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SKEW *ABC* ENERGY OF DIGRAPHS

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Abstract. In this paper, skew ABC matrix and its energy are introduced for digraphs. Firstly, some fundamental spectral features of the skew ABC matrix of digraphs are established. Then some upper and lower bounds are presented for the skew ABC energy of digraphs. Further extremal digraphs are determined attaining these bounds.

1. Introduction and Preliminaries

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. An edge joining vertices v_i and v_j is denoted by $v_iv_j \in E(G)$ and degree of a vertex v_i is denoted by d_i. The atom-bond connectivity index ABC of G is introduced by Estrada et al. [6] as

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}},$$

which is a significant predictive index in the studies about the heat of formation in alkanes (see [8]- [6]), for further information about mathematical and chemical applications about atom-bond connectivity index, also see ([9]- [11]- [13]- [15]- [27]). The concept of graph energy is defined as sum of the absolute values of the eigenvalues of a graph by Gutman [16]. The energy of a graph has been widely studied by many mathematicians and chemists, as it has close links with chemistry (see [17]). So, several kinds of graph energy are introduced and examined such as Laplacian energy, Randić energy, distance energy, Zagreb energy, etc.

Estrada [7] defined the generalized ABC matrix $S_\alpha(G) = \left(s_{ij}^\alpha\right)$ of order n, where the $(i,j)-th$ entry is $\left(\frac{d_i+d_j-2}{d_id_j}\right)^\alpha$, if $v_iv_j \in E(G)$ and 0, otherwise. If $\alpha = \frac{1}{2}$, the

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generalized ABC matrix is called as ABC matrix of a graph and will be denoted by $\Omega(G)$. Let ς_i be the eigenvalues of $\Omega(G)$ (also called ABC eigenvalues of G). ABC energy of a graph is defined by $E\Omega(G) = \sum_{i=1}^{n} |\varsigma_i|$. As $\Omega(G)$ is a real symmetric matrix, the ABC eigenvalues of G are real numbers. Recently, some bounds have presented for the ABC eigenvalues and ABC energy of graphs by Chen [5] and Ghorbani et al. [12].

Let $\vec{G}$ be a digraph with vertex set $V(\vec{G}) = \{v_1, v_2, ..., v_n\}$ and arc set $\Gamma(\vec{G})$. An arc from v_i to v_j is denoted by $v_i \to v_j$. Throughout this paper, all the digraphs are simple and do not have loops and if there is an arc from v_i to v_j, then there is not an arc from v_j to v_i. Hence, a digraph $\vec{G}$ without orientation gives the underlying graph G is simple.

Graph energy concept is extended to digraphs in [22]. Then the skew Laplacian energy of a digraph is defined by Adiga et al. [3] and new definitions are proposed for the skew Laplacian energy (see [2]- [4]). The skew energy of a digraph is defined by Adiga et al. [1] as $ES(\vec{G}) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of the skew adjacency matrix $S(\vec{G})$ of order n. Let $S(\vec{G})=(s_{ij})$, where the $(i,j)-th$ entry is 1, if $v_i \to v_j$; -1, if $v_j \to v_i$ and 0, otherwise. Since λ_i $(1 \leq i \leq n)$ are purely imaginary numbers, the singular values of $S(\vec{G})$ equal to the absolute values of λ_i. For recent studies about kinds of skew energy, also see the survey in [21] and the references therein.

The Randić index is introduced as "branching index" by $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$ in [24]. The general Randić index of a graph is defined by $R_\gamma(G) = \sum_{uv \in E(G)} (d_u d_v)^\gamma$ in [20]. R_{-1} is called as modified second Zagreb index. The skew Randić energy of a digraph is introduced by Gu et al. [14] and some bounds are presented for this energy kind. Inspired by the studies of the skew energy kinds of graphs, we will introduce skew ABC matrix of a digraph and its energy.

Skew ABC matrix of a simple digraph $\vec{G}$ is $\Omega_s=\Omega_s(\vec{G})=(b_{ij})$ of order n and we define the $(i,j)-th$ entry of Ω_s as

$$b_{ij} = \begin{cases} \left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}} & \text{if } v_i \to v_j \\ -\left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}} & \text{if } v_j \to v_i \\ 0 & \text{otherwise,} \end{cases}$$

where d_i and d_j are the degrees of the corresponding vertices in the underlying graph G. The skew ABC matrix of a simple digraph can be considered as a weighted skew adjacency matrix with $\left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}}$ weights.

Let $\{\vartheta_1, \vartheta_2, ..., \vartheta_n\}$ be eigenvalues of the skew ABC matrix of $\vec{G}$, namely be skew ABC eigenvalues. Since $\Omega_s(\vec{G})$ is a skew symmetric matrix, the skew ABC eigenvalues are purely imaginary numbers. We can define skew ABC energy of a digraph as

$$E\Omega_s(\vec{G}) = \sum_{j=1}^{n} |\vartheta_j| .$$

This paper is only concerned with the mathematical aspects of the skew ABC energy of digraphs. The rest of the paper is composed of two sections. In the next section, the spectral features of the skew ABC matrix of digraphs are presented. In the last section, some upper and lower bounds are obtained for the skew energy and the extremal digraphs are determined attaining these bounds.

2. Skew ABC Eigenvalues

In this section we consider some fundamental spectral properties of the skew ABC matrix of digraphs.

Proposition 1. *Let $\vec{G}$ be a digraph of order n with no isolated vertices. If $\phi(\vec{G};\vartheta) = \det(\vartheta I_n - \Omega_S) = c_0\vartheta^n + c_1\vartheta^{n-1} + ... + c_n$ is the characteristic polynomial of $\Omega_s(\vec{G})$, then*
(i) $c_0 = 1$, $c_1 = 0$,
(ii) $c_2 = n - 2R_{-1}(G)$,
(iii) $c_j = 0$, *for all odd* j.

Proof. (i) Let $tr(.)$ stands for trace of a matrix. Obviously we have $c_0 = 1$ and $c_1 = \sum_{j=1}^{n} \vartheta_j = tr(\Omega_s) = 0$.

(ii) c_2 equals to the sum of the determinants of all 2×2 principal submatrices of $\Omega_s(\vec{G})$, thus

$$\begin{aligned} c_2 &= \sum_{j<k} \det\begin{pmatrix} 0 & b_{jk} \\ b_{kj} & 0 \end{pmatrix} = \sum_{j<k} -b_{jk}b_{kj} = \sum_{j<k} (b_{jk})^2 = \sum_{v_jv_k \in E(G)} \frac{d_j + d_k - 2}{d_jd_k} \\ &= \sum_{v_jv_k \in E(G)} \frac{d_j + d_k}{d_jd_k} - 2\sum_{v_jv_k \in E(G)} \frac{1}{d_jd_k} \\ &= n - 2R_{-1}(G), \end{aligned}$$

where $R_{-1}(G) = \sum_{uv \in E(G)} \frac{1}{d_ud_v}$.

(iii) Let j be odd. c_j equals to the sum of the determinants of all $j \times j$ principal submatrices of $\Omega_s(\vec{G})$ is 0 as a principal submatrix of a skew symmetric matrix is skew symmetric. □

Proposition 2. *Let $\vec{G}$ be a digraph of order $n(\geq 3)$ with no isolated vertices and $\{i\vartheta_1, i\vartheta_2, ..., i\vartheta_n\}$ be the skew ABC eigenvalues of $\vec{G}$ such that $\vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n$. Then*

(i) $\vartheta_j = -\vartheta_{n+1-j}$ for all $1 \leq j \leq n$. If n is even, then $\vartheta_{\frac{n}{2}} \geq 0$ and if n is odd, then $\vartheta_{\frac{n+1}{2}} = 0$.

(ii) $\sum_{j=1}^{n} |\vartheta_j|^2 = 2\left(n - 2R_{-1}(G)\right).$

Proof. (i) The proof is clear.

(ii) Obviously we have

$$\sum_{j=1}^{n} (i\vartheta_j)^2 = tr((\Omega_s)^2) = \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk}b_{kj} = -\sum_{j=1}^{n}\sum_{k=1}^{n} (b_{jk})^2 = -2\left(n - 2R_{-1}(G)\right),$$

which completes the proof. □

From Proposition 1 and Proposition 2, we also have

$$\sum_{1\leq i<j\leq n} \vartheta_i\vartheta_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n}\vartheta_i\right)^2 - \sum_{i=1}^{n}\vartheta_i^2\right] = 2R_{-1}(G) - n.$$

$Sp\left(\Omega_s(\vec{G})\right)$ denotes the skew ABC spectrum of $\vec{G}$ which is a multiset consist of eigenvalues (with multiplicities) of $\Omega_s(\vec{G})$. Also, $Sp\left(\Omega(G)\right)$ is the ABC spectrum of the underlying graph G.

Example 1. *Let $\vec{C_4}$ be a directed cycle of order 4 with the arc set $\{(1,2), (2,3), (3,4), (4,1)\}$. The skew ABC spectrum of $\vec{G}$ is $Sp\left(\Omega_s(\vec{C_4})\right) = \{-\frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2}, \frac{1}{2}i\sqrt{2}, \frac{1}{2}\sqrt{2}\}$ and the skew ABC energy of $\vec{C_4}$ is $E\Omega_s(\vec{C_4}) = 2\sqrt{2}$. Consider the underlying graph C_4. The ABC spectrum of C_4 is $Sp\left(\Omega(C_4)\right) = \{-\sqrt{2}, 0^{(2)}, \sqrt{2}\}$. Hence, $E\Omega_s(\vec{C_4}) = E\Omega(C_4)$.*

Example 2. *Let $\vec{K}_{p,q}$ $(p, q \neq 1)$ be a complete bipartite digraph in which the vertex set is a disjoint union $A \cup B$ with $|A| = p$ and $|B| = q$. Consider the elementary orientation that is, orienting all the edges from A to B and writing the elements of A firstly, form the matrix $\Omega_s(\vec{K}_{p,q}) = \begin{pmatrix} 0_p & \beta J_{p\times q} \\ -\beta J_{q\times p} & 0_q \end{pmatrix}$, where $\beta = \sqrt{\frac{p+q-2}{pq}}$ and J_n is the order n matrix with all entries are 1.*

$$\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) = \det\begin{pmatrix} \vartheta I_p & -\beta J_{p\times q} \\ \beta J_{q\times p} & \vartheta I_q \end{pmatrix}.$$

Since ϑI_p is nonsingular, then

$$\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) = \det(\vartheta I_p)\det\left(\vartheta I_q + \beta J_{q\times p}(\vartheta I_p)^{-1}\beta J_{p\times q}\right)$$

$$= \det(\vartheta I_p)\det\left(\vartheta I_q + \beta J_{q\times p}\frac{1}{\vartheta}I_p\beta J_{p\times q}\right),$$

(see [23]). Recall $J_{q\times p}J_{p\times q} = pJ_q$, *thus*

$$\begin{aligned}\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) &= \vartheta^p \det\left(\vartheta I_q + \frac{\beta^2}{\vartheta}pJ_q\right)\\ &= \vartheta^{p-q}\det\left(\vartheta^2 I_q + \beta^2 pJ_q\right).\end{aligned}$$

$\beta^2 pJ_q$ *has eigenvalues* $\beta^2 pq$ *of multiplicity* 1 *and* 0 *of multiplicity* $q-1$, *since* $Sp(J_q) = \{q, 0^{(q-1)}\}$. *Then*

$$\phi\left(\Omega_s(\vec{K}_{p,q});\vartheta\right) = \vartheta^{p+q-2}\left(\vartheta^2 + \beta^2 pq\right),$$

and $\Omega_s(\vec{K}_{p,q})$ *has eigenvalues* $-\beta\sqrt{pq}i$, $\beta\sqrt{pq}i$ *and* 0 *of multiplicity* $p+q-2$, *i.e.,* $\sqrt{p+q-2}i$, $-\sqrt{p+q-2}i$ *and* 0 *of multiplicity* $p+q-2$, *hence*

$$E\Omega_s(\vec{K}_{p,q}) = 2\sqrt{p+q-2},$$

and $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = \left\{-\sqrt{p+q-2}i,\ 0^{(n-2)}, \sqrt{p+q-2}i\right\}$. *It is seen that there is an orientation such that* $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = iSp(\Omega(K_{p,q}))$. *Orienting all the edges from* B *to* A *and writing the elements of* B *firstly, form the matrix* $\Omega_s\left(\vec{K}_{p,q}\right) = \begin{pmatrix} 0_q & \beta J_{q\times p} \\ -\beta J_{p\times q} & 0_p\end{pmatrix}$. *Obviously, carrying out the process above gives the same skew* ABC *eigenvalues.*

The relationship between the skew spectrum of a digraph and spectrum of its underlying graph is firstly analyzed in [25]. By Example 2, it is concluded that there is an orientation such that $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = iSp\left(\Omega\left(K_{p,q}\right)\right)$. An analogous relation that can be seen in Theorem 1, exists between the skew ABC spectrum and ABC spectrum.

Lemma 1 ([25])**.** *If* $A = \begin{pmatrix} 0 & Y \\ Y^T & 0\end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & Y \\ -Y^T & 0\end{pmatrix}$ *are two real matrices, then* $Sp(B) = iSp(A)$.

Theorem 1. *G is a bipartite graph if and only if there is an orientation such that* $Sp\left(\Omega_s(\vec{G})\right) = iSp(\Omega(G))$.

Proof. If G is bipartite, then by suitable labelling the vertices, the ABC matrix of G takes the form $\Omega(G) = \begin{pmatrix} 0 & Y \\ Y^T & 0\end{pmatrix}$. Let $\vec{G}$ be an orientation such that the skew ABC matrix is of the form $\Omega_s(\vec{G}) = \begin{pmatrix} 0 & Y \\ -Y^T & 0\end{pmatrix}$. By Lemma 1, the proof is obvious.

Conversely, assume that $Sp\left(\Omega_s(\vec{G})\right)=iSp(\Omega(G))$ for some orientation. As $\Omega_s(\vec{G})$ is a real skew symmetric matrix, $Sp\left(\Omega_s(\vec{G})\right)$ has only pure imaginary eigenvalues, thus the skew ABC eigenvalues are symmetric with respect to the real axis. Hence, $Sp\left(\Omega_s(\vec{G})\right) = -iSp(\Omega(G))$ is symmetric about the imaginary axis. So, G is bipartite. □

3. Bounds for the Skew ABC Energy

In this section, we intend to obtain bounds for the skew ABC energy of digraphs by using the mathematical inequalities and properties of the skew ABC eigenvalues and examine the equality case of these bounds. In recent studies, many bounds are presented for $R_{-1}(G)$. Using these bounds, one can also obtain different bounds for the skew ABC energy of digraphs by combining the bounds will be presented in this section. Now, we consider the bounds for $R_{-1}(G)$ in [19] and [26]. Throughout this section, it is assumed that $\{i\vartheta_1, i\vartheta_2, ..., i\vartheta_n\}$ be the skew ABC eigenvalues of $\vec{G}$ with $\vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n$. Moreover K_n denotes the complete graph of order n and $G = \left(\frac{n}{2}\right) K_2$ stands for the vertex-disjoint union of $\frac{n}{2}$ copies of K_2.

Theorem 2 ([26]). *If G is a graph of order $n(\geq 2)$ with no isolated vertices with maximum vertex degree Δ and minimum vertex degree δ, then*

$$\frac{n}{2\Delta} \leq R_{-1}(G) \leq \frac{n}{2\delta}, \tag{1}$$

with equality if and only if G is regular.

Theorem 3 ([19]). *If G is a graph of order n with no isolated vertices, then*

$$\frac{n}{2(n-1)} \leq R_{-1}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}$$

Equality in lower bound holds if and only if $G = K_n$. Equality in upper bound holds if and only if either (i) $G = \left(\frac{n}{2}\right) K_2$ when n is even or (ii) $G = K_{1,2} \cup \frac{n-3}{2} K_2$ when n is odd.

Initially, we can give the following upper bound involving $R_{-1}(G)$ and n for the skew ABC energy of digraphs.

Theorem 4. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices, then*

$$E\Omega_s(\vec{G}) \leq \sqrt{2n(n - 2R_{-1}(G))}. \tag{3}$$

with equality if $|\vartheta_i| = |\vartheta_j|$ for all $1 \leq i \neq j \leq n$.

Proof. Applying Cauchy-Schwarz inequality and using Proposition 2 yields

$$E\Omega_s(\vec{G}) = \sum_{i=1}^{n} |\vartheta_i| \leq \sqrt{\sum_{i=1}^{n} |\vartheta_i|^2}\sqrt{n} \tag{4}$$

$$= \sqrt{2n\left(n-2R_{-1}(G)\right)}.$$

Equality case is obvious from the equality in (4). □

Using the lower bound of (1) in (3), we can obtain a new upper bound in terms of n and Δ as follows.

Corollary 1. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ and $\Delta\,(\geq 1)$ is the maximum vertex degree of the underlying graph G, then*

$$E\Omega_s\left(\vec{G}\right) \leq n\sqrt{2\left(1-\frac{1}{\Delta}\right)}, \qquad (5)$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K_2}$.

Proof. From (1) and (3), clearly we get $E\Omega_s\left(\vec{G}\right) \leq \sqrt{2n\left(n-\frac{n}{\Delta}\right)}$, so the proof is obvious. We will focus on the equality case. Equality holds in (5) if and only if equality holds in (4), namely $|\vartheta_i| = |\vartheta_j|$ for all $1 \leq i \neq j \leq n$ and G is regular. Thus $\vartheta_1=\vartheta_2 = ... = \vartheta_n = 0$ that is, $\Omega_s\left(\vec{G}\right) = 0$ and we have n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K_2}$, for an arbitrary orientation. □

The following bound presents a relationship between the skew ABC energy of a digraph and ABC energy of complete graph K_n.

Corollary 2. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices, then*

$$E\Omega_s\left(\vec{G}\right) \leq \left(\frac{n}{2\sqrt{n-1}}\right) E\Omega\left(K_n\right). \qquad (6)$$

Proof. If $G = K_n$, then K_n has two distinct ABC eigenvalues such that $\sqrt{2n-4}$ of multiplicity 1 and $-\frac{\sqrt{2n-4}}{n-1}$ of multiplicity $n-1$ (see Proposition 3.1, [5]). Then $E\Omega\left(K_n\right) = 2\sqrt{2n-4}$. Using this fact with (2) and (3)

$$\begin{aligned} E\Omega_s\left(\vec{G}\right) &\leq \sqrt{2n\left(n-2R_{-1}(G)\right)} \\ &\leq \sqrt{2n\left(\frac{n^2-2n}{n-1}\right)} \\ &= \frac{n}{\sqrt{n-1}}\sqrt{2n-4} \\ &= \left(\frac{n}{2\sqrt{n-1}}\right) E\Omega\left(K_n\right) \end{aligned}$$

yields the result. □

The following theorem presents a new upper and lower bound in terms of $\det\left(\Omega_s\left(\vec{G}\right)\right)$, $R_{-1}(G)$ and n.

Theorem 5. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices and $p = det\left(\Omega_s(\vec{G})\right)$, then*

$$\sqrt{2\left(n-2R_{-1}\left(G\right)\right)+n\left(n-1\right)p^{\frac{2}{n}}} \leq E\Omega_s(\vec{G}) \leq \sqrt{2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}}}, \tag{7}$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$.

Proof. Recall the arithmetic-geometric mean inequality in [18], where $x_1, x_2, ..., x_n$ are non-negative numbers and

$$\begin{aligned} n\left[\frac{1}{n}\sum_{j=1}^{n}x_j - \left(\prod_{j=1}^{n}x_j\right)^{\frac{1}{n}}\right] &\leq n\sum_{j=1}^{n}x_j - \left(\sum_{j=1}^{n}\sqrt{x_j}\right)^2 \qquad (8) \\ &\leq n\left(n-1\right)\left[\frac{1}{n}\sum_{j=1}^{n}x_j - \left(\prod_{j=1}^{n}x_j\right)^{\frac{1}{n}}\right], \end{aligned}$$

with equality if and only if $x_1 = x_2 = ... = x_n$. Choosing $x_j = |\vartheta_j|^2$ in (8) yields

$$nK \leq n\sum_{j=1}^{n}|\vartheta_j|^2 - \left(\sum_{j=1}^{n}|\vartheta_j|\right)^2 \leq n\left(n-1\right)K,$$

where $K = \frac{1}{n}\sum_{j=1}^{n}|\vartheta_j|^2 - \left(\prod_{j=1}^{n}|\vartheta_j|^2\right)^{\frac{1}{n}}$. Hence

$$nK \leq 2n\left(n-2R_{-1}\left(G\right)\right) - \left(E\Omega_s(\vec{G})\right)^2 \leq n\left(n-1\right)K. \tag{9}$$

From Proposition 2, we have $K = \frac{1}{n}\left[2\left(n-2R_{-1}\left(G\right)\right)\right] - p^{\frac{2}{n}}$, where $p = det\left(\Omega_s(\vec{G})\right)$. From the left hand side of (9), we obtain

$$\left(E\Omega_s(\vec{G})\right)^2 \leq 2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}},$$

i.e.,

$$E\Omega_s(\vec{G}) \leq \sqrt{2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}}}.$$

From the right hand side of (9)

$$2n\left(n-2R_{-1}\left(G\right)\right) - n\left(n-1\right)K \leq \left(E\Omega_s(\vec{G})\right)^2.$$

As $n\left(n-1\right)K = 2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right) - n\left(n-1\right)p^{\frac{2}{n}}$, we have

$$E\Omega_s(\vec{G}) \geq \sqrt{2\left(n-2R_{-1}\left(G\right)\right)+n\left(n-1\right)p^{\frac{2}{n}}}.$$

Note that if n is odd, then $p=0$. Consequently, we have

$$\sqrt{2\left(n-2R_{-1}\left(G\right)\right)} \leq E\Omega_s\left(\overrightarrow{G}\right) \leq \sqrt{2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)}.$$

The equality holds in (7) if and only if $\left|\vartheta_1\right|^2 = \left|\vartheta_2\right|^2 = ... = \left|\vartheta_n\right|^2$. Thus $\vartheta_1 = \vartheta_2 = ... = \vartheta_n = 0$. So, $\Omega_s\left(\overrightarrow{G}\right) = 0$ and we have n is even and $\overrightarrow{G} = \left(\frac{n}{2}\right)\vec{K}_2$ for an arbitrary orientation. □

Lemma 2 ([10])**.** *If $x_1, x_2, ..., x_n \geq 0$ and $r_1, r_2, ..., r_n \geq 0$ such that $\sum\limits_{j=1}^{n} r_j = 1$, then*

$$\sum_{j=1}^{n} x_j r_j - \prod_{j=1}^{n} x_j^{r_j} \geq nr\left(\frac{1}{n}\sum_{j=1}^{n} x_j - \prod_{j=1}^{n} x_j^{\frac{1}{n}}\right), \tag{10}$$

where $r = \min\left\{r_1, r_2, ..., r_n\right\}$. Equality holds if and only if $x_1 = x_2 = ... = x_n$.

Finally, we give a new lower bound involving $\det\left(\Omega_s\left(\overrightarrow{G}\right)\right)$, $\left|\vartheta_1\right|$ and n.

Theorem 6. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices and $p = det\left(\Omega_s\left(\vec{G}\right)\right)$, then*

$$E\Omega_s\left(\vec{G}\right) \geq \left|\vartheta_1\right| + 2\left(n-1\right)\left[\frac{p^{\frac{2n-1}{2n(n-1)}}}{\left|\vartheta_1\right|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}}\right] \tag{11}$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$.

Proof. Setting $x_j = \left|\vartheta_j\right|$ for $j = 1, 2, ..., n$, $r_1 = \frac{1}{2n}$, $r_j = \frac{2n-1}{2n(n-1)}$ for $j = 2, ..., n$ and $r = \frac{1}{2n}$ in (10), we obtain

$$\begin{aligned} &\left(\frac{\left|\vartheta_1\right|}{2n} + \frac{2n-1}{2n\left(n-1\right)}\sum_{j=2}^{n}\left|\vartheta_j\right|\right) - \left|\vartheta_1\right|^{\frac{1}{2n}}\prod_{j=2}^{n}\left|\vartheta_j\right|^{\frac{2n-1}{2n(n-1)}} \\ \geq\ & n\frac{1}{2n}\left(\frac{1}{n}\sum_{j=1}^{n}\left|\vartheta_j\right| - \prod_{j=1}^{n}\left|\vartheta_j\right|^{\frac{1}{n}}\right) \\ =\ & \frac{1}{2n}E\Omega_s\left(\vec{G}\right) - \frac{1}{2}p^{\frac{1}{n}}. \end{aligned}$$

Note that $\left|\vartheta_1\right|^{\frac{1}{2n}}\prod\limits_{j=2}^{n}\left|\vartheta_j\right|^{\frac{2n-1}{2n(n-1)}} = \left|\vartheta_1\right|^{-\frac{1}{2(n-1)}}\prod\limits_{j=1}^{n}\left|\vartheta_j\right|^{\frac{2n-1}{2n(n-1)}} = \frac{p^{\frac{2n-1}{2n(n-1)}}}{\left|\vartheta_1\right|^{\frac{1}{2(n-1)}}}$ and $\sum\limits_{j=2}^{n}\left|\vartheta_j\right| = E\Omega_s\left(\vec{G}\right) - \left|\vartheta_1\right|$, thus

$$\left[\frac{1}{2n} - \frac{2n-1}{2n\left(n-1\right)}\right]\left|\vartheta_1\right| + \left[\frac{2n-1}{2n\left(n-1\right)} - \frac{1}{2n}\right]E\Omega_s\left(\vec{G}\right) \geq \frac{p^{\frac{2n-1}{2n(n-1)}}}{\left|\vartheta_1\right|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}},$$

then

$$-\frac{1}{2\left(n-1\right)}\left|\vartheta_{1}\right|+\frac{1}{2\left(n-1\right)}E\Omega_{s}\big(\vec{G}\big)\geq\frac{p^{\frac{2n-1}{2n(n-1)}}}{\left|\vartheta_{1}\right|^{\frac{1}{2(n-1)}}}-\frac{1}{2}p^{\frac{1}{n}}.$$

Hence, we have

$$E\Omega_{s}\big(\vec{G}\big)\geq\left|\vartheta_{1}\right|+2\left(n-1\right)\left[\frac{p^{\frac{2n-1}{2n(n-1)}}}{\left|\vartheta_{1}\right|^{\frac{1}{2(n-1)}}}-\frac{1}{2}p^{\frac{1}{n}}\right].$$

If n is odd, then $E\Omega_s\left(\vec{G}\right)\geq\left|\vartheta_1\right|$. The equality holds in (11) if and only if $|\vartheta_1| = |\vartheta_2|=...=|\vartheta_n|$, then $\vartheta_1=\vartheta_2 = ... = \vartheta_n = 0$. So, we have $p = 0$ and $\Omega_s\left(\vec{G}\right) = 0$, that is, n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$ for an arbitrary orientation. □

Conclusion

In recent studies, the *ABC* matrix and *ABC* energy of graphs have introduced. This paper expands these concepts to skew *ABC* matrix and skew *ABC* energy of digraphs. The skew *ABC* matrix of a digraph is defined and its spectral features are established. Further, some upper and lower bounds for the skew *ABC* energy of digraphs are presented with extremal digraphs attaining these bounds.

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