. In the 3-dimensional Euclidean space E^3, a curve $\alpha(s)$ is called Bertrand curve if there exists a curve γ different from α with the same principal normal line as α. Bertrand partner curves are important and fascinating examples of offset curves used in computer-aided design [13]. The classical characterization for the Bertrand curve is that a curve $\alpha(s)$ is a Bertrand curve if and only if its curvature functions $\kappa(s)$, $\tau(s)$ satisfy the condition $a\kappa(s)+b\tau(s)=1$, where a, b are real constant numbers. And, the parametric form of the Bertrand mate of $\alpha(s)$ is defined by $\gamma(s)=\alpha(s)+\lambda N(s)$, where $\lambda \neq 0$ is constant and $N(s)$ is unit principal normal line of α [17]. It is interesting that for $n \geq 4$, there exists no Bertrand curves in this form. This fact was proved by Matsuda and Yorozu [12]. Considering this fact, in the same paper, they have defined a new type of associated curves called (1,3)-Bertrand curves in E^4.

Moreover, another well-known type of associated curve pairs is involute-evolute curve couple. These curves were first studied by Huygens in his work [8]. Classically,

2020 *Mathematics Subject Classification.* 53A04.
Keywords. Direction curve, (1,3)-Bertrand-direction curve, (0,2)-involute-direction curve.
mehmetonder197999@gmail.com
0000-0002-9354-5530.

©2022 Ankara University
Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics

an evolute of a given curve is defined as the locus of the centers of curvatures of the curve, which is the envelope of the normal of reference curve. Fuchs defined an involute of a given curve as a curve for which all tangents of reference curve are normal [3]. In the same study, equation of enveloping curve of the family of normal planes for space curve has been also defined. Gere and Zupnik studied involute-evolute curves by considering a curve composed of two arcs with common evolute [6]. Fukunaga and Takahashi defined evolutes and involutes of fronts in the plane and introduced some properties of these curves [4,5]. Later, Yu, Pei and Cui considered evolutes of fronts on Euclidean 2-sphere [18]. Özyılmaz and Yılmaz studied involute-evolute of W-curves in Euclidean 4-space E^4 [16]. Li and Sun studied evolutes of fronts in the Minkowski Plane [9].

Recently, Hanif and Hou have defined generalized involute and evolute curves in E^4[7]. They have obtained necessary and sufficient conditions for a curve to have a generalized involute or evolute curve. Another study of generalized involute-evolute curves has been given by Öztürk, Arslan and Bulca [15]. They have given characterization of involute curves of order k of a given curve in E^n and also introduced some results on these type of curves in E^3 and E^4.

Furthermore, Choi and Kim have defined a new type of associated curves in E^3 called principal normal (binormal) direction-curve and principal normal (binormal) donor-curve [2]. Similarly, Macit and Düldül have defined W-direction curve and W-donor curve in E^3, where W is unit Darboux vector of the reference curve [10]. Later, the author has defined Bertrand direction curves, Mannheim direction curves and involute-evolute direction curves in E^3 and introduced relations between those curves and some special curves such as helices and slant helices [14].

In this study, first, we define (1,3)-Bertrand-direction curves and introduce the relations between the Frenet apparatus of these curves. We show that a curve with non-constant first curvature κ does not have (1,3)-Bertrand-direction curve. Later, we give that no C^∞-special Frenet curve in E^4 is an (1,3)-evolute-donor curve.

2. Preliminaries

Let $\alpha : I \to E^4$ be a regular curve, i.e., $\|\alpha'(t)\| \neq 0$, where I is subset of real numbers set $\mathbb{R}$ and $\|\alpha'(t)\|$ denotes the norm of tangent vector $\alpha'(t)$ in the Euclidean 4-space E^4. This norm is defined by $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$ where $\langle x, x\rangle$ is the Euclidean inner(dot) product and $x = (x_1, x_2, x_3, x_4)$ is a vector in E^4. The curve $\alpha(t)$ is called unit speed if $\|\alpha'(t)\| = 1$. The parameter of a unit speed curve is represented by s and called arc-length parameter. The curve $\alpha(s)$ is called special Frenet curve if there exist differentiable functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ on I and differentiable orthonormal frame field $\{T, N, B_1, B_2\}$ along $\alpha(s)$ such that:

i) Following Frenet formulas hold

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B_1, \\ B_1' &= -\tau N + \sigma B_2, \\ B_2' &= -\sigma B_1. \end{aligned} \tag{1}$$

ii) The orthonormal frame field $\{T, N, B_1, B_2\}$ has positive orientation.

iii) The functions $\kappa(s)$, $\tau(s)$ are positive and the function $\sigma(s)$ does not vanish. The unit vector fields T, N, B_1 and B_2 are called tangent, principal normal, first binormal and second binormal of $\alpha(s)$ and the functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are called first, second and third curvatures of $\alpha(s)$, respectively [11].

If we take $T = n_1$, $N = n_2$, $B_1 = n_3$, $B_2 = n_4$, the term "special" means that the vector field n_{i+1}, $(1 \leq i \leq 3)$ is inductively defined by the vector fields n_i and n_{i-1} and the positive functions κ and τ [12]. For this, the Frenet apparatus of a special Frenet curve have been determined by the following steps:

(1) $\alpha'(s) = T(s)$
(2) $\kappa(s) = \|T'(s)\| > 0$, $N(s) = \frac{1}{\kappa(s)} T'(s)$.
(3) $\tau(s) = \|N'(s) + \kappa(s)T(s)\| > 0$, $B_1(s) = \frac{1}{\tau(s)} \left(N'(s) + \kappa(s)T(s)\right)$
(4) $B_2(s) = \varepsilon \frac{1}{\left\|{B_1}'(s)+\tau(s)N(s)\right\|} \left({B_1}'(s) + \tau(s)N(s)\right)$, where $\varepsilon = \pm 1$ is chosen as the frame $\{T, N, B_1, B_2\}$ has positive orientation and $\sigma(s) = \langle B_1'(s), B_2(s)\rangle$ does not vanish.

All these 4 steps should be checked that the curve $\alpha(s)$ is a special Frenet curve [11].

The plane spanned by the vectors T, B_1 is called the Frenet (0,2)-plane and the plane spanned by the vectors N, B_2 is called the Frenet (1,3)-normal plane of α[7,12]

Definition 1. *([12]) A C^∞-special Frenet curve $\alpha : I \to E^4$ is called a (1,3)-Bertrand curve if there exits another C^∞-special Frenet curve $\beta : J \to E^4$ and a C^∞-mapping $\varphi : I \to J$ such that the Frenet (1,3)-normal planes of α and β at the corresponding points coincide. The parametric representation of β is $\beta(\varphi(s)) = \alpha(s) + zN(s) + tB_2(s)$, where z, t are constant real numbers.*

Theorem 1. *([12]) If $n \geq 4$, then no C^∞-special Frenet curve in E^n is a Bertrand curve.*

Definition 2. *([7]) Let $\alpha(s)$ and $\gamma(\bar{s})$ be two regular curves in E^4 such that $\bar{s} = f(s)$ is the arc-length parameter of $\gamma(\bar{s})$. If the Frenet (0,2)-plane of α and Frenet (1,3)-plane of γ at the corresponding points coincide, then α is called (1,3)-evolute curve of γ and γ is called (0,2)-involute curve of α. The (0,2)-involute curve γ has the parametric form $\gamma(s) = \alpha(s) + (c - s)T(s) + kB_1(s)$, where c, k are real constants.*

Let $I \subset \mathbb{R}$ be an open interval. For a unit speed special Frenet curve $\alpha : I \to E^4$, let define a vector valued function $X(s)$ as follows

$$X(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s), \tag{2}$$

where p, l, r and n are differentiable scalar functions of s. Let $X(s)$ be unit, i.e.,

$$p^2(s) + l^2(s) + r^2(s) + n^2(s) = 1, \tag{3}$$

holds. Then the definitions of X-donor curve and X-direction curve in E^4 are given as follows.

Definition 3. *Let α be a special Frenet curve in E^4 and $X(s)$ be a unit vector valued function as given in* (2)*. The integral curve $\gamma : I \to E^4$ of $X(s)$ is called an X-direction curve of α. The curve α having γ as an X-direction curve is called the X-donor curve of γ in E^4.*

3. (1,3)-Bertrand-Direction Curves in E^4

In this section, we define (1,3)-Bertrand-direction curves and (1,3)-Bertrand-donor curves for special Frenet curves and introduce necessary and sufficient conditions for these curve pairs.

Definition 4. *Let $\alpha = \alpha(s)$ be a special Frenet curve in E^4 with arc-length parameter s and $X(s)$ be a unit vector field as given in* (2)*. Let special Frenet curve $\beta(\bar{s}) : I \to E^4$ be an X-direction curve of α. The Frenet frames and curvatures of α and β be denoted by $\{T, N, B_1, B_2\}$, κ, τ, σ and $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$, $\bar{\kappa}$, $\bar{\tau}$, $\bar{\sigma}$, respectively, and let any Frenet vector of α does not coincide with any Frenet vector of β. If β is a (1,3)-Bertrand partner curve of α, then β is called (1,3)-Bertrand-direction curve of α and α is said to be (1,3)-Bertrand-donor curve of β.*

From Definition 4, it is clear that at the corresponding points of the curves, the planes spanned by $\{N, B_2\}$ and $\{\bar{N}, \bar{B}_2\}$ coincide. Then, we have,

$$sp\{N, B_2\} = sp\{\bar{N}, \bar{B}_2\}, \;\; sp\{T, B_1\} = sp\{\bar{T}, \bar{B}_1\}, \tag{4}$$

Moreover, since β is an integral curve of $X(s)$, we have $\frac{d\beta}{ds} = X(s)$. Also, since $X(s)$ is unit, the arc-length parameter $\bar{s}$ of β is obtained as

$$\bar{s} = \int_0^s \left\| \frac{d\beta}{ds} \right\| ds = \int_0^s ds = s \tag{5}$$

i.e., arc-length parameters of (1,3)-Bertrand-direction curves α and β are same. Thus, hereafter we will use prime for both curves to show the derivative with respect to s.

Theorem 2. *The special Frenet curve $\alpha : I \to E^4$ is a (1,3)-Bertrand-donor curve if and only if there exist non-zero constants r, μ, λ, p such that*

$$p^2 + r^2 = 1, \;\; \lambda^2 + \mu^2 = 1, \tag{6}$$

$$p\kappa - r\tau = \frac{\lambda}{\mu} r\sigma, \tag{7}$$

$$(p^2 - \lambda^2)\kappa - pr\tau \neq 0. \tag{8}$$

Proof. Let $X(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s)$ be a unit vector valued function and the special Frenet curve $\beta : I \to E^4$ be integral curve of $X(s)$ and also be a (1,3)-Bertrand-direction curve of α, where $p(s),\ l(s),\ r(s)$ and $n(s)$ are smooth scalar functions of arc-length parameter s. Then, we have

$$\bar{T}(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s). \tag{9}$$

From (4), it follows $\bar{T} \perp sp\{N, B_2\}$. Then, multiplying (9) with N and B_2, we have $l(s) = 0,\ n(s) = 0$, respectively, and (9) becomes

$$\bar{T}(s) = p(s)T(s) + r(s)B_1(s), \tag{10}$$

and from (10), it follows $p^2(s) + r^2(s) = 1$, since $\bar{T}$ is unit. Differentiating (10) with respect to s and using Frenet formulas (1), we get

$$\bar{\kappa}\bar{N} = p'T + (p\kappa - r\tau)N + r'B_1 + r\sigma B_2. \tag{11}$$

Multiplying (11) with T and B_1 and considering (4), we get $p' = 0,\ r' = 0$, respectively, i.e., p and r are constants. If p or r is zero, then Frenet vectors of α and β coincide. It follows that p and r are non-zero constants. Then, from (10), we get $p^2 + r^2 = 1$ and we have first equality in (6).

Now, (11) becomes

$$\bar{\kappa}\bar{N} = (p\kappa - r\tau)N + r\sigma B_2, \tag{12}$$

which gives

$$\bar{\kappa} = \sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}. \tag{13}$$

Let define

$$\lambda = \frac{p\kappa - r\tau}{\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}, \quad \mu = \frac{r\sigma}{\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}. \tag{14}$$

Then, (12) becomes

$$\bar{N} = \lambda N + \mu B_2, \quad \lambda^2 + \mu^2 = 1. \tag{15}$$

By Definition 4, any Frenet vector of α does not coincide with any Frenet vector of β. Thus, we have that $\lambda \neq 0,\ \mu \neq 0$. Differentiating the first equation in (15) with respect to s and considering Frenet formulas (1), it follows

$$-\bar{\kappa}\bar{T} + \bar{\tau}\bar{B}_1 = -\lambda\kappa T + \lambda' N + (\lambda\tau - \mu\sigma)B_1 + \mu' B_2. \tag{16}$$

Multiplying (16) with N and B_2, we get $\lambda' = 0,\ \mu' = 0$, respectively, i.e., $\lambda,\ \mu$ are real non-zero constants. So, we have $\lambda^2 + \mu^2 = 1$, which is the second equality in (6).

Moreover, from (13) and (14), we have

$$\bar{\kappa} = \frac{p\kappa - r\tau}{\lambda} = \frac{r\sigma}{\mu}. \tag{17}$$

Then, (17) gives us $p\kappa - r\tau = \frac{\lambda}{\mu} r\sigma$ and we obtain (7).

Now, writing (10) and (17) in (16), it follows

$$\lambda\bar{\tau}\bar{B}_1 = \left((p^2 - \lambda^2)\kappa - pr\tau\right) T + \left(pr\kappa + (\lambda^2 - r^2)\tau - \lambda\mu\sigma\right) B_1. \tag{18}$$

From (7), we have

$$\sigma = \frac{\mu(p\kappa - r\tau)}{\lambda r}. \tag{19}$$

Writing (19) in (18) and using (6), equality (18) becomes

$$\bar{\tau}\bar{B}_1 = A\left(T - \frac{p}{r}B_1\right), \tag{20}$$

where $A = \frac{(p^2-\lambda^2)\kappa - pr\tau}{\lambda}$. Since $\bar{B}_1 \neq 0$, we get $A \neq 0$, i.e., $(p^2 - \lambda^2)\kappa - pr\tau \neq 0$. Then we have (8).

Conversely, assume that relations (6), (7) and (8) hold for some non-zero constants r, μ, λ, p and α be a special Frenet curve with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures κ, τ, σ. Let define a vector valued function

$$X(s) = pT(s) + rB_1(s), \tag{21}$$

and let $\beta : I \to E^4$ be an integral curve of $X(s)$. We will show that β is a (1,3)-Bertrand-direction curve of α. Differentiating (21) with respect to s gives

$$\bar{\kappa}\bar{N} = (p\kappa - r\tau)N + r\sigma B_2. \tag{22}$$

Writing (7) in (22), it follows

$$\bar{\kappa}\bar{N} = r\sigma\left(\frac{\lambda}{\mu}N + B_2\right). \tag{23}$$

From (23), it follows,

$$\bar{\kappa} = \varepsilon_1\frac{r\sigma}{\mu}, \tag{24}$$

where $\varepsilon_1 = \pm 1$ such that $\bar{\kappa} > 0$. Writing (24) in (23) gives

$$\bar{N} = \varepsilon_1\left(\lambda N + \mu B_2\right). \tag{25}$$

Differentiating (25) with respect to s gives

$$\bar{N}' = \varepsilon_1\left(-\lambda\kappa T + (\lambda\tau - \mu\sigma)B_1\right). \tag{26}$$

Using (21), (24) and (26), we have

$$\bar{N}' + \bar{\kappa}\bar{T} = \frac{\varepsilon_1}{\mu}\left((pr\sigma - \lambda\mu\kappa)T + (r^2\sigma + \lambda\mu\tau - \mu^2\sigma)B_1\right). \tag{27}$$

Writing (7) in (27) and using (6), (27) becomes

$$\bar{N}' + \bar{\kappa}\bar{T} = \varepsilon_1\frac{(p^2 - \lambda^2)\kappa - pr\tau}{\lambda}\left(T - \frac{p}{r}B_1\right). \tag{28}$$

From (28) and (8), we have

$$\bar{\tau} = \left\|\bar{N}' + \bar{\kappa}\bar{T}\right\| = \varepsilon_2\frac{(p^2 - \lambda^2)\kappa - pr\tau}{\lambda r} \neq 0, \tag{29}$$

where $\varepsilon_2 = \pm 1$ such that $\bar{\tau} > 0$. Then,

$$\bar{B}_1 = \frac{1}{\bar{\tau}}\left(\bar{N}' + \bar{\kappa}\bar{T}\right) = \frac{\varepsilon_1}{\varepsilon_2}\left(rT - pB_1\right). \tag{30}$$

Considering (21), (25) and (30), we can define the unit vector $\bar{B}_2$ as

$$\bar{B}_2 = \frac{1}{\varepsilon_2}\left(\mu N - \lambda B_2\right),$$

that is

$$\bar{B}_2 = \frac{1}{\varepsilon_2\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}\left(r\sigma N - (p\kappa - r\tau)B_2\right), \tag{31}$$

and we have $\det(\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2) = 1$. Using (30) and (31), it follows

$$\bar{\sigma} = \left\langle \bar{B}_1', \bar{B}_2 \right\rangle = \varepsilon_1\left(\mu(r\kappa + p\tau) + p\lambda\sigma\right). \tag{32}$$

If we assume that $\bar{\sigma} = 0$, then we have $\mu(r\kappa + p\tau) = -p\lambda\sigma$. Multiplying that with r, we get $\mu(r^2\kappa + pr\tau) = -pr\lambda\sigma$. Since $r^2 = 1 - p^2$, the last equality becomes $\mu\left(-p(p\kappa - r\tau) + \kappa\right) = -pr\lambda\sigma$. Using (7), it follows $\mu\kappa = 0$, which is a contradiction since $\mu \neq 0$ and α is a special Frenet curve. Then, $\bar{\sigma} \neq 0$, i.e., β is a special Frenet curve. Moreover, since $r,\ \mu,\ \lambda,\ p$ are non-zero constants, from the equalities (21), (25), (30) and (31), it follows that no Frenet vectors of α and β coincide. Furthermore, since we obtain $sp\left\{N, B_2\right\} = sp\left\{\bar{N}, \bar{B}_2\right\}$, we have that β is (1,3)-Bertrand-direction curve of α.

□

Moreover, since α is a (1,3)-Bertrand curve, by Definition 1, its (1,3)-Bertrand partner curve β has the parametric form $\beta(s) = \alpha(s) + zN(s) + tB_2(s)$ where $z,\ t$ are constant real numbers. Differentiating that with respect to s and using the equality $\bar{T} = pT + rB_1$, we have $pT + rB_1 = (1 - z\kappa)T + (z\tau - t\sigma)B_1$ which gives that $\kappa z = 1 - p$. If $z = 0$, we get $p = 1$. But this is a contradiction since $p^2 + r^2 = 1$ and $r \neq 0$. Then, $\kappa = (1 - p)/z$ is a non-zero positive constant and we have the followings.

Corollary 1. *No C^∞-special Frenet curve in E^4 with non-constant first curvature κ is a (1,3)-Bertrand-donor curve.*

Corollary 2. *If the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-Bertrand-donor curve, then there exists a linear relation $c_1\tau + c_2\sigma = \kappa$ where $c_1,\ c_2,\ \kappa \neq 0$ are constants and κ, τ, σ are Frenet curvatures of α.*

Corollary 3. *Let β be (1,3)-Bertrand-direction curve of α. Then the relations between Frenet apparatus are given as follows*

$$\bar{T} = pT + rB_1,\ \bar{N} = \varepsilon_1\left(\lambda N + \mu B_2\right),\ \bar{B}_1 = \frac{\varepsilon_1}{\varepsilon_2}\left(rT - pB_1\right),\ \bar{B}_2 = \frac{1}{\varepsilon_2}\left(\mu N - \lambda B_2\right), \tag{33}$$

$$\bar{\kappa} = \varepsilon_1 \frac{r\sigma}{\mu} > 0, \ \ \bar{\tau} = \varepsilon_2 \frac{(p^2 - \lambda^2)\kappa - pr\tau}{\lambda r} > 0, \ \ \bar{\sigma} = \varepsilon_1 \left(\mu(r\kappa + p\tau) + p\lambda\sigma \right), \quad (34)$$

where $r, \ \mu, \ \lambda, \ p$ *are non-zero real constants and* $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

Since we have $p^2 + r^2 = 1, \ \ \lambda^2 + \mu^2 = 1$, from (33) we also have,

$$T = p\bar{T} + \frac{\varepsilon_1}{\varepsilon_2} r\bar{B}_1, N = \varepsilon_1\lambda\bar{N} + \varepsilon_2\mu\bar{B}_2, B_1 = r\bar{T} - \frac{\varepsilon_1}{\varepsilon_2} p\bar{B}_1, B_2 = \varepsilon_1\mu\bar{N} - \varepsilon_2\lambda\bar{B}_2. \quad (35)$$

Example 1. *Let consider unit speed special Frenet curve* $\alpha(s)$ *given by*

$$\alpha(s) = \frac{1}{\sqrt{2}} \left[\frac{1}{2}\sin 2s, \ -\frac{1}{2}\cos 2s, \ \ \frac{1}{3}\sin 3s, \ -\frac{1}{3}\cos 3s \right]. \quad (36)$$

The Frenet vectors of $\alpha(s)$ *are obtained as*

$$T(s) = \frac{1}{\sqrt{2}} \left(\cos 2s, \ \sin 2s, \ \cos 3s, \ \sin 3s \right), \quad (37)$$

$$N(s) = \frac{1}{\sqrt{13}} \left(-2\sin 2s, \ 2\cos 2s, \ -3\sin 3s, \ 3\cos 3s \right), \quad (38)$$

$$B_1(s) = \frac{1}{\sqrt{2}} \left(\cos 2s, \ \sin 2s, \ -\cos 3s, \ -\sin 3s \right), \quad (39)$$

$$B_2(s) = \frac{1}{\sqrt{13}} \left(-3\sin 2s, \ 3\cos 2s, \ 2\sin 3s, \ -2\cos 3s \right), \quad (40)$$

respectively. Then the curvatures are

$$\kappa = \frac{\sqrt{26}}{2}, \quad \tau = \frac{5\sqrt{26}}{26}, \ \ \sigma = \frac{6\sqrt{26}}{13}. \quad (41)$$

For real constants

$$r = \frac{1}{3}, \ p = \frac{2\sqrt{2}}{3}, \ \lambda = \frac{5 + 26\sqrt{2}}{\sqrt{\left(5 + 26\sqrt{2}\right)^2 + 144}}, \ \mu = \frac{12}{\sqrt{\left(5 + 26\sqrt{2}\right)^2 + 144}}, \quad (42)$$

the conditions (6), (7) *and* (8) *hold. Then* $\alpha(s)$ *is a (1,3)-Bertrand-donor curve. From* (33), *(1,3)-Bertrand-direction curve* β *of* $\alpha(s)$ *is obtained as*

$$\begin{aligned} \beta(s) = \tfrac{1}{3\sqrt{2}} \Big(& \tfrac{2\sqrt{2}+1}{2}\sin 2s + c_1, \ -\tfrac{2\sqrt{2}+1}{2}\cos 2s + c_2, \\ & + \tfrac{2\sqrt{2}-1}{3}\sin 3s + c_3, \ -\tfrac{2\sqrt{2}-1}{3}\cos 3s + c_4 \Big) \end{aligned} \quad (43)$$

where c_i; $(1 \leq i \leq 4)$ *are integration constants.*

4. Generalized Involute-Evolute-Direction Curves in E^4

In this section, we will consider a new type of curve pairs. In ref. [7], the authors defined (1,3)-evolute curve and (0,2)-involute curve in E^4 as given in Definition 2. Now, we will show that similar definitions for (1,3)-evolute curve and (0,2)-involute curve in E^4 as direction curves don't exist, i.e., there are no (0,2)-involute-direction curves and (1,3)-evolute-donor curves. For this purpose, let assume the converse, i.e., suppose that (0,2)-involute-direction curves and (1,3)-evolute-donor curves exist. Let $\alpha = \alpha(s)$ be a special Frenet curve in E^4 with arc-length parameter s and $X(s)$ be a unit vector field in the form Eq. (2). Let the special Frenet curve $\gamma(\bar{s}) : I \to E^4$ be an X-direction curve of α. The Frenet vectors and curvatures of α and γ be denoted by $\{T, N, B_1, B_2\}$, κ, τ, σ and $\left\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\right\}$, $\bar{\kappa}$, $\bar{\tau}$, $\bar{\sigma}$, respectively and let any Frenet vector of α does not coincide with any Frenet vector of γ. By the assumption, let γ be a (0,2)-involute curve of α. Since also γ is direction curve of α let we call γ as (0,2)-involute-direction curve of α and α as (1,3)-evolute-donor curve of γ. Then, the Frenet planes spanned by $\{T, B_1\}$ and $\left\{\bar{N}, \bar{B}_2\right\}$ coincide and we have,

$$sp\{T, B_1\} = sp\left\{\bar{N}, \bar{B}_2\right\}, \;\; sp\{N, B_2\} = sp\left\{\bar{T}, \bar{B}_1\right\}. \tag{44}$$

Similar to the (1,3)-Bertrand-direction curves, since γ is an integral curve of $X(s)$ and $X(s)$ is unit, for the arc-length parameter $\bar{s}$ of γ we have $\bar{s} = \int_0^s \left\| \frac{d\gamma}{ds} \right\| ds = \int_0^s ds = s$. Then, hereafter the prime will show the derivative with respect to s.

Theorem 3. *No C^∞-special Frenet curve in E^4 is a (1,3)-evolute-donor curve.*

Proof. First, we will show that if such curves exist, then the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-evolute-donor curve if and only if there exist non-zero constants b, d, x_1, x_2 such that

$$b^2 + d^2 = 1, \;\; x_1^2 + x_2^2 = 1, \tag{45}$$

$$d\sigma - b\tau = \frac{x_2}{x_1} b\kappa. \tag{46}$$

$$(d^2 - x_2^2)\kappa - x_1 x_2 \tau \neq 0. \tag{47}$$

For this purpose, let define a unit vector valued function $X(s)$ as $X(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s)$ where $a(s)$, $b(s)$, $c(s)$ and $d(s)$ are differentiable scalar functions of arc-length parameter s. Let the special Frenet curve $\gamma : I \to E^4$ be integral curve of $X(s)$ and also be (0,2)-involute-direction curve of $\alpha(s)$. Then, we have

$$\bar{T}(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s). \tag{48}$$

By assumption, $\bar{T} \perp sp\{T, B_1\}$. Then, taking the inner product of (48) with T and B_1, we have $a(s) = 0$, $c(s) = 0$, respectively, and (48) becomes

$$\bar{T}(s) = b(s)N + d(s)B_2, \;\; b^2(s) + d^2(s) = 1. \tag{49}$$

Now, differentiating the first equation in (49) with respect to s, it follows

$$\bar{\kappa}\bar{N} = -b\kappa T + b'N + (b\tau - d\sigma)B_1 + d'B_2. \tag{50}$$

Taking the inner product of (50) with N and B_2 and considering (44), we get $b' = 0$, $d' = 0$, respectively, i.e., b, d are non-zero constants. Also, we have $b^2 + d^2 = 1$, the first equality in (45).

Now, (50) becomes

$$\bar{\kappa}\bar{N} = -b\kappa T + (b\tau - d\sigma)B_1. \tag{51}$$

From (51), it follows

$$\bar{\kappa} = \sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}. \tag{52}$$

Let define

$$x_1 = \frac{-b\kappa}{\sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}}, \quad x_2 = \frac{b\tau - d\sigma}{\sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}}. \tag{53}$$

Then, (51) becomes

$$\bar{N} = x_1 T + x_2 B_1, \quad x_1^2 + x_2^2 = 1. \tag{54}$$

Since, any Frenet vector of α does not coincide with any Frenet vector of γ, we have $x_1 \neq 0$, $x_2 \neq 0$. Differentiating the first equation in (54) with respect to s, we get

$$-\bar{\kappa}\bar{T} + \bar{\tau}\bar{B}_1 = x_1'T + (x_1\kappa - x_2\tau)N + x_2'B_1 + x_2\sigma B_2. \tag{55}$$

Taking the inner product of (55) with T and B_1, we get $x_1' = 0$, $x_2' = 0$, respectively, i.e., x_1, x_2 are non-zero real constants. Then, from (54), we have the second equality in (45).

Moreover, from (52) and (53), it follows

$$x_1\bar{\kappa} = -b\kappa, \quad x_2\bar{\kappa} = b\tau - d\sigma, \tag{56}$$

which gives us $d\sigma - b\tau = \frac{x_2}{x_1}b\kappa$, we get (46).

Now, writing (49) and (56) in (55) gives

$$\bar{\tau}\bar{B}_1 = \frac{(d^2 - x_2^2)\kappa - x_1x_2\tau}{x_1}N + \frac{-bd\kappa + x_1x_2\sigma}{x_1}B_2. \tag{57}$$

From (46), we get

$$\sigma x_1 d = x_1 b\tau + x_2 b\kappa. \tag{58}$$

Writing (58) in (57) and using (46), we have,

$$\bar{\tau}\bar{B}_1 = \zeta\left(N - \frac{b}{d}B_2\right), \tag{59}$$

where

$$\zeta = \frac{(d^2 - x_2^2)\kappa - x_1x_2\tau}{x_1}. \tag{60}$$

Since $\bar{B}_1 \neq 0$, it should be $(d^2 - x_2^2)\kappa - x_1x_2\tau \neq 0$. Then we have (47).

Conversely, assume that relations (45), (46) and (47) hold for some non-zero constants $b,\ d,\ x_1,\ x_2$ and α be a special Frenet curve with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\kappa,\ \tau,\ \sigma$. Let define a vector valued function

$$X(s) = bN(s) + dB_2(s), \tag{61}$$

and let $\gamma : I \to E^4$ be an integral curve of $X(s)$. We will show that γ is a (0,2)-involute-direction curve of α. Since $\bar{T}(s) = X(s)$, differentiating (61) with respect to s gives

$$\bar{\kappa}\bar{N} = -b\kappa T + (b\tau - d\sigma)B_1. \tag{62}$$

Writing (46) in (62), we have

$$\bar{\kappa}\bar{N} = -b\kappa\left(T + \frac{x_2}{x_1}B_1\right). \tag{63}$$

From (63), it follows

$$\bar{\kappa} = \xi_1\frac{b\kappa}{x_1}, \tag{64}$$

where $\xi_1 = \pm 1$ such that $\bar{\kappa} > 0$. Writing (64) in (63) gives

$$\bar{N} = -\xi_1\left(x_1 T + x_2 B_1\right). \tag{65}$$

By differentiating (65) with respect to s, we get

$$\bar{N}' = -\xi_1\left((x_1\kappa - x_2\tau)N + x_2\sigma B_2\right). \tag{66}$$

Using (61), (64) and (66), we have

$$\bar{N}' + \bar{\kappa}\bar{T} = \frac{\xi_1}{x_1}\big((x_1x_2\tau + (x_2^2 - d^2)\kappa)N + (bd\kappa - x_1x_2\sigma)B_2\big). \tag{67}$$

Writing (46) in (67) and using (45), (67) becomes

$$\bar{N}' + \bar{\kappa}\bar{T} = \xi_1\frac{(x_2^2 - d^2)\kappa + x_1x_2\tau}{x_1}\left(N - \frac{b}{d}B_2\right). \tag{68}$$

From (68) and (47), we have

$$\bar{\tau} = \left\|\bar{N}' + \bar{\kappa}\bar{T}\right\| = \xi_2\frac{(x_2^2 - d^2)\kappa + x_1x_2\tau}{x_1 d} \neq 0, \tag{69}$$

where $\xi_2 = \pm 1$ such that $\bar{\tau} > 0$. Then, we get

$$\bar{B}_1 = \frac{1}{\bar{\tau}}\left(\bar{N}' + \bar{\kappa}\bar{T}\right) = \frac{\xi_1}{\xi_2}\left(dN - bB_2\right). \tag{70}$$

Considering (61), (65) and (70), we can define a unit vector

$$\bar{B}_2 = \frac{1}{\xi_2}\left(-x_2 T + x_1 B_1\right), \tag{71}$$

and the necessary condition $\det(\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2) = 1$ for the Frenet frame holds. Using (70) and (71), we obtain

$$\bar{\sigma} = \left\langle \bar{B}_1', \bar{B}_2\right\rangle = \xi_1\left(dx_2\kappa + x_1(d\tau + b\sigma)\right). \tag{72}$$

If we assume that $\bar{\sigma} = 0$, then we have $x_1(d\tau + b\sigma) = -dx_2\kappa$. Multiplying that with b, we get $x_1(bd\tau + b^2\sigma) = -bdx_2\kappa$. Since $b^2 = 1 - d^2$, the last equality becomes $x_1(-d(d\sigma - b\tau) + \sigma) = -bdx_2\kappa$. Using (46), it follows $x_1\sigma = 0$, which is a contradiction since $x_1 \neq 0$ and α is a special Frenet curve. Then, $\bar{\sigma} \neq 0$, i.e., γ is a special Frenet curve. Consequently, since $b,\ d,\ x_1,\ x_2$ are non-zero constants, from (61), (65), (70) and (71), we get $sp\{T, B_1\} = sp\{\bar{N}, \bar{B}_2\}$ and no Frenet vectors of α and γ coincide. So, we have that γ is (0,2)-involute-direction curve of α.

Furthermore, from Definition 2, the parametric form of γ is $\gamma(s) = \alpha(s) + (c - s)T(s) + kB_1(s)$ where $c,\ k$ are real constants. Differentiating that with respect to s and using the equality $\bar{T} = bN + dB_2$, we have

$$bN + dB_2 = ((c - s)\kappa - k\tau)\,N + k\sigma B_2$$

which gives that

$$\kappa(c - s) = b + k\tau, \ \ k\sigma = d. \tag{73}$$

From (45)-(47) and (73), we have that if the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-evolute-donor curve then there exists a linear relation

$$c_3\kappa + c_4\tau = \sigma \tag{74}$$

where c_3, c_4, σ are non-zero constants and κ, τ, σ are Frenet curvatures of α. From (74), we have that if κ (or respectively τ) is constant, then τ (or respectively κ) must be constant. But considering (73), it follows if the first curvature κ (or respectively τ) is constant, then τ (or respectively κ) is always non-constant which is a contradiction and that finishes the proof. □

5. Conclusions

There is no Bertrand curves in E^4 given by the classical definition that Bertrand curves have common principal normal lines. Then, a new type of Bertrand curves have been introduced in [12] and called (1,3)-Bertrand curves. We considered this definition with integral curves and define (1,3)-Bertrand-direction curves and (1,3)-Bertrand-donor curves. Necessary and sufficient conditions for a curve to be a (1,3)-Bertrand-donor curve have been introduced. Moreover, we investigated whether (1,3)-evolute-donor curves in E^4 exist and show that there is no (1,3)-evolute-donor curve in E^4.

Declaration of Competing Interests The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Bertrand, J., Mémoire sur la théorie des courbes à double courbure, Comptes Rendus 36, *Journal de Mathématiques Pures et Appliquées.*, 15 (1850), 332-350.

[2] Choi, J.H., Kim, Y.H., Associated curves of a Frenet curve and their applications, *Applied Mathematics and Computation*, 218 (2012), 9116-9124. https://doi.org/10.1016/j.amc.2012.02.064

[3] Fuchs, D., Evolutes and involutes of spatial curves, *American Mathematical Monthly*, 120(3) (2013), 217-231. https://doi.org/10.4169/amer.math.monthly.120.03.217

[4] Fukunaga, T., Takahashi, M., Evolutes and involutes of frontals in the euclidean plane, *Demonstratio Mathematica*, 48(2) (2015), 147-166. https://doi.org/10.1515/dema-2015-0015

[5] Fukunaga, T., Takahashi, M., Involutes of fronts in the Euclidean plane, *Beitrage zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 57(3) (2016), 637-653. https://doi.org/10.1007/s13366-015-0275-1

[6] Gere, B.H., Zupnik, D., On the construction of curves of constant width, *Studies in Applied Mathematics*, 22(1-4) (1943), 31-36.

[7] Hanif, M., Hou, Z.H., Generalized involute and evolute curve-couple in Euclidean space, *Int. J. Open Problems Compt. Math.*, 11(2) (2018), 28-39.

[8] Huygens, C., Horologium oscillatorium sive de motu pendulorum ad horologia aptato, *Demonstrationes Geometricae*, 1673.

[9] Li, Y., Sun, G.Y., Evolutes of fronts in the Minkowski Plane, *Mathematical Methods in the Applied Science*, 42(16) 2018, 5416-5426. https://doi.org/10.1002/mma.5402

[10] Macit, N., Düldül, M., Some new associated curves of a Frenet curve in E^3 and E^4, *Turk J Math.*, 38 (2014), 1023-1037. https://doi.org/10.3906/mat-1401-85

[11] Matsuda, H., Yorozu, S., On generalized Mannheim curves in Euclidean 4-space, *Nihonkai Math. J.*, 20 (2009), 33-56.

[12] Matsuda, H., Yorozu, S., Notes on Bertrand curves, *Yokohama Mathematical Journal*, 50 (2003), 41-58.

[13] Nutbourne, A.W., Martin, R.R., Differential Geometry Applied to Design of Curves and Surfaces, Ellis Horwood, Chichester, UK, 1988.

[14] Önder, M., Construction of curve pairs and their applications, *Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 91(1) 2021, 21-28. https://doi.org/10.1007/s40010-019-00643-2

[15] Öztürk, G., Arslan, K., Bulca, B., A Characterization of involutes and evolutes of a given curve in E^n. *Kyungpook Math. J.*, 58 (2018), 117-135.

[16] Özyılmaz, E., Yılmaz, S., Involute-evolute curve couples in the Euclidean 4-space, *Int. J. Open Problems Compt. Math.*, 2(2) (2009), 168-174.

[17] Struik, D.J., Lectures on Classical Differential Geometry, 2^{nd} ed. Addison Wesley, Dover, 1988.

[18] Yu, H., Pei, D., Cui, X., Evolutes of fronts on Euclidean 2-sphere, *J. Nonlinear Sci. Appl.*, 8 (2015), 678-686. http://dx.doi.org/10.22436/jnsa.008.05.20