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Research Article

New inequalities of Huygens-type involving tangent and sine functions

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Abstract

Using the estimations of the even-indexed Bernoulli number and Euler number this paper established some new inequalities for the three functions $2(\sin x)/x + (\tan x)/x$, $(\sin x)/x + 2(\tan(x/2))/(x/2)$ and $2x/\sin x + x/\tan x$ bounded by the powers of tangent function.

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1. Introduction

For $0 < x < \pi/2$, the Huygens inequality (see [1], [2]) is known as follows:

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3. \tag{1.1}$$

Neuman established the following inequality chain in [3]:

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2\frac{\tan(x/2)}{(x/2)} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3. \tag{1.2}$$

Chen and Cheung [4] obtained an improvement of (1.1) without proof.

Theorem 1.1. *If* $0 < |x| < \pi/2$, *then*

$$\frac{3}{20}x^3 \tan x < 2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 < \frac{16}{\pi^4}x^3 \tan x. \tag{1.3}$$

Furthermore, $3/20$ *and* $16/\pi^4$ *are the best constants in* (1.3).

The first purpose of this paper is to create a new bound for the function $2(\sin x)/x + (\tan x)/x - 3$ and obtain the following new inequality.

Theorem 1.2. *Let* $0 < |x| < \pi/2$. *Then*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} - 3 < \frac{3}{20}x^4\left(\frac{\tan x}{x}\right)^{15/14}, \tag{1.4}$$

where $3/20$ *is the best constant in* (1.4).

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The second purpose of this paper is to find inequalities similar to (1.3) and (1.4) for the other two functions $(\sin x)/x + 2(\tan(x/2))/(x/2)$ and $2x/(\sin x) + x/\tan x$ shown in (1.2).

Theorem 1.3. *Let $0 < x < \pi/2$. Then*

$$\frac{\sin x}{x} + 2\frac{\tan(x/2)}{(x/2)} - 3 < \frac{1}{40}x^3 \tan x, \tag{1.5}$$

where the constant $1/40$ in (1.5) is the best possible.

Theorem 1.4. *Let $0 < |x| < \pi/2$. Then*

$$\frac{\sin x}{x} + 2\frac{\tan(x/2)}{(x/2)} - 3 < \frac{1}{40}x^4 \left(\frac{\tan x}{x}\right)^{5/28}, \tag{1.6}$$

where the constant $1/40$ in (1.6) is the best possible.

Theorem 1.5. *Let $0 < x < \pi/2$. Then*

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 < \frac{1}{60}x^3 \tan x, \tag{1.7}$$

where the constant $1/60$ in (1.7) is the best possible.

Theorem 1.6. *Let $0 < |x| < \pi/2$. Then*

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 < \frac{1}{60}x^4 \left(\frac{\tan x}{x}\right)^{5/14}, \tag{1.8}$$

where the constant $1/60$ in (1.8) is the best possible.

2. Lemmas

In order to prove the main conclusions of this paper, we need the following lemmas.

Lemma 2.1. ([5],[6]) *Let $0 < |x| < \pi$. Then*

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!} x^{2n-1}. \tag{2.1}$$

Here, we give a new proof of Lemma 2.1.

Proof. In [6] or [7] we can find that

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \tag{2.2}$$

holds for all $x \in (-\pi, 0) \cup (0, \pi)$. Via

$$\frac{1}{\sin x} = \cot\frac{x}{2} - \cot x \tag{2.3}$$

we can arrive (2.1). □

Remark 2.2. When considering the function

$$\frac{x}{\sin x} + 2\frac{(x/2)}{\tan(x/2)}$$

to subdivide (1.2) one can find such an interesting fact

$$\frac{x}{\sin x} + 2\frac{(x/2)}{\tan(x/2)} = 2\frac{x}{\sin x} + \frac{x}{\tan x}$$

due to (2.3).

Lemma 2.3. *Let B_{2n} be the even-indexed Bernoulli numbers, we have the following power series expansions*

$$\begin{aligned}
\tan x &= \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}, \\
\tan^2 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}, \\
\tan^3 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1},
\end{aligned}$$

$$\begin{aligned}
\tan^4 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4} \\
&\quad - \sum_{n=2}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2}, \\
\sec^2 x \tan x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-3}, \\
\sec^2 x \tan^2 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4} \\
&\quad - \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2}, \\
\sec^2 x \tan^3 x &= \frac{1}{4}\sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)}{6\,(2n)!}|B_{2n}|x^{2n-5} \\
&\quad - \frac{1}{4}\sum_{n=2}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)\,(2n-2)}{3\,(2n)!}|B_{2n}|x^{2n-3}
\end{aligned}$$

hold for all $x \in (-\pi/2, \pi/2)$.

Proof. The power series expansion of $\tan x$ can be found in [6] or [7]. Based on this basic power series expansion, we can draw some series of conclusions. Via the relation

$$\tan^{k+1} x = \frac{1}{k}\left[\left(\tan^k x\right)' - k\tan^{k-1} x\right],\ k \geq 1,$$

we get

$$\begin{aligned}
\tan^2 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}, \\
\tan^3 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1},
\end{aligned}$$

$$\begin{aligned}
\tan^4 x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4} \\
&\quad - \sum_{n=2}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2}, \\
\tan^5 x &= \frac{1}{4}\left(\tan^4 x\right)' - \tan^3 x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{n=3}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)(2n-4)}{6(2n)!}|B_{2n}|x^{2n-5}\\
&-\frac{1}{4}\sum_{n=2}^{\infty}\frac{2^{2n+2}(2^{2n}-1)(2n-1)(2n-2)}{3(2n)!}|B_{2n}|x^{2n-3}\\
&-\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-3}+\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1}.
\end{aligned}
$$

Since

$$\sec^2 x\tan^k x=\frac{1}{k+1}\frac{d}{dx}\tan^{k+1}x, k\neq -1,$$

we can get the desired power series expansions of these three functions $\sec^2 x\tan x$, $\sec^2 x$ $\tan^2 x$, and $\sec^2 x\tan^3 x$. □

Lemma 2.4. *Let E_{2n} be the even-indexed Euler numbers, we have the following power series expansions*

$$
\begin{aligned}
\sec x &= \frac{1}{\cos x}=\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n},\\
\sec x\tan x &= \frac{\sin x}{\cos^2 x}=\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1},\\
\sec^3 x &= \frac{1}{\cos^3 x}=\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right],\\
\sec x\tan^2 x &= \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}-\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right],\\
\sec x\tan^3 x &= \frac{1}{6}\left[\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-3}-5\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}\right],\\
\tan x\sec^3 x &= \frac{\sin x}{\cos^4 x}=\frac{1}{6}\left[\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-3}+\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}\right],\\
\sec x\tan^4 x &= \sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right]\\
&+\frac{1}{24}\left[\sum_{n=1}^{\infty}\frac{(2n-1)(2n-2)(2n-3)|E_{2n}|}{(2n-1)!}x^{2n-4}+\sum_{n=0}^{\infty}\frac{2n(2n-1)|E_{2n}|}{(2n)!}x^{2n-2}\right]\\
&+\frac{3}{8}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right]
\end{aligned}
$$

hold for all $x\in(-\pi/2,\pi/2)$.

Proof. The power series expansion of $\tan x$ can be found in [6] or [7]. Based on this basic power series expansion, we can draw some series of conclusions.

$$
\begin{aligned}
\sec x\tan x &= \frac{\sin x}{\cos^2 x}=\left(\frac{1}{\cos x}\right)'=\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1},\\
\sec^3 x &= \frac{1}{\cos^3 x}=\frac{1}{2}\left[\left(\frac{\sin x}{\cos^2 x}\right)'+\frac{1}{\cos x}\right]\\
&= \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right],
\end{aligned}
$$

$$\begin{aligned}\tan x\sec^3 x &= \frac{\sin x}{\cos^4 x}=\frac{1}{3}\left(\frac{1}{\cos^3 x}\right)' \\ &= \frac{1}{6}\left[\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-3}+\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}\right], \\ \sec x\tan^2 x &= \sec^3 x-\sec x=\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}-\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right], \\ \sec x\tan^3 x &= \frac{1}{3}\left(\sec^3 x\right)'-(\sec x)' \\ &= \frac{1}{6}\left[\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-3}-5\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\cos^5 x} &= \frac{1}{4}\left[\frac{1}{3}\left(\frac{1}{\cos^3 x}\right)''+\frac{3}{\cos^3 x}\right] \\ &= \frac{1}{24}\left[\sum_{n=1}^{\infty}\frac{(2n-1)(2n-2)(2n-3)|E_{2n}|}{(2n-1)!}x^{2n-4}+\sum_{n=0}^{\infty}\frac{2n(2n-1)|E_{2n}|}{(2n)!}x^{2n-2}\right] \\ &\quad+\frac{3}{8}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right],\end{aligned}$$

then

$$\begin{aligned}\sec x\tan^4 x &= \frac{1}{\cos x}\frac{\sin^4 x}{\cos^4 x}=\frac{1}{\cos x}-\frac{2}{\cos^3 x}+\frac{1}{\cos^5 x} \\ &= \sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right] \\ &\quad+\frac{1}{24}\left[\sum_{n=1}^{\infty}\frac{(2n-1)(2n-2)(2n-3)|E_{2n}|}{(2n-1)!}x^{2n-4}+\sum_{n=0}^{\infty}\frac{2n(2n-1)|E_{2n}|}{(2n)!}x^{2n-2}\right] \\ &\quad+\frac{3}{8}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right].\end{aligned}$$

□

Lemma 2.5. ([6],[8],[9]) *For all integers $n \geq 1$, let B_{2n} be the even-indexed Bernoulli numbers. Then the double inequality*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-1}<|B_{2n}|<\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n-1}}{2^{2n-1}-1} \tag{2.4}$$

holds.

Lemma 2.6. ([6]) *For all integers $n \geq 1$, let E_{2n} be the even-indexed Euler numbers. Then the double inequality*

$$\frac{2^{2(n+1)}}{\pi^{2n+1}}\frac{3^{2n+1}}{3^{2n+1}+1}<\frac{|E_{2n}|}{(2n)!}<\frac{2^{2(n+1)}}{\pi^{2n+1}} \tag{2.5}$$

holds.

3. Proofs of main results

Now let us prove some results about the refinements and sharpness of the Huygens-type inequalities.

3.1. Proof of Theorem 1.1.

Let

$$F_1(x) = \frac{2\frac{\sin x}{x} + \frac{\tan x}{x} - 3}{x^3 \tan x},\ 0 < x < \frac{\pi}{2}.$$

Then

$$F_1'(x) = \frac{\sin x}{x^5 \tan^2 x} f_1(x),$$

where

$$f_1(x) = 3x^2 \frac{1}{\sin x} - 8\tan x - 2x\tan^2 x - 4\frac{\sin x}{\cos^2 x} + 3x^2 \frac{\sin x}{\cos^2 x} + 9x\sec x.$$

By substituting the expansions of power series of corresponding functions in Lemmas 2.1, 2.2 and 2.3 into the above formula, we can get

$$\begin{aligned}
f_1(x) &= 3x^2\left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1}\right) - 8\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} \\
&\quad -2x\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2} - 4\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad +3x^2\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} + 9x\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n} \\
&= 3x + 3\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n+1} - 8\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} \\
&\quad -2\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-1} - 4\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad +3\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n+1} + 9\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+1}5 \\
&= 3\sum_{n=3}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n+1} - 8\sum_{n=4}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} \\
&\quad -2\sum_{n=4}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-1} - 4\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad +3\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n+1} + 9\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+1}
\end{aligned}$$

$$\begin{aligned}
&= 3\sum_{n=3}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n+1} + 3\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n+1} + 9\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+1} \\
&\quad -8\sum_{n=4}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} - 2\sum_{n=4}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-1} \\
&\quad -4\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&= \sum_{n=3}^{\infty}\frac{3(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1} + \sum_{n=3}^{\infty}\frac{3|E_{2n}|}{(2n-1)!}x^{2n+1} + \sum_{n=3}^{\infty}\frac{9|E_{2n}|}{(2n)!}x^{2n+1} \\
&\quad -\sum_{n=4}^{\infty}\frac{8(2^{2n}-1)}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} - \sum_{n=4}^{\infty}\frac{2\cdot 2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-1}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{n=4}^{\infty}\frac{4|E_{2n}|}{(2n-1)!}x^{2n-1} \\
= & \sum_{n=3}^{\infty}\frac{3\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n+1}+\sum_{n=3}^{\infty}\frac{(6n+9)\,|E_{2n}|}{(2n)!}x^{2n+1} \\
& -\sum_{n=4}^{\infty}\frac{|B_{2n}|2^{2n}(2^{2n}-1)\,(4n+6)}{(2n)!}x^{2n-1}-\sum_{n=4}^{\infty}\frac{4|E_{2n}|}{(2n-1)!}x^{2n-1} \\
= & \sum_{n=4}^{\infty}\frac{3\left(2^{2n-2}-2\right)}{(2n-2)!}|B_{2n-2}|x^{2n-1}+\sum_{n=4}^{\infty}\frac{(6n+3)\,|E_{2n-2}|}{(2n-2)!}x^{2n-1} \\
& -\sum_{n=4}^{\infty}\frac{|B_{2n}|2^{2n}(2^{2n}-1)\,(4n+6)}{(2n)!}x^{2n-1}-\sum_{n=4}^{\infty}\frac{4|E_{2n}|}{(2n-1)!}x^{2n-1} \\
:= & \sum_{n=4}^{\infty}a_nx^{2n-1},
\end{aligned}
$$

where

$$
a_n=\frac{3\left(2^{2n-2}-2\right)|B_{2n-2}|}{(2n-2)!}+\frac{(6n+3)\,|E_{2n-2}|}{(2n-2)!}-\frac{2^{2n}(2^{2n}-1)\,(4n+6)\,|B_{2n}|}{(2n)!}-\frac{8n|E_{2n}|}{(2n)!}.
$$

By Lemmas 2.4 and 2.5 we have

$$
\begin{aligned}
a_n > & \frac{3\left(2^{2n-2}-2\right)}{(2n-2)!}\frac{2(2n-2)!}{(2\pi)^{2n-2}}\frac{2^{2n-2}}{2^{2n-2}-1}+\frac{(6n+3)\,2^{2n}}{\pi^{2n-1}}\frac{3^{2n-1}}{3^{2n-1}+1} \\
& -\frac{2^{2n}(2^{2n}-1)\,(4n+6)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-2}-\frac{8n\cdot 2^{2(n+1)}}{\pi^{2n+1}} \\
= & \frac{6\left(2^{2n-2}-2\right)}{\pi^{2n-2}\left(2^{2n-2}-1\right)}+\frac{(6n+3)\,2^{2n}}{\pi^{2n-1}}\frac{3^{2n-1}}{1+3^{2n-1}} \\
& -\frac{2\cdot 2^{2n}(2^{2n}-1)\,(4n+6)}{\pi^{2n}\left(2^{2n}-2\right)}-\frac{8n\cdot 2^{2(n+1)}}{\pi^{2n+1}} \\
:= & \frac{h_1(n)}{\pi^{2n+1}\left(2^{2n}-2\right)\left(2^{2n}-4\right)\left(3^{2n}+3\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(n) = & \left[\left(6\pi^2-32-8\pi\right)n+3\pi^2-12\pi\right]24^{2n} \\
& -2\left[\left(18\pi^2-20\pi-96\right)n+9\pi^2-3\pi^3-30\pi\right]12^{2n} \\
& -12\left[(8+2\pi)\,n+3\pi\right]8^{2n} \\
& +4\left[\left(12\pi^2-64-8\pi\right)n+6\pi^2-15\pi^3-12\pi\right]6^{2n} \\
& +6\left[(96+20\pi)\,n+3\pi^3+30\pi\right]4^{2n} \\
& +96\pi^3\cdot 3^{2n}-12\left[(64+8\pi)\,n+15\pi^3+12\pi\right]2^{2n}+288\pi^3 \\
:= & 6^{2n}p(n)+q(n)
\end{aligned}
$$

with

$$
\begin{aligned}
p(n) = & \left[\left(6\pi^2-32-8\pi\right)n+3\pi^2-12\pi\right]4^{2n} \\
& -2\left[\left(18\pi^2-20\pi-96\right)n+9\pi^2-3\pi^3-30\pi\right]2^{2n}
\end{aligned}
$$

$$
\begin{aligned}
&-12\left[(8+2\pi)n+3\pi\right]\left(\frac{4}{3}\right)^{2n}\\
&+4\left[\left(12\pi^2-64-8\pi\right)n+6\pi^2-15\pi^3-12\pi\right],\\
q(n) \;=\;& 6\left[(96+20\pi)n+3\pi^3+30\pi\right]4^{2n}\\
&+96\pi^3\cdot 3^{2n}-12\left[(64+8\pi)n+15\pi^3+12\pi\right]2^{2n}+288\pi^3.
\end{aligned}
$$

Obviously, $q(n) > 0$ holds for all $n \geq 4$. We need to prove that $p(n) > 0$ holds for all $n \geq 4$. Since for $n \geq 4$,

$$
\begin{aligned}
p(n+1)-16p(n) \;=\;& 32\left(3\pi^2-4\pi-16\right)4^{2n}\\
&+8\left[6\left(9\pi^2-48-10\pi\right)n-\left(9\pi^3+70\pi-9\pi^2-96\right)\right]2^{2n}\\
&+\frac{128}{3}\left[(32n+8\pi)n+11\pi-4\right]\left(\frac{4}{3}\right)^{2n}\\
&-\left[240n(\pi+2)(3\pi-8)-\left(900\pi^3+688\pi-312\pi^2-256\right)\right]\\
>\;& 0,
\end{aligned}
$$

and

$$
p(4)=1728\,216\pi^2-\frac{6186\,494\,864}{2187}\pi+1476\pi^3-\frac{17\,926\,532\,096}{2187}>18906>0,
$$

we obtain $p(n) > 0$ and $h_1(n) > 0$ holds for all $n \geq 4$. So $a_n > 0$ for $n \geq 4$. Then $f(x) > 0$ on $(0, \pi/2)$. Then $F_1'(x) > 0$ on $(0, \pi/2)$. Therefore the function $F_1(x)$ is increasing on $(0, \pi/2)$. At the same time,

$$
\lim_{x\to 0^+} F_1(x)=\frac{3}{20}\quad\text{and}\quad \lim_{x\to(\pi/2)^-} F_1(x)=\frac{16}{\pi^4}.
$$

The proof of Theorem 1.1 is completed. □

3.2. Proof of Theorem 1.2.

Let

$$
F_2(x)=\ln\left[\left(\frac{3}{20}\right)^{14}x^{41}\tan^{15}x\right]-14\ln\left(2\frac{\sin x}{x}+\frac{\tan x}{x}-3\right),\quad 0<x<\frac{\pi}{2}.
$$

Then

$$
F_2'(x)=\frac{\cos x}{x(\tan x)(2\sin x+\tan x-3x)}f_2(x),
$$

where

$$
\begin{aligned}
f_2(x) \;=\;& 55\sec^3 x-55\sec x+2x\tan x-45x^2\sec^3 x\\
&-123x\sec x\tan x+110\tan^2 x+x\tan x\sec^3 x+30x\tan^3 x.
\end{aligned}
$$

By substituting the expansions of power series of corresponding functions in Lemmas 2.2 and 2.3 into the above formula, we obtain that

$$
\begin{aligned}
f_2(x)=&\frac{55}{2}\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right]-55\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\\
&+2x\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1}-\frac{45}{2}x^2\left[\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right]\\
&-123x\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1}+110\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}
\end{aligned}
$$

$$+\frac{1}{6}x\left[\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-3}+\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}\right]$$
$$+30x\left[\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-3}-\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1}\right]$$
$$=\frac{55}{2}\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+\frac{55}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-55\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}$$
$$+2\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n}-\frac{45}{2}\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n}-\frac{45}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}$$
$$-123\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n}+110\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}$$
$$+\frac{1}{6}\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-2}+\frac{1}{6}\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}$$
$$+30\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-2}-30\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n}$$
$$=\frac{55}{2}\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+110\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}$$
$$+\frac{1}{6}\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-2}$$
$$+30\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-2}$$
$$+\frac{55}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-55\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}+2\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n}$$
$$-\frac{45}{2}\sum_{n=1}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n}-123\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n}+\frac{1}{6}\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}$$
$$-30\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n}-\frac{45}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}$$

$$=\frac{55}{2}\sum_{n=6}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-2}+110\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}$$
$$+\frac{1}{6}\sum_{n=6}^{\infty}\frac{(2n-1)(2n-2)|E_{2n}|}{(2n-1)!}x^{2n-2}+30\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-2}$$
$$+\frac{55}{2}\sum_{n=5}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-55\sum_{n=5}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}+2\sum_{n=5}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n}$$
$$-\frac{45}{2}\sum_{n=5}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n}-123\sum_{n=5}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n}+\frac{1}{6}\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}$$
$$-30\sum_{n=5}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n}-\frac{45}{2}\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}$$
$$=\sum_{n=6}^{\infty}\frac{55(2n)(2n-1)|E_{2n}|}{2(2n)!}x^{2n-2}+\sum_{n=6}^{\infty}\frac{2n(2n-1)(2n-2)|E_{2n}|}{6(2n)!}x^{2n-2}$$

$$
\begin{aligned}
&+\sum_{n=6}^{\infty}\frac{110\,(2n-1)\,(2^{2n}-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{15\,(2n-1)\,(2n-2)\,(2^{2n}-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}+\sum_{n=5}^{\infty}\frac{2\,(2^{2n}-1)\,2^{2n}}{(2n)!}|B_{2n}|x^{2n}\\
&-\sum_{n=5}^{\infty}\frac{30\,(2^{2n}-1)\,2^{2n}}{(2n)!}|B_{2n}|x^{2n}-\sum_{n=5}^{\infty}\frac{45\,(2n)\,(2n-1)\,|E_{2n}|}{2\,(2n)!}x^{2n}\\
&-\sum_{n=5}^{\infty}\frac{123\,(2n)\,|E_{2n}|}{(2n)!}x^{2n}+\sum_{n=5}^{\infty}\frac{(2n)\,|E_{2n}|}{6\,(2n)!}x^{2n}+\sum_{n=5}^{\infty}\frac{55|E_{2n}|}{2\,(2n)!}x^{2n}-\sum_{n=5}^{\infty}\frac{55|E_{2n}|}{(2n)!}x^{2n}\\
&-\sum_{n=4}^{\infty}\frac{45|E_{2n}|}{2\,(2n)!}x^{2n+2}\\
=&\sum_{n=6}^{\infty}\frac{n\,(2n+163)\,(2n-1)}{3}\frac{|E_{2n}|}{(2n)!}x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{10\times 2^{2n}\,(3n+8)\,(2n-1)\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}x^{2n-2}-\sum_{n=5}^{\infty}\frac{28\times 2^{2n}\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}x^{2n}\\
&-\sum_{n=5}^{\infty}\frac{(540n^2+1204n+165)}{6}\frac{|E_{2n}|}{(2n)!}x^{2n}-\sum_{n=4}^{\infty}\frac{45}{2}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=&\sum_{n=6}^{\infty}\left[\frac{n\,(2n+163)\,(2n-1)}{3}\frac{|E_{2n}|}{(2n)!}+\frac{10\times 2^{2n}\,(3n+8)\,(2n-1)\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}\right]x^{2n-2}\\
&-\sum_{n=5}^{\infty}\left[\frac{28\cdot 2^{2n}\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}+\frac{(540n^2+1204n+165)}{6}\frac{|E_{2n}|}{(2n)!}\right]x^{2n}-\sum_{n=4}^{\infty}\frac{45}{2}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=&\sum_{n=5}^{\infty}\left[\frac{(n+1)\,(2n+165)\,(2n+1)\,|E_{2n+2}|}{3\,(2n+2)!}\right.\\
&\left.+\frac{10\cdot 2^{2n+2}\,(3n+11)\,(2n+1)\,(2^{2n+2}-1)\,|B_{2n+2}|}{(2n+2)!}\right]x^{2n}\\
&-\sum_{n=5}^{\infty}\left[\frac{28\cdot 2^{2n}\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}+\frac{(540n^2+1204n+165)}{6}\frac{|E_{2n}|}{(2n)!}\right]x^{2n}-\sum_{n=5}^{\infty}\frac{45}{2}\frac{|E_{2n-2}|}{(2n-2)!}x^{2n}\\
:=&\sum_{n=5}^{\infty}b_nx^{2n},
\end{aligned}
$$

where

$$
\begin{aligned}
b_n=&\frac{(n+1)\,(2n+165)\,(2n+1)}{3}\frac{|E_{2n+2}|}{(2n+2)!}\\
&+\frac{10\cdot 2^{2n+2}\,(3n+11)\,(2n+1)\,(2^{2n+2}-1)\,|B_{2n+2}|}{(2n+2)!}\\
&-\left[\frac{28\cdot 2^{2n}\,(2^{2n}-1)\,|B_{2n}|}{(2n)!}+\frac{(540n^2+1204n+165)}{6}\frac{|E_{2n}|}{(2n)!}\right]-\frac{45}{2}\frac{|E_{2n-2}|}{(2n-2)!}.
\end{aligned}
$$

From Lemmas 2.4 and 2.5 we have

$$
b_n>\frac{(n+1)\,(2n+165)\,(2n+1)}{3}\frac{4^{n+2}}{\pi^{2n+3}}\left(\frac{1}{1+3^{-3-2n}}\right)
$$

$$
\begin{aligned}
&+\frac{10\cdot 2^{2n+2}\left(3n+11\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{2^{2n+2}}{2^{2n+2}-1}\\
&-\left[\frac{28\cdot 2^{2n}\left(2^{2n}-1\right)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-2}+\frac{\left(540n^2+1204n+165\right)}{6}\frac{4^{n+1}}{\pi^{2n+1}}\right]\\
&-\frac{45}{2}\frac{4^n}{\pi^{2n-1}}\\
=&\frac{(n+1)\left(2n+165\right)\left(2n+1\right)}{3}\frac{4^{n+2}}{\pi^{2n+3}}\frac{3^{3+2n}}{1+3^{3+2n}}\\
&+\frac{10\cdot 2^{2n+3}\left(3n+11\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{\pi^{2n+2}\left(2^{2n+2}-1\right)}\\
&-\left[\frac{28\cdot 2^{2n+1}\left(2^{2n}-1\right)}{\pi^{2n}\left(2^{2n}-2\right)}+\frac{\left(540n^2+1204n+165\right)}{6}\frac{4^{n+1}}{\pi^{2n+1}}\right]\\
&-\frac{45}{2}\frac{4^n}{\pi^{2n-1}}\\
:=&\frac{1}{6}\frac{2^{2n}h_2(n)}{\pi^{2n+3}\left(2^{2n}-2\right)\left(3^{2n+3}+1\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
h_2(n) &= u_1(n)6^{2n}-v_1(n)3^{2n}-w_1(n)2^{2n}+\delta_1(n)\\
&= \left[u_1(n)3^{2n}-v_1(n)\left(\frac{3}{2}\right)^{2n}-w_1(n)\right]2^{2n}+\delta_1(n)\\
&:= l(n)2^{2n}+\delta_1(n)
\end{aligned}
$$

with

$$
l(n) = u_1(n)3^{2n}-v_1(n)\left(\frac{3}{2}\right)^{2n}-w_1(n),
$$

$$
\begin{aligned}
u_1(n) &= 3456n^3-1296\left(45\pi^2-60\pi-224\right)n^2-432\left(301\pi^2-750\pi-994\right)n\\
&\quad -\left(3645\pi^4+9072\pi^3-142\,560-142\,560\pi+17\,820\pi^2\right),\\
v_1(n) &= 6912n^3-2592n^2\left(45\pi^2-60\pi-224\right)-864n\left(301\pi^2-750\pi-994\right)\\
&\quad -\left(35\,640\pi^2+9072\pi^3-285\,120\pi+7290\pi^4-285\,120\right),\\
w_1(n) &= 720\pi n^2\left(3\pi-4\right)+16\pi n\left(301\pi-750\right)+\left(135\pi^4+336\pi^3+660\pi^2-5280\pi\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_1(n) &= 1440\pi n^2\left(3\pi-4\right)+32\pi n\left(301\pi-750\right)+270\pi^4+336\pi^3+1320\pi^2-10\,560\pi\\
&> 0
\end{aligned}
$$

for $n \geq 5$. Now we will prove

$$
u_1(n)3^{2n}-v_1(n)\left(\frac{3}{2}\right)^{2n}-w_1(n)>0,\ n\geq 5.
$$

Since

$$
l(n+1)-9l(n)=u_2(n)3^{2n}+v_2(n)\left(\frac{3}{2}\right)^{2n}+\delta_2(n),
$$

where

$$
u_2(n) = 93\,312n^2-69\,984n\left(15\pi^2-20\pi-76\right)
$$

$$
\begin{aligned}
& +\left(3615\,840\pi-1695\,168\pi^2+6508\,512\right) \\
& > 0, \\
v_2(n) &= 46\,656n^3-5832n^2\left(135\pi^2-180\pi-664\right)-5832n\left(+211\pi^2-630\pi-538\right) \\
& -\frac{98\,415}{2}\pi^4-61\,236\pi^3+607\,014\pi^2+116\,640\pi-1329\,696 \\
& > 0, \\
\delta_2(n) &= 5760\pi n^2(3\pi-4)+32\pi n(1069\pi-2820) \\
& +1080\pi^4+2688\pi^3-1696\pi^2-27\,360\pi \\
& > 0
\end{aligned}
$$

for $n \geq 5$. So $l(n+1)-9l(n)>0$. In consideration of

$$
\begin{aligned}
l(5) &= \frac{1747\,530\,566\,925}{8}\pi-\frac{16\,037\,360\,937\,025}{128}\pi^2-\frac{34\,250\,862\,513}{64}\pi^3 \\
& -\frac{109\,984\,441\,275}{512}\pi^4+\frac{4704\,887\,031\,075}{8} \\
&\approx 2.648\,2\times 10^8>0,
\end{aligned}
$$

we obtain $l(n)>0$ for all $n\geq 5$. This leads to $h_2(n)>0$ and $b_n>0$ for $n\geq 5$. So $f_2(x)>0$ and $F_2'(x)>0$ on $(0,\pi/2)$. We come to a conclusion that the function $F_2(x)$ is increasing on $(0,\pi/2)$. Then $F_2(x)>0=F_2(0^+)$ holds for all $x\in(0,\pi/2)$. At the same time,

$$
\lim_{x\to 0^+}\frac{2\frac{\sin x}{x}+\frac{\tan x}{x}-3}{x^4\left(\frac{\tan x}{x}\right)^{15/14}}=\frac{3}{20},
$$

The proof of Theorem 2.1 is completed. □

3.3. Proof of Theorem 1.3.

Since

$$
\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1-\cos x}{\sin x},
$$

we have

$$
\frac{1}{40}x^3\tan x-\left[\frac{\sin x}{x}+2\frac{\tan(x/2)}{x/2}-3\right]=\frac{1}{40}x^3\tan x-\left[\frac{\sin x}{x}+\frac{4(1-\cos x)}{x\sin x}-3\right].
$$

Let

$$
F_3(x)=\frac{1}{40}x^3\tan x-\left[\frac{\sin x}{x}+\frac{4(1-\cos x)}{x\sin x}-3\right],\ 0<x<\frac{\pi}{2}.
$$

Then

$$
F_3'(x)=\frac{1}{40}\frac{\cos^3 x}{x^2}f_3(x),
$$

where

$$
\begin{aligned}
f_3(x) &= 160x\sec^2 x-80x\sec x-80x\sec^3 x-160\sec x\tan x+160\sec^2 x\tan x+40\tan^3 x \\
& +x^5\sec x\tan^2 x-40x\tan^2 x-80x\sec x\tan^2 x+3x^4\sec x\tan^3 x+x^5\sec x\tan^4 x.
\end{aligned}
$$

By substituting the expansions of power series of corresponding functions in Lemmas 2.2 and 2.3 into the above formula, we obtain that

$$
\begin{aligned}
f_3(x) &= \sum_{n=1}^{\infty}\frac{160(2^{2n}-1)(2n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}-\sum_{n=0}^{\infty}\frac{80|E_{2n}|}{(2n)!}x^{2n+1} \\
& -\sum_{n=1}^{\infty}\frac{40(2n-1)|E_{2n}|}{(2n-1)!}x^{2n-1}-\sum_{n=0}^{\infty}\frac{40|E_{2n}|}{(2n)!}x^{2n+1}-\sum_{n=1}^{\infty}\frac{160(2n)|E_{2n}|}{(2n)!}x^{2n-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=2}^{\infty}\frac{80\,(2n-1)\,(2n-2)\,(2^{2n}-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}\\
&+\sum_{n=2}^{\infty}\frac{40\,(2n-1)\,(2n-2)\,2^{2n}(2^{2n}-1)}{2\,(2n)!}|B_{2n}|x^{2n-3}-\sum_{n=1}^{\infty}\frac{40\left(2^{2n}-1\right)2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\\
&+\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{2\,(2n-1)!}x^{2n+3}-\sum_{n=0}^{\infty}\frac{|E_{2n}|}{2\,(2n)!}x^{2n+5}-\sum_{n=2}^{\infty}\frac{40(2^{2n}-1)\,(2n-1)\,2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\\
&-\sum_{n=1}^{\infty}\frac{40\,(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n-1}+\sum_{n=0}^{\infty}\frac{40|E_{2n}|}{(2n)!}x^{2n+1}+\sum_{n=2}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{2\,(2n-1)!}x^{2n+1}\\
&-\sum_{n=1}^{\infty}\frac{5\,(2n)\,|E_{2n}|}{2\,(2n)!}x^{2n+3}+\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+5}-\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n+3}-\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+5}\\
&+\sum_{n=1}^{\infty}\frac{(2n-1)\,(2n-2)\,(2n-3)\,|E_{2n}|}{24\,(2n-1)!}x^{2n+1}+\sum_{n=0}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{24\,(2n)!}x^{2n+3}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{n=6}^{\infty}\frac{80\,(2n-1)\,(2n-2)\,(2^{2n}-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}\\
&+\sum_{n=6}^{\infty}\frac{20\,(2n-1)\,(2n-2)\,(2^{2n}-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}\\
&+\sum_{n=5}^{\infty}\frac{160\,(2n-1)\,(2^{2n}-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}-\sum_{n=5}^{\infty}\frac{40\,(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n-1}\\
&-\sum_{n=5}^{\infty}\frac{160\,(2n)\,|E_{2n}|}{(2n)!}x^{2n-1}-\sum_{n=5}^{\infty}\frac{40\,(2n-1)\,(2^{2n}-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\\
&-\sum_{n=5}^{\infty}\frac{40\,(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n-1}-\sum_{n=5}^{\infty}\frac{40\left(2^{2n}-1\right)2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\\
&-\sum_{n=4}^{\infty}\frac{40|E_{2n}|}{(2n)!}x^{2n+1}-\sum_{n=4}^{\infty}\frac{80|E_{2n}|}{(2n)!}x^{2n+1}+\sum_{n=4}^{\infty}\frac{40|E_{2n}|}{(2n)!}x^{2n+1}\\
&+\sum_{n=4}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{2\,(2n-1)!}x^{2n+1}+\sum_{n=4}^{\infty}\frac{(2n-1)\,(2n-2)\,(2n-3)\,|E_{2n}|}{24\,(2n-1)!}x^{2n+1}\\
&+\sum_{n=3}^{\infty}\frac{(2n-1)\,|E_{2n}|}{2\,(2n-1)!}x^{2n+3}+\sum_{n=3}^{\infty}\frac{3\,(2n-1)\,|E_{2n}|}{8\,(2n-1)!}x^{2n+3}-\sum_{n=3}^{\infty}\frac{5\,(2n)\,|E_{2n}|}{2\,(2n)!}x^{2n+3}\\
&-\sum_{n=3}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n+3}+\sum_{n=3}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{24\,(2n)!}x^{2n+3}\\
&+\sum_{n=2}^{\infty}\frac{3|E_{2n}|}{8\,(2n)!}x^{2n+5}-\sum_{n=2}^{\infty}\frac{|E_{2n}|}{2\,(2n)!}x^{2n+5}+\sum_{n=2}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+5}-\sum_{n=2}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+5}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{n=6}^{\infty}\frac{200\times 2^{2n}\,(2n-1)\,(n-1)\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-3}\\
&+\sum_{n=5}^{\infty}\frac{80\times 2^{2n}\left(2^{2n}-1\right)(3n-2)}{(2n)!}|B_{2n}|x^{2n-1}-\sum_{n=5}^{\infty}\frac{160n\,(2n+1)\,|E_{2n}|}{(2n)!}x^{2n-1}\\
&+\sum_{n=4}^{\infty}\frac{9n-25n^2+12n^3+4n^4-480}{6}\frac{|E_{2n}|}{(2n)!}x^{2n+1}-\sum_{n=3}^{\infty}\frac{n\,(2n+29)}{6}\frac{|E_{2n}|}{(2n)!}x^{2n+3}\\
&-\sum_{n=2}^{\infty}\frac{1}{8}\frac{|E_{2n}|}{(2n)!}x^{2n+5}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=4}^{\infty} \frac{200\times 2^{2n+4}(2n+3)(n+1)\left(2^{2n+4}-1\right)}{(2n+4)!}|B_{2n+4}|x^{2n+1} \\
&\quad +\sum_{n=4}^{\infty} \frac{80\times 2^{2n+2}\left(2^{2n+2}-1\right)(3n+1)}{(2n+2)!}|B_{2n+2}|x^{2n+1} \\
&\quad -\sum_{n=4}^{\infty} \frac{160(n+1)(2n+3)|E_{2n+2}|}{(2n+2)!}x^{2n+1}+\sum_{n=4}^{\infty} \frac{9n-25n^2+12n^3+4n^4-480}{6}\frac{|E_{2n}|}{(2n)!}x^{2n+1} \\
&\quad -\sum_{n=4}^{\infty} \frac{(n-1)(2(n-1)+29)}{6}\frac{|E_{2n-2}|}{(2n-2)!}x^{2n+1}-\sum_{n=4}^{\infty} \frac{1}{8}\frac{|E_{2n-4}|}{(2n-4)!}x^{2n+1} \\
&:= \sum_{n=4}^{\infty} c_n x^{2n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
c_n &= \frac{200\times 2^{2n+4}(2n+3)(n+1)\left(2^{2n+4}-1\right)}{(2n+4)!}|B_{2n+4}|+\frac{80\times 2^{2n+2}\left(2^{2n+2}-1\right)(3n+1)}{(2n+2)!}|B_{2n+2}| \\
&\quad -\frac{160(n+1)(2n+3)|E_{2n+2}|}{(2n+2)!}+\frac{9n-25n^2+12n^3+4n^4-480}{6}\frac{|E_{2n}|}{(2n)!} \\
&\quad -\frac{(n-1)(2(n-1)+29)}{6}\frac{|E_{2n-2}|}{(2n-2)!}-\frac{1}{8}\frac{|E_{2n-4}|}{(2n-4)!}.
\end{aligned}
$$

From Lemmas 2.4 and 2.5 we have

$$
\begin{aligned}
c_n &> \frac{200\cdot 2^{2n+4}(2n+3)(n+1)\left(2^{2n+4}-1\right)}{(2n+4)!}\frac{2(2n+4)!}{(2\pi)^{2n+4}}\frac{2^{2n+4}}{2^{2n+4}-1} \\
&\quad +\frac{80\cdot 2^{2n+2}\left(2^{2n+2}-1\right)(3n+1)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{2^{2n+2}}{2^{2n+2}-1} \\
&\quad -\frac{160(n+1)(2n+3)}{1}\frac{4^{n+2}}{\pi^{2n+3}}+\frac{9n-25n^2+12n^3+4n^4-480}{6}\frac{4^{n+1}}{\pi^{2n+1}}\frac{1}{1+3^{-1-2n}} \\
&\quad -\frac{(n-1)(2(n-1)+29)}{6}\frac{4^{n}}{\pi^{2n-1}}-\frac{1}{8}\frac{4^{n-1}}{\pi^{2n-3}} \\
&= \frac{200\cdot 2^{2n+5}(2n+3)(n+1)}{\pi^{2n+4}}+\frac{80\cdot 2^{2n+3}(3n+1)}{\pi^{2n+2}} \\
&\quad -\frac{160\cdot 2^{2(n+2)}(n+1)(2n+3)}{\pi^{2n+3}}+\frac{9n-25n^2+12n^3+4n^4-480}{6}\frac{2^{2(n+1)}}{\pi^{2n+1}}\frac{3^{1+2n}}{1+3^{1+2n}} \\
&\quad -\frac{(n-1)(2(n-1)+29)}{6}\frac{2^{2n}}{\pi^{2n-1}}-\frac{1}{8}\frac{2^{2(n-1)}}{\pi^{2n-3}} \\
&:= \frac{1}{96}\frac{2^{2n}h_3(n)}{\pi^{2n+4}\left(3^{2n+1}+1\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
h_3(n) &= u_3(n)3^{2n}-v_3(n), \\
u_3(n) &= 768\pi^3n^4+2304\pi^3n^3-96n^2\left(15\,360\pi+50\pi^3+\pi^5-38\,400\right) \\
&\quad +48n\left(-76\,800\pi+11\,520\pi^2+36\pi^3-25\pi^5+192\,000\right) \\
&\quad -\left(9\pi^7-1296\pi^5+92\,160\pi^3-184\,320\pi^2+2211\,840\pi-5529\,600\right), \\
v_3(n) &= 32n^2\left(15\,360\pi+\pi^5-38\,400\right)-80n\left(38\,400-15\,360\pi+2304\pi^2-5\pi^5\right) \\
&\quad -\left(61\,440\pi^2+432\pi^5-3\pi^7+1843\,200-737\,280\pi\right).
\end{aligned}
$$

By mathematical induction we can prove the inequality

$$3^{2n} > \frac{v_3(n)}{u_3(n)} \tag{3.1}$$

holds for $n \geq 4$. First, The inequality (3.1) is obviously true for $n = 4$. Let's assume that (3.1) holds for $n = m$, that is,

$$3^{2m} > \frac{v_3(m)}{u_3(m)}$$

holds. In the following we shall prove that (3.1) holds for $n = m + 1$. Since

$$3^{2(m+1)} = 9 \cdot 3^{2m} > 9 \cdot \frac{v_3(m)}{u_3(m)},$$

we can complete the proof of (3.1) as long as

$$\frac{9v_3(m)}{u_3(m)} > \frac{v_3(m+1)}{u_3(m+1)}. \tag{3.2}$$

In fact,

$$9v_3(m)u_3(m+1) - v_3(m+1)u_3(m) = \sum_{i=0}^{6} r_i\,(m-4)^i\,,$$

where

$$\begin{aligned}
r_0 &= 39\,757\,572\,734\,976\,000\pi - 10\,615\,744\,836\,403\,200\pi^2 + 876\,832\,476\,364\,800\pi^3 \\
&\quad +40\,143\,091\,138\,560\pi^4 + 1812\,399\,390\,720\pi^5 - 2527\,372\,247\,040\pi^6 + 170\,277\,765\,120\pi^7 \\
&\quad +7562\,013\,696\pi^8 + 136\,028\,160\pi^9 - 174\,238\,656\pi^{10} - 355\,392\pi^{12} - 49\,696\,965\,918\,720\,000, \\
r_1 &= 27\,108\,424\,286\,208\,000\pi - 6910\,872\,492\,441\,600\pi^2 + 379\,995\,173\,683\,200\pi^3 \\
&\quad +71\,051\,661\,803\,520\pi^4 - 380\,094\,382\,080\pi^5 - 1634\,567\,454\,720\pi^6 + 87\,787\,929\,600\pi^7 \\
&\quad +10\,286\,349\,312\pi^8 + 26\,542\,080\pi^9 - 107\,286\,912\pi^{10} - 81\,216\pi^{12} - 33\,885\,530\,357\,760\,000, \\
r_2 &= 6803\,832\,176\,640\,000\pi - 1633\,689\,796\,608\,000\pi^2 + 7396\,982\,784\,000\pi^3 + 39\,078\,199\,296\,000\pi^4 \\
&\quad -847\,211\,397\,120\pi^5 - 359\,792\,640\,000\pi^6 + 12\,858\,163\,200\pi^7 + 4938\,654\,720\pi^8 \\
&\quad -18\,947\,328\pi^{10} - 2304\pi^{12} - 8504\,790\,220\,800\,000, \\
r_3 &= 739\,271\,245\,824\,000\pi - 164\,161\,703\,116\,800\pi^2 - 18\,830\,013\,235\,200\pi^3 + 10\,141\,197\,926\,400\pi^4 \\
&\quad -258\,861\,957\,120\pi^5 - 30\,576\,476\,160\pi^6 + 424\,673\,280\pi^7 + 1157\,652\,480\pi^8 \\
&\quad -930\,816\pi^{10} - 924\,089\,057\,280\,000, \\
r_4 &= 28\,991\,029\,248\,000\pi - 5798\,205\,849\,600\pi^2 - 3508\,273\,152\,000\pi^3 + 1403\,309\,260\,800\pi^4 \\
&\quad -29\,491\,200\,000\pi^5 - 754\,974\,720\pi^6 + 142\,233\,600\pi^8 - 6144\pi^{10} - 36\,238\,786\,560\,000, \\
r_5 &= 101\,921\,587\,200\pi^4 - 254\,803\,968\,000\pi^3 - 1132\,462\,080\pi^5 + 8601\,600\pi^8, \\
r_6 &= 3019\,898\,880\pi^4 - 7549\,747\,200\pi^3 + 196\,608\pi^8.
\end{aligned}$$

Since $r_i > 0$ for $i = 0, 1, \cdots, 6$, we have (3.2) holds. This means (3.1) holds for $n \geq 4$ and $h_3(n) > 0$ for $n \geq 4$. So $c_n > 0$ holds for all $n \geq 4$ and $f_3(x) > 0$ and $F_3'(x) > 0$ for all $x \in (0, \pi/2)$. That is, $F_3(x)$ is increasing on $(0, \pi/2)$. So $F_3(x) > 0 = F_3(0^+)$ holds for all $x \in (0, \pi/2)$. At the same time,

$$\lim_{x \to 0^+} \frac{\frac{\sin x}{x} + 2^{\frac{\tan(x/2)}{x/2}} - 3}{x^3 \tan x} = \frac{1}{40}.$$

The proof of Theorem 1.3 is completed. □

3.4. Proof of Theorem 1.4.

Since

$$\begin{aligned} \frac{\sin x}{x}+2\frac{\tan(x/2)}{x/2}-3 &< \frac{1}{40}x^4\left(\frac{\tan x}{x}\right)^{5/28} \\ \Longleftrightarrow\quad \frac{\sin x}{x}+\frac{4\,(1-\cos x)}{x\sin x}-3 &< \frac{1}{40}x^4\left(\frac{\tan x}{x}\right)^{5/28} \\ \Longleftrightarrow\quad \left(\frac{\sin x}{x}+\frac{4\,(1-\cos x)}{x\sin x}-3\right)^{28} &< \frac{1}{40^{28}}x^{112}\left(\frac{\tan x}{x}\right)^{5}, \end{aligned}$$

we let

$$F_4(x)=\ln\left(\frac{1}{40^{28}}x^{112}\right)+5\ln\frac{\tan x}{x}-28\ln\left[\frac{\sin x}{x}+\frac{4\,(1-\cos x)}{x\sin x}-3\right],\ 0<x<\frac{\pi}{2}.$$

Then

$$F_4'(x)=\frac{\cos^3 x}{(\tan x)\,(x\sin x)^2\left[\frac{\sin x}{x}+\frac{4(1-\cos x)}{x\sin x}-3\right]}f_4(x),$$

where

$$\begin{aligned} f_4(x) &= 132x\sec^2 x\tan x-76x\sec x\tan x-15x^2\sec x\tan^2 x-540\sec x\tan^2 x \\ &\quad -56x\sec^3 x\tan x+540\sec^2 x\tan^2 x-23x\tan^3 x+135\tan^4 x+5x\tan^5 x \\ &\quad -15x^2\sec x\tan^4 x-397x\sec x\tan^3 x+20x\sec^2 x\tan^3 x. \end{aligned}$$

By substituting the expansions of power series of corresponding functions in Lemmas 2.2 and 2.3 into the above formula, we obtain that

$$\begin{aligned} f_4(x)=&66\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{(2n)!}|B_{2n}|x^{2n-2}-76\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n} \\ &-\frac{15}{2}\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n}+\frac{15}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}-270\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n-2} \\ &+270\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-\frac{28}{3}\sum_{n=2}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{(2n-1)!}x^{2n-2}-\frac{28}{3}\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n} \\ &+540\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4} \\ &-540\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2} \\ &-23\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2} \\ &+23\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n} \\ &+135\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4} \\ &-135\sum_{n=2}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2} \\ &-5\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}+5\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n} \end{aligned}$$

$$
\begin{aligned}
&+\frac{5}{4}\sum_{n=3}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&-\frac{5}{4}\sum_{n=2}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)\,(2n-2)}{3\,(2n)!}|B_{2n}|x^{2n-2}-15\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
&+15\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n}+15\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}-\frac{15}{24}\sum_{n=1}^{\infty}\frac{(2n-1)\,(2n-2)\,(2n-3)\,|E_{2n}|}{(2n-1)!}x^{2n-2}\\
&-\frac{15}{24}\sum_{n=0}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{(2n)!}x^{2n}-\frac{45}{8}\sum_{n=1}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n}-\frac{45}{8}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
&-\frac{397}{6}\sum_{n=2}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{(2n-1)!}x^{2n-2}+397\cdot\frac{5}{6}\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}\\
&+5\sum_{n=3}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&-20\sum_{n=2}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)\,(2n-2)}{3\,(2n)!}|B_{2n}|x^{2n-2}\\
=&\frac{5}{4}\sum_{n=7}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&+540\sum_{n=7}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&+135\sum_{n=7}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&+5\sum_{n=7}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)}{6\,(2n)!}|B_{2n}|x^{2n-4}\\
&-540\sum_{n=6}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2}-23\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}\\
&-\frac{15}{24}\sum_{n=6}^{\infty}\frac{(2n-1)\,(2n-2)\,(2n-3)\,|E_{2n}|}{(2n-1)!}x^{2n-2}-135\sum_{n=6}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3\,(2n)!}|B_{2n}|x^{2n-2}\\
&-5\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}\\
&-\frac{5}{4}\sum_{n=6}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)\,(2n-2)}{3\,(2n)!}|B_{2n}|x^{2n-2}\\
&+66\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{(2n)!}|B_{2n}|x^{2n-2}-270\sum_{n=6}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n-2}\\
&-\frac{28}{3}\sum_{n=6}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{(2n-1)!}x^{2n-2}-\frac{397}{6}\sum_{n=6}^{\infty}\frac{(2n-1)\,(2n-2)\,|E_{2n}|}{(2n-1)!}x^{2n-2}\\
&-20\sum_{n=6}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)\,(2n-2)}{3\,(2n)!}|B_{2n}|x^{2n-2}-76\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}\\
&-\frac{15}{2}\sum_{n=5}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n}+270\sum_{n=5}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}-\frac{28}{3}\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}\\
&+23\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+5\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+15\sum_{n=5}^{\infty}\frac{(2n-1)\,|E_{2n}|}{(2n-1)!}x^{2n}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{15}{24}\sum_{n=5}^{\infty}\frac{2n(2n-1)|E_{2n}|}{(2n)!}x^{2n}-\frac{45}{8}\sum_{n=5}^{\infty}\frac{(2n-1)|E_{2n}|}{(2n-1)!}x^{2n}+397\cdot\frac{5}{6}\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}\\
&+\frac{15}{2}\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}-15\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}+15\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}-\frac{45}{8}\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=&\sum_{n=7}^{\infty}\frac{25(n-1)(n+52)(2n-1)(2n-3)\left(2^{2n}-1\right)2^{2n}}{6(2n)!}|B_{2n}|x^{2n-4}\\
&+\sum_{n=6}^{\infty}\frac{2(2n-1)(71n-611)\left(2^{2n}-1\right)2^{2n}}{3(2n)!}|B_{2n}|x^{2n-2}\\
&-\sum_{n=6}^{\infty}\frac{n(2n-1)\left(579n+10n^2+491\right)}{2}\frac{|E_{2n}|}{(2n)!}x^{2n-2}\\
&+\sum_{n=5}^{\infty}\frac{28\cdot 2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+\sum_{n=5}^{\infty}\frac{\left(977n+10n^2+540\right)|E_{2n}|}{2(2n)!}x^{2n}\\
&+\sum_{n=4}^{\infty}\frac{15}{8}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=&\sum_{n=6}^{\infty}\frac{25n(n+53)(2n+1)(2n-1)\left(2^{2n+2}-1\right)2^{2n+2}}{6(2n+2)!}|B_{2n+2}|x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{2(2n-1)(71n-611)\left(2^{2n}-1\right)2^{2n}}{3(2n)!}|B_{2n}|x^{2n-2}\\
&-\sum_{n=6}^{\infty}\frac{n(2n-1)\left(579n+10n^2+491\right)}{2}\frac{|E_{2n}|}{(2n)!}x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{28\cdot 2^{2n-2}\left(2^{2n-2}-1\right)}{(2n-2)!}|B_{2n-2}|x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{\left[977(n-1)+10(n-1)^2+540\right]|E_{2n-2}|}{2(2n-2)!}x^{2n-2}\\
&+\sum_{n=6}^{\infty}\frac{15}{8}\frac{|E_{2n-4}|}{(2n-4)!}x^{2n-2}\\
:=&\sum_{n=6}^{\infty}d_nx^{2n-2},
\end{aligned}
$$

where

$$
\begin{aligned}
d_n =\ & \frac{25}{6}\frac{2^{2n+2}(n)(n+53)(2n+1)(2n-1)\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|\\
&+\frac{2}{3}\frac{2^{2n}(2n-1)(71n-611)\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|-\frac{n(2n-1)\left(579n+10n^2+491\right)}{2}\frac{|E_{2n}|}{(2n)!}\\
&+\frac{28\times 2^{2n-2}\left(2^{2n-2}-1\right)}{(2n-2)!}|B_{2n-2}|+\frac{1}{2}\left(977(n-1)+10(n-1)^2+540\right)\frac{|E_{2n-2}|}{(2n-2)!}\\
&+\frac{15}{8}\frac{|E_{2n-4}|}{(2n-4)!}.
\end{aligned}
$$

By Lemmas 2.4 and 2.5 we have

$$
d_n > \frac{25}{6}\frac{2^{2n+2}(n)(n+53)(2n+1)(2n-1)\left(2^{2n+2}-1\right)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{2^{2n+2}}{2^{2n+2}-1}
$$

$$
\begin{aligned}
&+\frac{2}{3}\frac{2^{2n}(2n-1)(71n-611)\left(2^{2n}-1\right)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-1} \\
&-\frac{n(2n-1)\left(579n+10n^2+491\right)}{2}\frac{4^{n+1}}{\pi^{2n+1}} \\
&+\frac{28\times 2^{2n-2}\left(2^{2n-2}-1\right)}{(2n-2)!}\frac{2(2n-2)!}{(2\pi)^{2n-2}}\frac{2^{2n-2}}{2^{2n-2}-1} \\
&+\frac{\left(977(n-1)+10(n-1)^2+540\right)4^n}{2\pi^{2n-1}}\left(\frac{1}{1+3^{1-2n}}\right) \\
&+\frac{15}{8}\frac{|E_{2n-4}|}{(2n-4)!}\frac{4^{n-1}}{\pi^{2n-3}}\left(\frac{1}{1+3^{3-2n}}\right) \\
=\;&\frac{25}{6}\frac{2^{2n+3}n(n+53)(2n+1)(2n-1)}{\pi^{2n+2}}+\frac{2}{3}\frac{2^{2n+1}(2n-1)(71n-611)}{\pi^{2n}} \\
&-\frac{n(2n-1)\left(579n+10n^2+491\right)}{2}\frac{2^{2(n+1)}}{\pi^{2n+1}}+\frac{28\times 2^{2n-1}}{\pi^{2n-2}} \\
&+\frac{\left(977(n-1)+10(n-1)^2+540\right)2^{2n}}{2\pi^{2n-1}}\left(\frac{3^{2n-1}}{1+3^{2n-1}}\right) \\
&+\frac{15}{8}\frac{2^{2(n-1)}}{\pi^{2n-3}}\frac{3^{2n-3}}{1+3^{2n-3}} \\
:=\;&\frac{1}{96}\frac{2^{2n}h_4(n)}{\pi^{2n+2}\left(3^{2n}+27\right)\left(3^{2n}+3\right)},
\end{aligned}
$$

where

$$h_4(n)=u_4(n)\,9^{2n}+v_4(n)\,3^{2n}+\delta_4(n)$$

with

$$
\begin{aligned}
u_4(n) &= 1280n^4(10-3\pi)-256\times 9^{2n}n^3(861\pi-2650)-32n^2\left(2418\pi-568\pi^2-15\pi^3+100\right) \\
&\quad -16n\left(10\,344\pi^2-5892\pi-2871\pi^3+10\,600\right)+\pi^2\left(-20\,496\pi+1344\pi^2+45\pi^3+78\,208\right), \\
v_4(n) &= 38\,400n^4(10-3\pi)-7680n^3(861\pi-2650)-480n^2\left(4836\pi-1136\pi^2-27\pi^3+200\right) \\
&\quad -48n\left(+103\,440\pi^2-58\,920\pi-25\,839\pi^3+106\,000\right) \\
&\quad +3\pi^2\left(45\pi^3-184\,464\pi+13\,440\pi^2+782\,080\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_4(n) &= 103\,680n^4(10-3\pi)-20\,736n^3(861\pi-2650)-5184n^2\left(1209\pi-284\pi^2+50\right) \\
&\quad -5184n\left(-1473\pi+2586\pi^2+2650\right)+6334\,848\pi^2+108\,864\pi^4.
\end{aligned}
$$

Since $u_4(n)>0$, $v_4(n)>0$, and $\delta_4(n)>0$ for $n\geq 6$, we have $h_4(n)>0$ and $d_n>0$ for $n\geq 6$. This means $f_4(x)>0$ and $F_4'(x)>0$ on $(0,\pi/2)$. So $F_4(x)$ is increasing on $(0,\pi/2)$. Then $F_4(x)>0=F_4(0^+)$ holds for all $x\in(0,\pi/2)$. At the same time,

$$\lim_{x\to 0^+}\frac{\frac{\sin x}{x}+2\frac{\tan(x/2)}{x/2}-3}{x^4\left(\frac{\tan x}{x}\right)^{5/28}}=\frac{1}{40}.$$

The proof of Theorem 1.4 is completed. □

3.5. Proof of Theorem 1.5.

Let

$$F_5(x)=\frac{1}{60}x^3\tan x-\left(2\frac{x}{\sin x}+\frac{x}{\tan x}-3\right),\ 0<x<\frac{\pi}{2}.$$

Then

$$F_5'(x) = \frac{1}{60}\frac{1}{\tan^2 x} f_5(x),$$

where

$f_5(x) = 60x - 60\tan x + 60x\tan^2 x + 3x^2\tan^3 x + x^3\tan^4 x - 120\sec x\tan x + 120x\sec x + x^3\tan^2 x.$

By substituting the expansions of power series of corresponding functions in Lemmas 2.2 and 2.3 into the above formula, we obtain that

$$\begin{aligned}
f_5(x) &= 60x - 60\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} + 60x\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2} \\
&\quad +3x^2\left[\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}\right] \\
&\quad +x^3\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!}|B_{2n}|x^{2n-4} \\
&\quad -x^3\sum_{n=2}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3(2n)!}|B_{2n}|x^{2n-2} - 120\sum_{n=1}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad +120x\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n} + x^3\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2} \\
&= -60\sum_{n=4}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1} + 60\sum_{n=4}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-1} \\
&\quad +\sum_{n=4}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!}|B_{2n}|x^{2n-1} - 120\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad +3\sum_{n=4}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-1} - 3\sum_{n=3}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n+1} \\
&\quad +\sum_{n=3}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n+1} - \sum_{n=3}^{\infty}\frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3(2n)!}|B_{2n}|x^{2n+1} \\
&\quad +120\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+1} \\
&= \sum_{n=4}^{\infty}\frac{2}{3}\frac{\left(2^{2n}-1\right)2^{2n}(n-1)\left(5n+2n^2+177\right)}{(2n)!}|B_{2n}|x^{2n-1} - 120\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad -\sum_{n=3}^{\infty}\frac{2}{3}\frac{(n+4)2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n+1} + 120\sum_{n=3}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+1} \\
&= \sum_{n=4}^{\infty}\frac{2}{3}\frac{\left(2^{2n}-1\right)2^{2n}(n-1)\left(5n+2n^2+177\right)}{(2n)!}|B_{2n}|x^{2n-1} - 120\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n-1)!}x^{2n-1} \\
&\quad -\sum_{n=4}^{\infty}\frac{2}{3}\frac{(n+3)2^{2n-2}\left(2^{2n-2}-1\right)}{(2n-2)!}|B_{2n-2}|x^{2n-1} + 120\sum_{n=4}^{\infty}\frac{|E_{2n-2}|}{(2n-2)!}x^{2n-1} \\
&:= \sum_{n=4}^{\infty}e_n x^{2n-1},
\end{aligned}$$

where

$$\begin{aligned}
e_n &= \frac{2}{3}\frac{(2^{2n}-1)2^{2n}(n-1)(5n+2n^2+177)}{(2n)!}|B_{2n}| - 120\frac{2n|E_{2n}|}{(2n)!} \\
&\quad -\frac{2}{3}\frac{(n+3)2^{2n-2}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}| + 120\frac{|E_{2n-2}|}{(2n-2)!}.
\end{aligned}$$

By Lemmas 2.4 and 2.5 we have

$$\begin{aligned} e_n &> \frac{2}{3}\frac{\left(2^{2n}-1\right)2^{2n}\left(n-1\right)\left(5n+2n^2+177\right)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-1}-120\cdot 2n\cdot\frac{4^{n+1}}{\pi^{2n+1}} \\ &\quad -\frac{2}{3}\frac{(n+3)\,2^{2n-2}\left(2^{2n-2}-1\right)}{(2n-2)!}\frac{2(2n-2)!}{(2\pi)^{2n-2}}\frac{2^{2n-2}}{2^{2n-2}-2}+120\frac{4^n}{\pi^{2n-1}}\frac{1}{1+3^{1-2n}} \\ &= \frac{2}{3}\frac{2^{2n+1}\left(n-1\right)\left(5n+2n^2+177\right)}{\pi^{2n}}-\frac{240n\cdot 2^{2(n+1)}}{\pi^{2n+1}}-\frac{1}{3}\frac{(n+3)\,2^{2n}\left(2^{2n-2}-1\right)}{\pi^{2n-2}\left(2^{2n-2}-2\right)} \\ &\quad +\frac{120\cdot 2^{2n}3^{2n-1}}{\pi^{2n-1}\left(1+3^{2n-1}\right)} \\ &= \frac{1}{3}\frac{2^{2n}h_5(n)}{\pi^{2n+1}\left(2^{2n}-8\right)\left(3^{2n}+3\right)}, \end{aligned}$$

where

$$\begin{aligned} h_5(n) &= \left[8\pi n^3+12\pi n^2+n\left(688\pi-\pi^3-2880\right)+360\pi^2-708\pi-3\pi^3\right]6^{2n} \\ &\quad -\left[64\pi n^3+96\pi n^2-4n\left(\pi^3-1376\pi+5760\right)+\left(2880\pi^2-5664\pi-12\pi^3\right)\right]3^{2n} \\ &\quad +\left[24\pi n^3+36\pi n^2-3n\left(\pi^3-688\pi+2880\right)-9\pi^3-2124\pi\right]2^{2n} \\ &\quad -\left[192\pi n^3+288\pi n^2+12n\left(1376\pi-\pi^3-5760\right)-36\pi^3+16\,992\pi\right] \\ &:= \left[u_5(n)2^{2n}-v_5(n)\right]3^{2n}+\left[u_6(n)2^{2n}-v_6(n)\right] \end{aligned}$$

with

$$\begin{aligned} u_5(n) &= 8\pi n^3+12\pi n^2+n\left(688\pi-\pi^3-2880\right)+360\pi^2-708\pi-3\pi^3, \\ v_5(n) &= 64\pi n^3+96\pi n^2-4n\left(\pi^3-1376\pi+5760\right)+\left(2880\pi^2-5664\pi-12\pi^3\right), \\ u_6(n) &= 24\pi n^3+36\pi n^2-3n\left(\pi^3-688\pi+2880\right)-9\pi^3-2124\pi, \\ v_6(n) &= 192\pi n^3+288\pi n^2+12n\left(1376\pi-\pi^3-5760\right)-36\pi^3+16\,992\pi. \end{aligned}$$

It can be proved that

$$2^{2n}>\frac{v_5(n)}{u_5(n)},\ 2^{2n}>\frac{v_6(n)}{u_6(n)}$$

hold for $n\geq 4$ by mathematical induction. We only prove the former, that is,

$$2^{2n}>\frac{v_5(n)}{u_5(n)} \tag{3.3}$$

holds for all $n\geq 4$. First, The inequality (3.3) is obviously true for $n=4$. Let's assume that (3.3) holds for $n=m$, that is,

$$2^{2m}>\frac{v_5(m)}{u_5(m)}$$

holds. In the following we shall prove that (3.3) holds for $n=m+1$. Since

$$2^{2(m+1)}=4\cdot 2^{2m}>4\cdot\frac{v_5(m)}{u_5(m)},$$

we can complete the proof of (3.3) as long as

$$\frac{4v_5(m)}{u_5(m)}>\frac{v_5(m+1)}{u_5(m+1)}.$$

In fact,

$$\begin{aligned}
&4v_5(m)u_5(m+1)-v_5(m+1)u_5(m)\\
=&\begin{pmatrix} 41\,969\,664\pi^2-2064\,476\,160\pi+61\,966\,080\pi^3+2269\,056\pi^4-100\,800\pi^5+672\pi^6 \\ +3981\,312\,000 \end{pmatrix}\\
&+(m-4)\begin{pmatrix} 156\,119\,040\pi^2-1261\,578\,240\pi+23\,500\,800\pi^3-449\,680\pi^4-12\,960\pi^5 \\ +180\pi^6+1791\,590\,400 \end{pmatrix}\\
&+(m-4)^2\left(58\,625\,664\pi^2-251\,873\,280\pi+2177\,280\pi^3-75\,552\pi^4+12\pi^6+199\,065\,600\right)\\
&+(m-4)^3\left(8651\,520\pi^2-21\,565\,440\pi+138\,240\pi^3-6320\pi^4\right)\\
&+(m-4)^4\left(836\,736\pi^2-1105\,920\pi-288\pi^4\right)\\
&+46\,080\pi^2(m-4)^5+1536\pi^2(m-4)^6\\
>&0
\end{aligned}$$

holds for all $m \geq 4$ due to the coefficients of $(m-4)$ power are all greater than 0.

So $h_5(n) > 0$ and $e_n > 0$ for $n \geq 4$. This means $f_5(x) > 0$ and $F_5'(x) > 0$ on $(0, \pi/2)$. So $F_5(x)$ is increasing on $(0, \pi/2)$. Then $F_5(x) > 0 = F_5(0^+)$ holds for all $x \in (0, \pi/2)$. At the same time,

$$\lim_{x\to 0^+} \frac{2\frac{x}{\sin x}+\frac{x}{\tan x}-3}{x^3\tan x}=\frac{1}{60}.$$

The proof of Theorem 1.5 is completed. □

3.6. Proof of Theorem 1.6.

Let

$$F_6(x)=\ln\left(\frac{1}{60^{14}}\right)+56\ln x+5\ln\left(\frac{\tan x}{x}\right)-14\ln\left(2\frac{x}{\sin x}+\frac{x}{\tan x}-3\right),\ 0<x<\frac{\pi}{2}.$$

Then

$$F_6'(x)=\frac{1}{x\left(\tan^2 x\right)\left(2\frac{x}{\sin x}+\frac{x}{\tan x}-3\right)}f_6(x),$$

where

$$f_6(x)=-15x\tan^3 x+19x^2\tan^2 x+22x\tan x+19x^2-153\tan^2 x+10x^2\frac{1}{\cos^3 x}+74x\frac{\sin x}{\cos^2 x}+28\frac{x^2}{\cos x}.$$

By substituting the expansions of power series of corresponding functions in Lemmas 2.1, 2.2 and 2.3 into the above formula, we obtain that

$$\begin{aligned}
f_6(x) =& -15x\left[\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-3}-\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}\right]\\
&+19x^2\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}+22x\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}+19x^2\\
&-153\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}\\
&+10x^2\sum_{n=2}^{\infty}\frac{2^{2n-1}(2^{2n}-1)(2n-1)(2n-2)}{(2n)!}|B_{2n}|x^{2n-3}\\
&+74x\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}+28x^2\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\\
=&\ 19\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n}-15\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-2}
\end{aligned}$$

$$
\begin{aligned}
&+15\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+22\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+19x^2\\
&-153\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}\\
&+10x^2\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{(2n)!}x^{2n-2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\right]\\
&+74x\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n-1}+28x^2\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}\\
=\;&19\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n}-15\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}\\
&+15\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+22\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+19x^2\\
&-153\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}+5\sum_{n=1}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{(2n)!}x^{2n}\\
&+5\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}+74\sum_{n=1}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}+28\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=\;&19\sum_{n=5}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n}-15\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}\\
&+15\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+22\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}\\
&-153\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}
\end{aligned}
$$

$$
\begin{aligned}
&+5\sum_{n=5}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{(2n)!}x^{2n}+5\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}+74\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}+28\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=\;&-15\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)\,(2n-2)}{2\,(2n)!}|B_{2n}|x^{2n-2}-153\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}\\
&+19\sum_{n=5}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|x^{2n}+15\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}\\
&+22\sum_{n=5}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n}+5\sum_{n=5}^{\infty}\frac{2n\,(2n-1)\,|E_{2n}|}{(2n)!}x^{2n}+74\sum_{n=5}^{\infty}\frac{2n|E_{2n}|}{(2n)!}x^{2n}\\
&+5\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}+28\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}\\
=\;&-\sum_{n=6}^{\infty}\frac{3\times 2^{2n}\,(5n+46)\,(2n-1)\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-2}\\
&+\sum_{n=5}^{\infty}\frac{2\times 2^{2n}\left(2^{2n}-1\right)(19n+9)}{(2n)!}|B_{2n}|x^{2n}+\sum_{n=5}^{\infty}\frac{2n\,(10n+69)\,|E_{2n}|}{(2n)!}x^{2n}\\
&+33\sum_{n=4}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n+2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{n=5}^{\infty}\frac{3\times 2^{2n+2}\left(5\left(n+1\right)+46\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|x^{2n} \\
&\quad +\sum_{n=5}^{\infty}\frac{2\times 2^{2n}\left(2^{2n}-1\right)\left(19n+9\right)}{(2n)!}|B_{2n}|x^{2n}+\sum_{n=5}^{\infty}\frac{2n\left(10n+69\right)|E_{2n}|}{(2n)!}x^{2n} \\
&\quad +33\sum_{n=5}^{\infty}\frac{|E_{2n-2}|}{(2n-2)!}x^{2n} \\
&:= \sum_{n=5}^{\infty} j_n x^{2n},
\end{aligned}
$$

where

$$
\begin{aligned}
j_n &= -\frac{3\times 2^{2n+2}\left(5\left(n+1\right)+46\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|+\frac{2\times 2^{2n}\left(2^{2n}-1\right)\left(19n+9\right)}{(2n)!}|B_{2n}| \\
&\quad +\frac{2n\left(10n+69\right)|E_{2n}|}{(2n)!}+33\frac{|E_{2n-2}|}{(2n-2)!}.
\end{aligned}
$$

By Lemmas 2.4 and 2.5 we have

$$
\begin{aligned}
j_n &> -\frac{3\times 2^{2n+2}\left(5\left(n+1\right)+46\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{2^{2n+2}}{2^{2n+2}-2} \\
&\quad +\frac{2\times 2^{2n}\left(2^{2n}-1\right)\left(19n+9\right)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-1}+\frac{2n\left(10n+69\right)4^{n+1}}{\pi^{2n+1}}\frac{3^{1+2n}}{1+3^{1+2n}} \\
&\quad +33\times\frac{4^{n}}{\pi^{2n-1}}\frac{3^{2n-1}}{1+3^{2n-1}} \\
&= -\frac{6\times 2^{2n+2}\left(5\left(n+1\right)+46\right)\left(2n+1\right)\left(2^{2n+2}-1\right)}{\pi^{2n+2}\left(2^{2n+2}-2\right)}+\frac{2^{2n+2}\left(19n+9\right)}{\pi^{2n}} \\
&\quad +\frac{2n\left(10n+69\right)}{1}\frac{4^{n+1}}{\pi^{2n+1}}\frac{3^{1+2n}}{1+3^{1+2n}}+33\times\frac{4^{n}}{\pi^{2n-1}}\frac{3^{2n-1}}{1+3^{2n-1}} \\
&:= \frac{2^{2n}h_6(n)}{\pi^{2n+2}\left(3^{2n}+3\right)\left(3^{2n+1}+1\right)\left(2^{2n+1}-1\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
h_6(n) &= \alpha(n)18^{2n}-\beta(n)9^{2n}-\gamma(n)6^{2n}-\eta(n)3^{2n}-\theta(n)2^{2n}+\mu(n) \\
&= \left[\alpha(n)9^{2n}-\beta(n)\left(\frac{9}{2}\right)^{2n}-\gamma(n)3^{2n}-\eta(n)\left(\frac{3}{2}\right)^{2n}-\theta(n)\right]2^{2n}+\mu(n) \\
&:= s(n)2^{2n}+\mu(n)
\end{aligned}
$$

with

$$
\begin{aligned}
\alpha(n) &= 480n^2\left(\pi-3\right)-24n\left(642-138\pi-19\pi^2\right)+198\pi^3+216\pi^2-7344, \\
\beta(n) &= 120n^2\left(2\pi-3\right)+12n\left(138\pi+19\pi^2-321\right)+\left(108\pi^2+99\pi^3-1836\right), \\
\gamma(n) &= 480n^2\left(10-3\pi\right)+16n\left(3210-621\pi-95\pi^2\right)+\left(24\,480-66\pi^3-720\pi^2\right), \\
\eta(n) &= 240n^2\left(3\pi-5\right)+8n\left(621\pi+95\pi^2-1605\right)-\left(6120-33\pi^3-360\pi^2\right), \\
\theta(n) &= 1440n^2+24n\left(642-19\pi^2\right)+\left(7344-216\pi^2\right), \\
\mu(n) &= 360n^2+12n\left(321-19\pi^2\right)+1836-108\pi^2.
\end{aligned}
$$

Obviously, $\mu(n) > 0$ for $n \geq 5$. Next we prove that $s(n) > 0$ for $n \geq 5$. Since

$$\begin{aligned}
s(n+1) - 81s(n) &= (77\,760n\,(\pi - 3) + 38\,880\pi - 116\,640)\, 9^{2n} \\
&\quad + \left[\begin{array}{c} 7290n^2\,(2\pi - 3) + 243n\,(374\pi + 57\pi^2 - 903) \\ +\frac{243}{4}\,(-80\pi + 108\pi^2 + 99\pi^3 - 1716) \end{array} \right] \left(\frac{9}{2}\right)^{2n} \\
&\quad + \left[\begin{array}{c} 34\,560n^2\,(10 - 3\pi) + 10\,944n\,(330 - 10\pi^2 - 63\pi) \\ +1719\,360 + 12\,960\pi - 51\,840\pi^2 - 4752\pi^3 \end{array} \right] 3^{2n} \\
&\quad + \left(\begin{array}{c} 18\,900n^2\,(3\pi - 5) + 90n\,(4311\pi + 665\pi^2 - 11\,175) \\ -\left(479\,250 + 1620\pi - 28\,350\pi^2 - \frac{10\,395}{4}\pi^3\right) \end{array} \right) \left(\frac{3}{2}\right)^{2n} \\
&\quad +115\,200n^2 + \left(1229\,760 - 36\,480\pi^2\right) n + \left(586\,080 - 17\,280\pi^2\right) \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
s(5) &= 101\,972\,543\,992\,367\,160\pi + 8911\,886\,197\,652\,256\pi^2 + 706\,952\,510\,871\,453\pi^3 \\
&\quad -429\,827\,126\,609\,873\,520 \\
&\approx 4.058\,2 \times 10^{14} > 0,
\end{aligned}$$

we have $h_6(n) > 0$ for $n \geq 5$. Then $j_n > 0$ for $n \geq 5$. This means $f_6(x) > 0$ and $F_6'(x) > 0$ on $(0, \pi/2)$. So $F_6(x)$ is increasing on $(0, \pi/2)$. Then $F_6(x) > 0 = F_6(0^+)$ holds for all $x \in (0, \pi/2)$. At the same time,

$$\lim_{x \to 0^+} \frac{2\frac{x}{\sin x} + \frac{x}{\tan x} - 3}{x^4 \left(\frac{\tan x}{x}\right)^{5/14}} = \frac{1}{60}.$$

The proof of Theorem 1.6 is completed. □

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