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Existence of Warped Product Submanifolds of Almost Hermitian Manifolds

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ABSTRACT

This paper has two goals; the first is to generalize results for the existence and nonexistence of warped product submanifolds of almost Hermitian manifolds, accordingly a self-contained reference of such submanifolds is offered to save efforts of other researchers, which is the second goal. At the end of the paper a list of warped products is tabulated whether exist or not. Moreover, a discrete example of CR-warped product submanifold in Kaehler manifold is constructed. For further research direction, we addressed a couple of open problems arose from the results of this paper.

Keywords: *CR-warped products, Kaehler, nearly Kaehler, general warped product, doubly warped product, second fundamental form, totally geodesic.*
AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C40; 53C42; 53B25.*

1. Introduction

Warped products have been playing some important roles in the theory of general relativity as they have been providing the best mathematical models of our universe for now; that is, the warped product scheme was successfully applied in general relativity and semi-Riemannian geometry in order to build basic cosmological models for the universe. For instance, the Robertson-Walker spacetime, the Friedmann cosmological models and the standard static spacetime are given as warped product manifolds. For more cosmological applications, warped product manifolds provide excellent setting to model spacetime near black holes or bodies with large gravitational force. For example, the relativistic model of the Schwarzschild spacetime that describes the outer space around a massive star or a black hole admits a warped product construction [16].

In an attempt to construct manifolds of negative curvatures, R.L. Bishop and O'Neill [3] introduced the notion of *warped product manifolds* as follows: Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_{N_1} and g_{N_2}, respectively, and $f > 0$ a C^∞ function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \mapsto N_1$ and $\pi_2 : N_1 \times N_2 \mapsto N_2$. Then, the *warped product* $\tilde{M}^m = N_1 \times_f N_2$ is the Riemannian manifold $N_1 \times N_2 = (N_1 \times N_2, \tilde{g})$ equipped with a Riemannian structure such that $\tilde{g} = g_{N_1} + f^2 g_{N_2}$.

A warped product manifold $\tilde{M}^m = N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. For a nontrivial warped product $N_1 \times_f N_2$, we denote by $\mathfrak{D}_1$ and $\mathfrak{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathfrak{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathfrak{D}_2$ is obtained by tangent vectors of N_2 via the vertical lift.

Since our goal to search about existence and nonexistence of warped product submanifolds in almost Hermitian manifolds, we hypothesize the following two problems. The first is for single warped products

Problem 1. *Prove existence or nonexistence of single warped product submanifolds of almost Hermitian manifolds.*

The second problem is for doubly warped products

Problem 2. *Prove existence or nonexistence of doubly warped product submanifolds of almost Hermitian manifolds.*

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The present paper is organized as follows: After the introduction, we present in Section 2, the preliminaries, basic definitions and formulas. In Section 3, we provide basic results, which are necessary and useful to the next section. In Section 4, we generalize theorems for existence and nonexistence warped product submanifolds for single and doubly warped product submanifolds in almost hermitian manifolds. In Section 5, we discuss the CR-warped product submaifolds and generic warped products in Kaehler manifolds and construct an example and a table summarizing the main results of the paper. In the final section, we address two open problems related to the obtained results in this paper.

2. Preliminaries

At first, let us recall the following important two facts regarding Riemannian submanifolds, [10].

Definition 2.1. Let M^n and $\tilde{M}^m$ be differentiable manifolds. A differentiable mapping $\varphi : M^n \longrightarrow \tilde{M}^m$ is said to be an *immersion* if $d\varphi_x : T_xM^n \to T_{\varphi(x)}\tilde{M}^m$ is injective for all $x \in M^n$. If, in addition, φ is a homeomorphism onto $\varphi(M^n) \subset \tilde{M}^m$, where $\varphi(M^n)$ has the subspace topology induced from $\tilde{M}^m$, we say that φ is an *embedding*. If $M^n \subset \tilde{M}^m$ and the inclusion $\boldsymbol{i} : M^n \subset \tilde{M}^m$ is an embedding, we say that M^n is a submanifold of $\tilde{M}^m$.

It can be seen that if $\varphi : M^n \to \tilde{M}^m$ is an immersion, then $n \leq m$; the difference $m - n$ is called the *codimension* of the immersion φ.

For most local questions of geometry, it is the same to work with either immersions or embeddings. This comes from the following proposition which shows that every immersion is locally (in a certain sense) an embedding.

Proposition 2.1. *Let $\varphi : M^n \longrightarrow \tilde{M}^m$, $n \leq m$, be an immersion of the differentiable manifold M^n into the differentiable manifold $\tilde{M}^m$. For every point $x \in M^n$, there exists a neighborhood $\mathfrak{u}$ of x such that the restriction $\varphi|\mathfrak{u} \to \tilde{M}^m$ is an embedding.*

Now, we turn our attention to the differential geometry of the submanifold theory. First, let M^n be n-dimensional Riemannian manifold isometrically immersed in an m-dimensional Riemannian manifold $\tilde{M}^m$. Since we are dealing with a local study, then, by Proposition 2.1, we may assume that M^n is embedded in $\tilde{M}^m$. On this infinitesimal scale, Definition 2.1 guarantees that M^n is a *Riemannian submanifold* of some nearby points in $\tilde{M}^m$ with induced Riemannian metric g. Then, *Gauss* and *Weingarten formulas* are, respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\tilde{\nabla}_X \zeta = -A_\zeta X + \nabla^\perp_X \zeta \tag{2.2}$$

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^\perp M^n)$, where $\tilde{\nabla}$ and ∇ denote respectively the Levi-Civita and the *induced* Levi-Civita connections on $\tilde{M}^m$ and M^n, and $\Gamma(TM^n)$ is the module of differentiable sections of the vector bundle TM^n. $\nabla^\perp$ is the *normal connection* acting on the normal bundle $T^\perp M^n$.

Here, g denotes the *induced Riemannian metric* from $\tilde{g}$ on M^n. For simplicity's sake, the inner products which are carried by g, $\tilde{g}$ or any other induced Riemannian metric are performed via g. However, most of the inner products which will be applied in this thesis are equipped with g, other situations are rarely considered.

Here, it is well-known that the *second fundamental form* h and the *shape operator* A_ζ of M^n are related by

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta) \tag{2.3}$$

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^\perp M^n)$, [2], [16].

Geometrically, M^n is called a *totally geodesic* submanifold in $\tilde{M}^m$ if h vanishes identically. Particularly, the *relative null space*, $\mathcal{N}_x$, of the submanifold M^n in the Riemannian manifold $\tilde{M}^m$ is defined at a point $x \in M^n$ by [5] as

$$\mathcal{N}_x = \{X \in T_xM^n : h(X, Y) = 0 \quad \forall\, Y \in T_xM^n\}. \tag{2.4}$$

In a different line of thought, and for any $X \in \Gamma(TM^n)$, $\zeta \in \Gamma(T^\perp M^n)$ and a $(1, 1)$ tensor field ψ on $\tilde{M}^m$, we write

$$\psi X = PX + FX, \tag{2.5}$$

and

$$\psi N = t\zeta + f\zeta, \tag{2.6}$$

where PX, $t\zeta$ are the tangential components and FX, $f\zeta$ are the normal components of ψX and $\psi\zeta$, respectively, [4]. In the sake of following the common terminology, the tensor field ψ is replaced by J in almost Hermitian manifolds. However, the covariant derivatives of the tensor fields ψ, P and F are respectively defined as [2]

$$(\tilde{\nabla}_X \psi)Y = \tilde{\nabla}_X \psi Y - \psi \tilde{\nabla}_X Y, \tag{2.7}$$

$$(\tilde{\nabla}_X P)Y = \tilde{\nabla}_X PY - P\tilde{\nabla}_X Y \tag{2.8}$$

and

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\tilde{\nabla}_X Y. \tag{2.9}$$

Likewise, we consider a local field of orthonormal frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ on $\tilde{M}^m$, such that, restricted to M^n, $\{e_1, \cdots, e_n\}$ are tangent to M^n and $\{e_{n+1}, \cdots, e_m\}$ are normal to M^n. Then, the *mean curvature vector* $\vec{H}(x)$ is introduced as [2], [16]

$$\vec{H}(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.10}$$

On one hand, we say that M^n is a *minimal submanifold* of $\tilde{M}^m$ if $\vec{H} = 0$. On the other hand, one may deduce that M^n is totally umbilical in $\tilde{M}^m$ if and only if $h(X, Y) = g(X, Y)\vec{H}$, for any $X,\ Y \in \Gamma(TM^n)$ [8], where H and h are the mean curvature vector and the second fundamental form, respectively [7].

Let $\tilde{M}^{2m}$ be a real C^∞ manifold endowed with an almost complex structure J, i.e. J is a tensor field of type (1,1) such that, at every point $x \in \tilde{M}^{2m}$ we have $J^2 = -I$. Then, the pair $(\tilde{M}^{2m}, J)$ is called an almost complex manifold (see, for example [2], [12]). In addition, if the almost complex manifold $(\tilde{M}^{2m}, J)$ is furnished with a compatible Riemannian metric $\tilde{g}$, i.e., $\tilde{g}(JX, JY) = \tilde{g}(X, Y)$ for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$, then $(\tilde{M}^{2m}, J, \tilde{g})$ is called an almost Hermitian manifold.

It is known that the vanishing of the Nijenhuis tensor on almost Hermitian manifolds gives rise to a particular special class of almost Hermitian manifolds called Hermitian manifolds. The Hermitian manifold $(\tilde{M}^{2m}, J, \tilde{g})$ allows one to endow $\tilde{M}^{2m}$ with an alternating 2-form w given by

$$w(X, Y) = \tilde{g}(X, JY)$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. This 2-form is called the *associated Kaehler form*. Thus, $\tilde{g}$ now is called a Kaehler metric. In particular, $(\tilde{M}^{2m}, J, \tilde{g})$ becomes a *Kaehler manifold* if w is closed, i.e., $dw = 0$. Equivalently, we say that a Hermitian manifold $(\tilde{M}^{2m}, J, \tilde{g})$ is a Kaehlerian manifold if and only if the complex structure J is parallel with respect to $\tilde{\nabla}$, i.e., whenever the following condition is preserved

$$(\tilde{\nabla}_X J)Y = 0 \tag{2.11}$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$.

In a natural way, it is possible to weaken the condition in (2.11) by

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0 \tag{2.12}$$

for each $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. Every almost Hermitian manifold satisfying (2.12) is called *nearly Kaehler manifold* [2].

In [1], Bejancu initiated the study of the CR-submanifolds of almost Hermitian manifolds by generalizing complex (holomorphic) and totally real submanifolds. A submanifold M^n of an almost Hermitian manifold $\tilde{M}^{2m}$ is said to be a *CR-submanifold* if there exists on M^n a differentiable holomorphic distribution $\mathfrak{D}_T$ whose orthogonal complementary distribution $\mathfrak{D}^\perp$ is totally real i.e., $J\mathfrak{D}_T \subseteq TM^n$ and $J\mathfrak{D}^\perp \subseteq T^\perp M^n$.

Denote by μ the maximal J-invariant subbundle of the normal bundle $T^\perp M^n$. Then it is well-known that the normal bundle $T^\perp M^n$ admits the following decomposition

$$T^\perp M^n = J\mathfrak{D}^\perp \oplus \mu. \tag{2.13}$$

On a Kaehler manifold $\tilde{M}^{2m}$, the warped product $N_T \times_f N_\perp$ is called a *CR-warped product*, if the submanifolds N_T and $N_\perp$ are integral manifolds of $\mathfrak{D}_T$ and $\mathfrak{D}^\perp$, respectively.

3. Basic Lemmas

To relate the calculus of $N_1 \times N_2$ to that of its factors the crucial notion of *lifting* is introduced as follows. If $f \in \mathfrak{F}(N_1)$, the *lift* of f to $N_1 \times N_2$ is $\tilde{f} = f \circ \pi_1 \in \mathfrak{F}(N_1 \times N_2)$. If $X_p \in T_p(N_1)$ and $q \in N_2$, then the *lift* $X_{(p,q)}$ of X_p to (p,q) is the unique vector in $T_{(p,q)}(N_1)$ such that $d\pi_1(X_{(p,q)}) = X_p$. If $X \in \Gamma(TN_1)$ the *lift* of X to $N_1 \times N_2$ is the vector field X whose value at each (p,q) is the lift of X_p to (p,q). The set of all such *horizontal lifts* X is denoted by $\mathcal{L}(N_1)$. Functions, tangent vectors and vector fields on N_2 are lifted to $N_1 \times N_2$ in the same way using the projection π_2. Note that $\mathcal{L}(N_1)$ and symmetrically the *vertical lifts* $\mathcal{L}(N_2)$ are vector subspaces of $\Gamma\big(T(N_1 \times N_2)\big)$, [16].

We recall the following two general results for warped products [16].

Proposition 3.1. *On $\tilde{M}^m = N_1 \times_f N_2$, if $X,\ Y \in \mathcal{L}(N_1)$ and $Z,\ W \in \mathcal{L}(N_2)$, then*

(i) $\tilde{\nabla}_X Y \in \mathcal{L}(N_1)$ *is the lift of* $\tilde{\nabla}_X Y$ *on* N_1.
(ii) $\tilde{\nabla}_X Z = \tilde{\nabla}_Z X = (Xf/f)Z$.
(iii) $(\tilde{\nabla}_Z W)^{\perp} = h_{N_2}(Z,W) = -\big(g_{N_2}(Z,W)/f\big)\nabla(f)$.
(iv) $(\tilde{\nabla}_Z W)^T \in \mathcal{L}(N_2)$ *is the lift of* $\nabla^{N_2}_Z W$ *on* N_2,

where g_{N_2}, h_{N_2} and ∇^{N_2} are, respectively, the induced Riemannian metric on N_2, the second fundamental form of N_2 as a submanifold of $\tilde{M}^m$ and the induced Levi-Civita connection on N_2.

It is obvious that, the above proposition leads to the following geometric conclusion.

Corollary 3.1. *The leaves $N_1 \times q$ of a warped product are totally geodesic; the fibers $p \times N_2$ are totally umbilical.*

Clearly, the totally geodesy of the leaves follows from (i), while (iii) implies that the fibers are totally umbilical in $\tilde{M}^m$. It is significant to say that, this corollary is one of the key ingredients of this work. Since all our considered submanifolds are warped products.

Here, it is well-known that the *second fundamental form* σ and the *shape operator* A_ξ of M^n are related by

$$g(A_\xi X, Y) = g(\sigma(X,Y), \xi) \tag{3.1}$$

for all $X, Y \in \Gamma(TM^n)$ and $\xi \in \Gamma(T^{\perp}M^n)$ (for instance, see [2], [16]).

4. Existence and Nonexistence of Warped Product Submanifolds in Almost Hermitian Manifolds

This section has two significant purposes. The first one is to provide special case solutions for Problems 1 and 2, that is to see whether a warped product exists or not in almost Hermitian manifolds. In the existence case, we prove some preparatory characteristic results which are necessary for subsequent sections, and this is the second purpose. Some new examples are given to assert the existence of some important warped product manifolds.

For a submanifold M^n in an almost Hermitian manifold $\tilde{M}^{2m}$ let $\mathcal{P}_X Y$ denote the tangential component and $\mathcal{Q}_X Y$ the normal one of $(\tilde{\nabla}_X J)Y$ in $\tilde{M}^{2m}$, where $X,\ Y \in \Gamma(TM^n)$.

In order to make it a self-contained reference of warped product submanifolds for immersibility and nonimmersibility problems, we hypothesize most of our statements in the current and the next section for almost Hermitian manifolds, and for warped product submanifolds of type $N_T \times_f N_2$, where N_T and N are holomorphic and Riemannian submanifolds. Meaning that, a lot of particular case results are included in the theorems of the next section.

We begin by considering a warped product submanifold in almost Hermitian manifolds such that one of the factors is holomorphic.

Theorem 4.1. *Every warped product submanifold $M^n = N \times_f N_T$ in almost Hermitian manifolds $\tilde{M}^{2m}$ possesses the following*

(i) $g(\mathcal{P}_X Z, W) = 0$;

The operators $\perp$, T and $\nabla(f)$ refer to the normal projection, the tangential projection and the gradient of f, respectively.

(ii) $g(\mathcal{P}_Z X, JZ) - g(\mathcal{P}_{JZ} X, Z) = -2(X \ln f)||Z||^2$,

for every vector fields $X \in \Gamma(TN)$ and $Z,\ W \in \Gamma(TN_T)$ such that N and N_T are Riemannian and invariant submanifolds of $\tilde{M}^{2m}$, respectively.

Proof. Taking X and Z as in hypothesis, it is clear that

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X JZ - J\tilde{\nabla}_X Z.$$

Since $Z \in \Gamma(TN_T)$, Proposition 3.1 (ii) implies that $\nabla_X JZ = J\nabla_X Z = (X \ln f)JZ$. Thus, making use of (2.1), we get

$$(\tilde{\nabla}_X J)Z = h(X, JZ) - Jh(X, Z).$$

Taking the inner product with W, we get (i). For the second part, and by taking advantage of (2.1), (2.2) and Proposition 3.1 (ii), we can write

$$(\tilde{\nabla}_Z J)X + (\tilde{\nabla}_X J)Z = (PX \ln f)Z + h(PX, Z) - A_{FX}Z$$
$$+\nabla^{\perp}_Z FX - (X \ln f)JZ - 2Jh(X, Z) + h(X, JZ).$$

Taking the inner product with JZ in the above equation gives

$$g(\mathcal{P}_Z X + \mathcal{P}_X Z, JZ) = -g(h(Z, JZ), FX) - (X \ln f)||Z||^2.$$

If we substitute JZ for Z in the above equation, then we have

$$-g(\mathcal{P}_{JZ} X + \mathcal{P}_X JZ, Z) = g(h(Z, JZ), FX) - (X \ln f)||Z||^2.$$

By these two equations, we get

$$g(\mathcal{P}_Z X + \mathcal{P}_X Z, JZ) - g(\mathcal{P}_{JZ} X + \mathcal{P}_X JZ, Z) = -2(X \ln f)||Z||^2.$$

Finally, we may apply statement (i) in the above equation to get (ii). □

In particular, if we assume the ambient manifold $\tilde{M}^{2m}$ to be either Kaehler or nearly Kaehler in the theorem above, the nonexistence of proper warped products of the type $N \times_f N_T$ immediately follows. Using (2.12) in statement (ii) gives

$$g(\mathcal{P}_X Z, JZ) - g(\mathcal{P}_X JZ, Z) = 2(X \ln f)||Z||^2,$$

if one applies statement (i) on the left hand side of the above equation, he automatically gets $X \ln f = 0$, for every $X \in \Gamma(TN)$. Obviously, this conclusion is true for Kaehler manifolds also. Hence, we can state the following

Corollary 4.1. *Warped product submanifolds with holomorphic second factor are Riemannian products, in both Kaehler and nearly Kaehler manifolds.*

It is worth pointing out that, the previous corollary generalizes many nonexistence results in this field, (see, for example [6], [11] and [17]).

By reversing the two factors of the warped product in Theorem 4.1, we present the following corresponding theorem for doubly warped product submanifolds.

Theorem 4.2. *Let $M^n =_{f_2} N_T \times_{f_1} N$ be a doubly warped product submanifold in an almost Hermitian manifold $\tilde{M}^{2m}$. Then,*

$$g(\mathcal{P}_X Z, JX) - g(\mathcal{P}_{JX} Z, X) = -2(Z \ln f_2)||X||^2,$$

for vector fields $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN)$, where N and N_T are Riemannian and invariant submanifolds of $\tilde{M}^{2m}$, respectively.

Proof. Taking X and Z as in hypothesis. By (**??**), (2.1) and (2.2), it is straightforward to carry out the following calculations

$$(\tilde{\nabla}_X J)Z = (X \ln f_1)PZ + (PZ \ln f_2)X + h(X, PZ) - A_{FZ}X$$
$$+\nabla^{\perp}_X FZ - (X \ln f_1)JZ - (Z \ln f_2)JX - Jh(X, Z).$$

If we take the inner product with JX in the above equation, then

$$g(\mathcal{P}_X Z, JX) = -g(h(X, JX), FZ) - (Z \ln f_2)||X||^2.$$

By replacing JX with X in the above equation we deduce that

$$-g(\mathcal{P}_{JX} Z, X) = g(h(X, JX), FZ) - (Z \ln f_2)||X||^2.$$

Thus, the assertion follows from the above two equations. □

The following corollary can be directly obtained from (2.11) and Theorem 4.2.

Corollary 4.2. *A doubly warped product submanifold with holomorphic first factor is trivial in Kaehler manifolds.*

Combining Corollaries 4.1 and 4.2 together, one can directly get the next prominent result.

Corollary 4.3. *In Kaehler manifolds, there is no proper doubly warped product submanifold such that one of its factors is holomorphic.*

For doubly warped product submanifolds with one of the factors holomorphic, we have already had a negative answer from the preceding corollary. However, the situation is not the same with (singly) warped product submanifolds of holomorphic first factor, and thus we present one of the basic characteristic theorems for subsequent chapters.

Theorem 4.3. *Let $M^n = N_T \times_f N$ be a warped product in an almost Hermitian manifold $\tilde{M}^{2m}$. Then, the following hold:*

(i) $g(\mathcal{P}_X Z, Y) = -g(h(X, Y), FZ)$;

(ii) $g(\mathcal{P}_Z X, Z) = (JX \ln f)||Z||^2 + g(h(X, Z), FZ)$;

(iii) $g(\mathcal{P}_Z X, Y) = 0$;

(iv) $g(\mathcal{P}_Z X, W) + g(\mathcal{P}_W X, Z) = 2(JX \ln f)g(Z, W) + g(h(X, Z), FW) + g(h(X, W), FZ)$;

(v) $g(\mathcal{P}_Z X - \mathcal{P}_X Z, W) - g(\mathcal{P}_W X, Z) = 2(X \ln f)g(Z, PW)$;

(vi) $g(\mathcal{P}_X Z, W) + g(\mathcal{P}_X W, Z) = 0$;

(vii) $g(\mathcal{Q}_X X, J\zeta) + g(\mathcal{Q}_{JX} JX, J\zeta) = -g(h(X, X), \zeta) - g(h(JX, JX), \zeta)$,

for any vector fields $X,\ Y \in \Gamma(TN_T)$, $Z,\ W \in \Gamma(TN)$ and $\zeta \in \Gamma(\nu)$.

Proof. For X and Z as above, we have

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X JZ - J\tilde{\nabla}_X Z. \tag{4.1}$$

Equivalently,

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - J\tilde{\nabla}_X Z. \tag{4.2}$$

Taking the inner product with Y in the above equation gives (i) immediately. Now, by reversing the roles of X and Z in (4.1), it follows

$$(\tilde{\nabla}_Z J)X = \tilde{\nabla}_Z JX - J\tilde{\nabla}_Z X. \tag{4.3}$$

Taking the inner product with Z in the above equation implies (ii). Subtracting the equation above from (4.2), taking into consideration that h is a symmetric form and $\nabla_X Z = \nabla_Z X$, we immediately get

$$(\tilde{\nabla}_X J)Z - (\tilde{\nabla}_Z J)X = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - \tilde{\nabla}_Z JX.$$

Taking the inner product with JY in the above equation yields

$$g(\mathcal{P}_X Z, JY) - g(\mathcal{P}_Z X, JY) = -g(h(X, JY), FZ).$$

Replacing JY by Y in the above equation, gives

$$g(\mathcal{P}_Z X, Y) - g(\mathcal{P}_X Z, Y) = g(h(X,Y), FZ).$$

Applying statement (i) in the above equation proves statement (iii).
Taking the inner product with W in (4.3), we will obtain

$$g(\mathcal{P}_Z X, W) = (JX \ln f) g(Z, W) + (X \ln f) g(Z, PW) + g(h(X,Z), FW). \tag{4.4}$$

By interchanging the rules of Z and W in the above equation, and due to the fact that $g(Z, PW)$ is skew-symmetric with respect to Z and W, the following holds

$$g(\mathcal{P}_W X, Z) = (JX \ln f) g(Z, W) - (X \ln f) g(Z, PW) + g(h(X,W), FZ). \tag{4.5}$$

If we add (4.4) and (4.5) together, then (iv) follows. While by subtracting (4.5) from (4.4) we immediately reach

$$g(\mathcal{P}_Z X, W) - g(\mathcal{P}_W X, Z) = 2(X \ln f) g(Z, PW) + g(h(X,Z), FW) - g(h(X,W), FZ). \tag{4.6}$$

Moreover, one can take the inner product in (4.2) with W to obtain

$$g(\mathcal{P}_X Z, W) = g(h(X,Z), FW) - g(h(X,W), FZ). \tag{4.7}$$

Hence, if we subtract (4.7) from (4.6), we get (v). On the other hand, by using the polarization identity of Z and W in (v), we obtain

$$g(\mathcal{P}_W X - \mathcal{P}_X W, Z) - g(\mathcal{P}_Z X, W) = -2(X \ln f) g(Z, PW).$$

By using statement (v) and the above equation, statement (vi) follows directly.
For (vii), notice that

$$(\tilde{\nabla}_X J)X = \tilde{\nabla}_X JX - J\tilde{\nabla}_X X.$$

First, we take the inner product in the above equation with $J\zeta$ to get

$$g(\mathcal{Q}_X X, J\zeta) = g(h(JX, X), J\zeta) - g(h(X,X), \zeta).$$

After that, we replace JX by X in the above equation to derive

$$g(\mathcal{Q}_{JX} JX, J\zeta) = -g(h(JX, X), J\zeta) - g(h(JX, JX), \zeta).$$

Hence (vii) can be obtained by adding the above two equations. This completes the proof. □

In [1], Bejancu initiated the study of the CR-submanifolds of almost Hermitian manifolds by generalizing invariant (holomorphic) and anti-invariant (totally real) submanifolds. He called a submanifold M^n of an almost Hermitian manifold $\tilde{M}^{2m}$ a *CR-submanifold* if there exists on M^n a differentiable holomorphic distribution $\mathfrak{D}_T$ whose orthogonal complementary distribution $\mathfrak{D}_\perp$ is totally real. In other words, M^n is said to be a CR-submanifold if it is endowed with a pair of orthogonal complementary distributions $(\mathfrak{D}_T, \mathfrak{D}_\perp)$, satisfying the following conditions:

(i) $TM^n = \mathfrak{D}_T \oplus \mathfrak{D}_\perp$

(ii) $\mathfrak{D}_T$ is a holomorphic distribution, i.e., $J\mathfrak{D}_T \subseteq TM^n$

(iii) $\mathfrak{D}_\perp$ is a totally real distribution, i.e., $J\mathfrak{D}_\perp \subseteq T^\perp M^n$.

Denote by ν the maximal J-invariant subbundle of the normal bundle $T^\perp M^n$. Then it is well-known that the normal bundle $T^\perp M^n$ admits the following decomposition

$$T^\perp M^n = F\mathfrak{D}_\perp \oplus \nu. \tag{4.8}$$

In Kaehler manifolds $\tilde{M}^{2m}$, the warped product $N_T \times_f N_\perp$ is called a *CR-warped product* submanifold, if the submanifolds N_T and $N_\perp$ are integral manifolds of $\mathfrak{D}_T$ and $\mathfrak{D}_\perp$, respectively. The following prominent nonexistence fact generalizes many nonexistence results in Kaehler manifolds, (see, for example [11] and [17]).

Corollary 4.4. *In Kaehler manifolds, there is no warped product of type $N_T \times_f N$ other than CR-warped products.*

Proof. We want to show that N is a totally real submanifold when the first factor is holomorphic. Equivalently, it suffices to prove that $PZ = 0$ for every $Z \in \Gamma(TN)$. Evidently, using (2.11) in Theorem 4.3 (v), we deduce that $X \ln f = 0$ or $g(PZ, W) = 0$, for arbitrary vector fields Z and W tangent to the second factor. This implies either $N_T \times_f N$ is a Riemannian product or $PZ = 0$ for every $Z \in \Gamma(TN)$. Hence if the second factor is not totally real submanifold, then $N_T \times_f N$ is trivial. □

In Kaehler manifolds, a characterization theorem for the CR-warped product submanifold of the type $N_T \times_f N_\perp$ is proved in [6]. Here, we construct a concrete example asserting the existence of such warped product submanifold.

Example 4.1. Let $\mathbb{R}^6$ be equipped with the canonical complex structure J, with its Cartesian coordinates $(x_1, \cdots, x_6)$. Then a 3-dimensional submanifold M^3 of $\mathbb{R}^6$ is given by

$$x_1 = t\cos\theta, \quad x_2 = s\cos\theta, \quad x_5 = t\sin\theta, \quad x_6 = s\sin\theta, \quad x_3 = x_4 = 0.$$

It is clear that M^3 is well-defined with a tangent bundle TM^3 spanned by Z_1, Z_2 and Z_3, such that

$$Z_1 = \cos\theta\,\frac{\partial}{\partial x_1} + \sin\theta\,\frac{\partial}{\partial x_5}, \quad Z_2 = \cos\theta\,\frac{\partial}{\partial x_2} + \sin\theta\,\frac{\partial}{\partial x_6},$$

$$Z_3 = -t\sin\theta\,\frac{\partial}{\partial x_1} - s\sin\theta\,\frac{\partial}{\partial x_2} + t\cos\theta\,\frac{\partial}{\partial x_5} + s\cos\theta\,\frac{\partial}{\partial x_6}.$$

Therefore, $\mathfrak{D}_T =$ span $\{Z_1,\ Z_2\}$, and $\mathfrak{D}_\perp =$ span $\{Z_3\}$ are holomorphic and totally real distributions, respectively. Thus, M^3 is a CR-submanifold of $\mathbb{R}^6$. Since it is not difficult to see that $\mathfrak{D}_T$ is integrable, then we can denote the integral manifolds of $\mathfrak{D}_T$ and $\mathfrak{D}_\perp$ respectively by N_T and $N_\perp$. Based on the above tangent bundle, the metric tensor g of M^3 is expressed by

$$g = 2dt^2 + 2ds^2 + (t^2 + s^2)d\theta^2$$
$$= g_{N_T} + (t^2 + s^2)g_{N_\perp}.$$

Obviously, g is a warped metric tensor on M^3. Consequently, M^3 is a CR-warped product submanifold of type $N_T \times_f N_\perp$ in $\mathbb{R}^6$, with warping function $f = \sqrt{t^2 + s^2}$. By means of Gauss formula, we obtain that

$$h(Z_1, Z_1) = h(Z_2, Z_2) = 0.$$

This means that M is a $\mathfrak{D}_T$-minimal warped product in $\mathbb{R}^6$.

The following result describes locally a relation of the coefficients of the second fundamental form.

Corollary 4.5. *Let $M^n = N_T \times_f N$ be a warped product submanifold in Kaehler or in nearly Kaehler manifolds $\tilde{M}^{2m}$. Then, we have*

$$\sum_{\substack{A,B=1\\ A\neq B}}^{n_2} g(h(X, e_A), Fe_B) = 0,$$

where $e_1, \cdots, e_{n_2}$ form a local orthonormal frame fields of $\Gamma(TN)$, and X is any vector field tangent to the first factor.

Proof. Using (2.11) or (2.12) with parts (ii) and (v) of Theorem 4.3 gives

$$-2(JX \ln f)g(Z, W) = g(h(X, Z), FW) + g(h(X, W), FZ),$$

for $X \in \Gamma(TN_T)$ and $Z,\ W \in \Gamma(TN)$. Take any two distinct orthogonal unit vectors, say e_v and e_u, from the above frame. Let $Z = e_v$ and $W = e_u$ in the above equation. Then $g(h(X, e_v), Fe_u) = -g(h(X, e_u), Fe_v)$, which gives the result. □

It is reasonable to include the following key result at the end of this section, which plays fascinating roles in subsequent chapters.

Proposition 4.1. *Let $M^n = N_T \times_f N$ be isometrically immersed in nearly Kaehler manifolds. Then, the following are fulfilled:*

(i) $g(h(X,Y),FZ)=0$;

(ii) $g(h(X,Z),FZ)=-(JX\ln f)||Z||^2$;

(iii) $g(h(X,X),\zeta)+g(h(JX,JX),\zeta)=0$;

(iv) $g(h(X,Z),FW)=\frac{1}{3}(X\ln f)g(PZ,W)-(JX\ln f)g(Z,W)$,

where the vector fields X, Y are tangent to the first factor, Z and W are tangent to the second factor and ζ is tangent to the normal subbundle ν.

Proof. In virtue of (2.12), the first statement follows directly by using parts (i) and (iii) of Theorem 4.3. The second statement is obtained from Theorem4.3 (vi), (ii). The third statement is clear from Theorem 4.3 (vii) and (2.12). For the last statement, we substitute $Z+W$ instead of Z in statement (ii) above, hence we get

$$g(h(X,Z),FW)+g(h(X,W),FZ)=-2(JX\ln f)g(Z,W), \tag{4.9}$$

for $X,\ Z$ and W as in the statement above.

Now, making use of (2.1), (2.2), (3.1), (2.12) and Proposition 3.1 (ii), we carry out the following calculations

$$\begin{aligned} g(h(X,Z),FW) &= g(h(X,Z),JW)=-g(Jh(X,Z),W)=g(J(\nabla_XZ-\tilde{\nabla}_XZ),W)\\ &=(X\ln f)g(PZ,W)-g(J\tilde{\nabla}_XZ,W)\\ &=(X\ln f)g(PZ,W)+g((\tilde{\nabla}_XJ)Z,W)-g(\tilde{\nabla}_XJZ,W)\\ &=(X\ln f)g(PZ,W)-g((\tilde{\nabla}_ZJ)X,W)-g(\tilde{\nabla}_XPZ,W)-g(\tilde{\nabla}_XFZ,W)\\ &=(X\ln f)g(PZ,W)+g(J\tilde{\nabla}_ZX)-g(\tilde{\nabla}_ZJX,W)-(X\ln f)g(PZ,W)+g(A_{FZ}X,W)\\ &=g(J\nabla_ZX,W)+g(Jh(X,Z),W)-(JX\ln f)g(Z,W)+g(h(X,W),FZ)\\ &=(X\ln f)g(PZ,W)-g(h(X,Z),FW)-(JX\ln f)g(Z,W)+g(h(X,W),FZ). \end{aligned}$$

This gives

$$2g(h(X,Z),FW)-g(h(X,W),FZ)=(X\ln f)g(PZ,W)-(JX\ln f)g(Z,W). \tag{4.10}$$

Thus, combining (4.9) and (4.10) together gives statement (iv) directly, which completes the proof. □

In what follows we summarize the immersibility and nonimmersibility cases of Kaehler and nearly Kaehler manifolds according to the preceding results.

Warped Product Submanifold	Kaehler	Nearly Kaehler
$N_\perp \times_f N_T$	X	X
$N_T \times_f N_\perp$	✓	✓
$N_\theta \times_f N_T$	X	X
$N_T \times_f N_\theta$	X	?
$N \times_f N_T$	X	X
$N_T \times_f N$	X	?
$N_\perp \times_f N_\theta$	X	?
$N_\theta \times_f N_\perp$	✓	✓

Table 1. Existence and nonexistence of proper warped product submanifolds in Kaehler and nearly Kaehler manifolds.

5. Research problems based on The Results of Previous Sections

Due to the results of this paper, we hypothesize a pair of open problems.
Firstly,

Problem 3. *Can we prove the existence or the nonexistence of semi-slant warped product submanifolds of the type $N_T \times_f N_\theta$ in nearly Kaehler manifold as an ambient manifold.*

Secondly, we ask:

Problem 4. *Can we prove the existence or the nonexistence of generic warped product submanifolds of the type $N_T \times_f N$ in nearly Kaehler manifold as an ambient manifold.*

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