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Understanding of Geometric Reflection: John's Learning Path For Geometric Reflection*

Geometrik Yansımayı Anlama: John'un Geometrik Yansıma İçin Öğrenme Yolu

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ABSTRACT: This study explores the development of a pre-service teacher's mental structure from a motion view to a mapping view of geometric reflection. Many pre-service secondary mathematics teachers' (PTs) understand geometric reflection as a motion rather than a mapping of a domain containing points in a plane relative to a reflection line, which is an essential understanding needed for teaching mathematics. Dubinsky's action, process, object and schema (APOS) framework to document the transition of the PT's (John's) mental structures from a motion to a mapping view. Data from interview transcripts, videos, and written artifacts were analyzed using. Results indicated that John's initial motion view of geometric reflection informed his evolving mapping view by developing sub-concepts of the reflection line, domain and plane. It is argued that the mapping view evolves from the motion view as the sub-concepts develop through successive challenges using figures and questioning. The study is a part of a larger study and was conducted with six PTs. However, it focuses on one of the PTs, John, who reached the mapping view of geometric reflection. The other PTs also demonstrated a similar mental structure.

Keywords: Geometric transformations, pre-service teachers, geometric reflection, motion view, mapping view.

ÖZ: Bu çalışma, bir öğretmen adayının geometrik yansımayı anlamada ki zihinsel yapısının, hareket perspektifinden eşleştirme perspektifine gelişiminin araştırılması amacıyla yapılmıştır. Birçok ortaöğretim matematik öğretmeni adayı, geometrik yansımayı anlamada önemli olan yansıma ekseni, yansımanın etki alanı ve düzlem alt konseptlerine göre eşleştirme perspektifi yerine hareket perspektifine sahiptirler. Dubinsky'nin eylem, süreç, nesne ve sema (APOS) teorik çerçevesi kullanılarak bir öğretmen adayının (John) geometrik yansımayıanlamada ki zihinsel yapılarının hareket perspektifinden eşleştirme perspektifine gelişimleri incelenmiştir. Veri toplama araçları olarak mülakat transkriptleri, videolar ve çalışma kağıtları analiz edilmiştir. Bulgulara göre, John'un geometrik yansımayı anlamada sahip olduğu yansıma ekseni, yansımanın etki alanı ve düzlem hareket perspektifinden eşleştirme perspektifine geçişini hazırlanan mülakat soruları ve mülakat sürecinde sorulan açık-uçlu sorular kolaylaştırmıştır. Çalışma daha büyük bir çalışmanın parçasıdır ve altı öğretmen adayı ile yürütülmüştür. Diğer katılımcılarda hareket perspektifinden eşleştirme perspektifine benzer zihinsel yapı gelişimleri gösterdikleri için, çalışmada bir katılımcının zihinsel yapı gelişimleri detaylarıyla anlatılmıştır.

Anahtar kelimeler: Geometrik dönüşümler, öğretmen adayları, geometrik yansıma, hareket perspektifi, eşleştirme perspektifi.

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The study of geometric transformations (e.g., translation, reflection, rotation) has gained importance in recent years (Akarsu, 2018; Flanagan, 2001; Glass, 2001; Yanik & Flores, 2009). According to the National Council of Teachers of Mathematics [NCTM] (2000), it is important for students to learn and apply geometric transformations and symmetry in order to deeply examine mathematical situations. Studying and understanding these topics can improve their mathematical thinking, reasoning, and problem-solving skills by enabling them to understand such topics as functions, symmetry, and congruence (Clements et al., 1997; Hollebrands, 2003; Portnoy et al., 2006; Yanik, 2011). The most important geometric transformation to understand is geometric reflection because it plays a crucial role in forming other geometric transformations (e.g., translation and rotation) (Yanik, 2006). For instance, if a student were asked to reflect a triangle using two parallel lines of reflection, the student would need to identify that after two reflections, the first image and final image would be geometric translations. This means that the composition of two geometric reflections produces other geometric transformations such as geometric translations. To effectively teach the concept of geometric reflection, teachers need to understand its meaning and role in geometric transformation.

The term geometric reflection has several meanings in mathematics, of which the main distinction is between the motion view and the mapping view (Akarsu, 2018; Edwards, 2003; Hollebrands, 2003; Yanik, 2011). According to the motion view, a geometric reflection refers to a transformation representing a flip of figures around a point, line, or plane (Boyd et al., 2004). According to the mapping view, which is informed by formal mathematics and derived from the definition of function, a geometric reflection on a plane is, as Martin (1982) describes, "a one-to-one correspondence from the set of points in the plane onto itself" (p. 1). From this viewpoint, understanding the domain and range of a geometric reflection involves understanding that all points in the plane are mapped to other points in the plane, so the transformation is not limited to a figure or a point. Instead, as Yanik and Flores (2009) state, one maps "all points in the plane to other points in the plane rather than moving images/points from their original locations to different locations" (p. 42).

To understand the difference between geometric reflection from a motion view and a mapping view, three important sub-concepts must be considered: reflection line, domain, and plane (Flanagan, 2001; Yanik, 2006). PTs with a motion view do not take equidistance and perpendicularity properties into account when they use a reflection line to reflect a geometric figure; they define domain only in terms of given points or geometric figures, and they consider the points or geometric figures as separate from the plane. On the other hand, PTs with a mapping view do take the properties of equidistance and perpendicularity into account when they use the reflection line to perform a geometric reflection; they consider the domain as a whole plane consisting of infinitely many points; and they consider the points or geometric figures as subsets of, not separate from, the plane.

While recent studies emphasize the importance of having a mapping view of geometric reflection, studies show that most students, including PTs, generally have a motion view which is likely to result in inadequate knowledge for understanding and, therefore, teaching geometric reflection (Hollebrands, 2003; Yanik, 2006). But despite the growing literature that supports the necessity for PTs to have a mapping view of geometric reflection, there has been little research on how to help them acquire it (Hollebrands, 2003; Yanik, 2006). To address this issue, I hypothesized that having a motion view of geometric reflection is a prerequisite to developing a mapping view of geometric reflection. Therefore, the research questions in the study are as follows:

1. How does a motion view of geometric reflection develop into a mapping view for a secondary mathematics PT?
2. What factors facilitate PTs' development of a motion view of geometric reflections into a mapping view?

Theoretical Framework

To investigate and analyze PTs' mental structures with regard to understanding geometric reflection, the Action, Process, Object and Schema (APOS) theoretical framework was used. This framework provides useful guidance for researchers to analyze the level of students' learning of a mathematical subject. Within this framework, I used Genetic Decomposition (GD) as a hypothetical model to identify the mental structures needed by PTs to learn a mathematical subject (Dubinsky, 1991). Specifically, a GD is preliminary model for describing how the mental structures of action, process and object form a schema for a mathematical concept. An inference drawn from the literature is that the mental structures of action and process are necessary and sufficient to create a mental structure for a mapping view of geometric reflection (Hollebrands, 2003; Yanik, 2006). Therefore, in this study I focused on the mental structures of action and process of the theory.

An action is a mental structure that is the first stage in understanding a concept. It starts with the organization or change of mental structures that have been built previously. When forming new mental structures at the action stage, external clues (e.g., a formula or similar example etc.) are needed. These clues should be used to clearly represent the formation of new mental structures (e.g., process) step by step. Therefore, the process of forming new mental structures is not possible without the initial formation and internalization of the mental structure of action (Dubinsky, 1991).

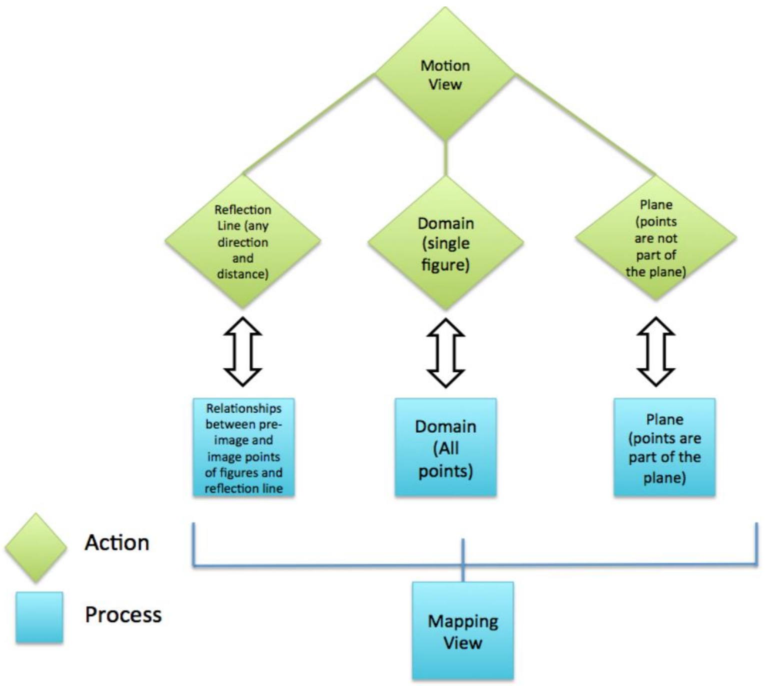
According to Flanagan (2001), first, students with the action structure of the reflection line that defines the geometric reflection have difficulty understanding the role of the reflection line (e.g., understanding the relations between corresponding pre-image and image points and the reflection line or the use of the equidistance and perpendicular properties of geometric reflection). Second, students with an action structure of domain assume that a geometric reflection is applied to a single point or figure. Third, students with an action structure of plane assume they may simply perform a reflection as a movement rather than mapping the plane onto itself. This view is erroneous because the plane is a set of infinite points, and geometric figures are not separate from the plane but a subset of points on it.

According to Dubinsky et al. (2005), the process mental structure is formed as a result of the iteration and interiorization of the mental structure of action. While a new mental structure is being formed, all the steps can be imagined without the need to be explicitly stated. According to Flanagan (2001), first, students with a process structure of reflection line know the role of the reflection line (e.g., understand the relations between corresponding pre-image and image points and the reflection line and the use of the equidistance and perpendicular properties of geometric reflection). Second, students with the process structure of domain have started to think about all points in the plane rather than only the labeled points on a single figure within it. Third, students with the process structure of plane think of the geometric figures as a part of the plane rather than as separate from it.

I created a preliminary genetic decomposition (PGD) using the literature on mathematics education (Flanagan, 2001; Yanik & Flores, 2009) and inferences drawn from it. The PGD indicates my hypothesis about the PT's action and process structure of geometric reflection based on the motion and mapping view (see Figure 1). The PGD was used to design a series of interviews to develop the geometric reflection tasks to collect research data. Flanagan (2001) and Yanik (2006) have shown that the formation of a geometric reflection mental structure depends on the coordination of the mental structures of action and of process. Accordingly, as seen in Figure 1, I hypothesized that to make a meaningful transition from a motion view to a mapping view, a PT must undergo changes in his/her mental structures of action and process. However, the motion view of the reflection line, domain and plane indicates a different mental structure of action and of process from that of the mapping view of reflection line, domain and plane. For example, a PT with an action structure of motion view considers the domain as a single point or the figure, whereas a PT with a process structure of mapping view considers the domain as a whole plane. However, the existing mental structure of action associated with a motion view can develop into a new mental structure to reach a process structure of mapping view of the domain, as well as of the reflection line and plane. As illustrated in Figure 1, the formation of the mapping view depends on the structuring and organizing of all the relevant mental structures. I hypothesized that the PTs need an action structure of motion view before developing a process structure of a mapping view.

Figure 1

Preliminary Genetic Decomposition



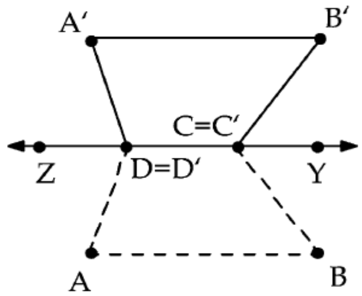
Research on Students' and Pre-Service Mathematics Teachers' Understanding of the Concept of Geometric Reflection

Studies of students' and pre-service teachers' understanding of geometric reflection show they have similar difficulties with accurately describing the role of the reflection line, considering the domain as a single figure or assemblage of points, and using the operational definition of the plane (Boulter & Kirby, 1994; Dixon, 1997; Hollebrands, 2003, 2004; Mhlolo & Schafer, 2013; Yanik, 2006). These studies document that both students and pre-service teachers have a motion view rather than a mapping view of geometric reflection.

Hollebrands (2003) implemented a 7-week unit to investigate how 4 tenth-grade students understood the concepts of translation, rotation, reflection, and dilation using the Geometer's Sketchpad (Jackiw, 2001). She found that all the students' understanding of domain indicated a motion view of geometric reflection. For instance, when asked to explore all points on a quadrilateral ABCD that would be fixed over the reflection line j (see Figure 2), a student said that "there were no fixed points in the figure except for the ones on line j, which is not a point of ABCD" (p. 62). This response indicated that the student focused on only the points on the figure rather than all points in the plane as the domain. Hence, it could be inferred that the student had an action structure of domain in understanding geometric reflection. To have a process structure of the domain, the student would need to consider the domain as comprising all points in the plane.

Figure 2

Student's Drawing of a Reflection on a Reflection Line (p. 62)



Mhlolo and Schafer (2013) examined 235 eleventh grade students' preconceptions about geometric reflection. When asked to reflect several points on a Cartesian plane over an oblique reflection line, none of the students reflected the points correctly by using the properties of equidistance and perpendicularity, from which it could be inferred that they had an action structure rather than a process structure of the reflection line in their understanding of geometric reflection.

Yanik (2006) conducted an 8-week teaching experiment to explore the development of four PTs' understanding of translation, reflection, rotation, and dilation using Geometer's Sketchpad and found they all had a motion view of geometric reflection based on their understanding of the role of the reflection line, their consideration of domain as a discrete figure, and using the operational definition of the plane. Specifically, the PTs had difficulty determining how to use the reflection line and understanding the properties of equidistance and perpendicularity, which are important for identifying how to position the points or figure correctly.

Studies show that both students and PTs have a motion view rather than a mapping view of geometric reflection based on their understanding of the role of the reflection line, domain, and plane (Flanagan, 2001; Yanik, 2006). One reason for this problem might be the lack of content in textbooks (Zorin, 2011). In this section, I discuss my examination of the treatment of geometric reflection in terms of motion and mapping views in the middle-grade (6, 7, 8) textbooks of the National Science Foundation (NSF) funded Curriculum of Mathematics Project (CMP) Textbook series (including teacher's guides and student's editions). These books were selected based on their popularity in the USA.

Analysis of the definitions of geometric reflection in CMP textbooks indicates that they support a motion view rather than a mapping view. For instance, the CMP3 teacher's guide describes geometric reflection as occurring "if a reflection in a line maps the figure exactly onto itself" (p. 13). From this description, a student would likely infer that a geometric reflection involves only the given figure rather than all points in the plane. In this way, CMP textbooks may promote students' understanding of the domain as a single figure rather than as all points in the plane in performing a geometric reflection. Thus, while these textbooks were meant to support conceptual development, their focus on inductive reasoning may inadvertently support a motion rather than a mapping view of reflection. This suggests that explicitly providing formal mathematical definitions of concepts in textbooks rather than approaching them implicitly is important for helping students understand mathematical concepts (Tossavainen et al., 2017).

Methods

Participants

In the original study, six PTs in the Curriculum and Instruction Department of a large, public, Midwestern University in the United States were identified as potential participants for the study. After an initial interview with each, four PTs were selected based on their willingness and ability to explain their thought processes. These four participants were all in their last year of an undergraduate program in mathematics education. By the end of the third interview in the original study, all four participants had reached a mapping view of geometric reflection and demonstrated a similar transformation of mental structuring. Thus, it was considered sufficient to focus on mental structure development of one of the participants for the present case study.

Settings for Interviews

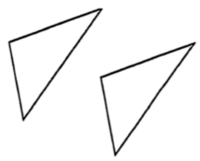
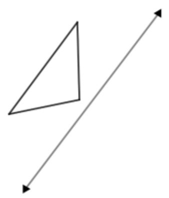
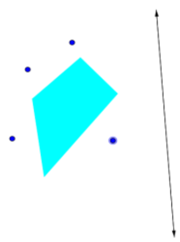
Each participant was interviewed individually three times to investigate their reasoning and thought processes in depth. The initial interview questions were identical for each participant, and the subsequent exploratory interview questions were individualized. All interview sessions were video-recorded, with one video camera focused on interactions between the participant and the researcher. A second video camera focused on the participant's work area and his/her utterances, gestures, manipulations, and speech characteristics.

Interviews

The purpose of the initial interviews was to gather background knowledge about participants in order to choose PTs to be further interviewed to collect evidence of their mental structures as they moved from a motion view to a mapping view of geometric reflection and developed an understanding of the sub-concepts of reflection line, domain, and plane. Some example tasks were shown in Table 1.

Table 1

Example Interview Tasks

The Task is About	Relevant Tasks
Defining geometric reflections	• Using your own words and any pictures or diagrams you need to express your ideas, define the term of geometric reflection. • Can you give an example of geometric reflection?
Recognizing geometric reflections	• Is this a reflection? Why or why not? 
Describing the role of reflection line	• Explain how you determined where to a place the figure. • What do you think about the role of reflection line? 
Describing domain	• What points did you reflect? • Are there any other points being reflected besides these points? • Are there any points inside the figure? If yes, where do these points inside the figure go when you reflect them? 
Describing plane	• Can you describe the plane? • Is there a movement you perform a geometric reflection?

In the first explanatory interviews, I explored PTs' mental structures associated with a motion view of geometric reflection line and domain. I designed the interview questions to determine PTs' thinking in terms of actions and processes. I hypothesized that to reach a mapping view, PTs must have the process structure of reflection line, domain, and plane. Specifically, I asked the PTs to identify whether two figures constituted a reflection and to justify their stance. Whether or not the participants responded correctly, I asked probing questions to better understand their reasoning. In addition, I gave the PTs a series of open and closed figures on which to perform a reflection of a line, triangle, trapezoid, rectangle, etc. I asked the PTs to perform a geometric reflection and explain how the image in the geometric reflection was positioned in the plane. This task targeted the participants' use of the reflection line, which was positioned in a variety of ways (e.g., vertical, horizontal, and oblique) relative to the bottom of the page.

For the second explanatory interview, I gave the PTs a set of tasks adapted from Yanik (2006), along with additional tasks I had created. The main purpose of the tasks was to collect evidence of PTs' mental structures for actions and processes with regard to the reflection line, domain, and plane. For example, I explored the concept of domain, and I asked, "Is there any point outside the figure?" If the answer was "yes," I asked, "Where do the points outside the figure go when you have a reflection?" If the PT's explanation of the position of the points in performing a geometric reflection took into account points both inside and outside the figure (i.e., the PT had begun to consider all points in the domain rather than view the domain as a single figure), I hypothesized that the PT was at the process structure for the concept of domain.

For the third explanatory interview, the PTs were given a set of tasks adapted from Flanagan (2001) and Yanik (2006) and other tasks that I created. The main purpose of these tasks was to gather evidence of the PTs' current mental structures of actions or processes of a mapping view and to determine how they coordinated and interrelated their schemas for the concepts of reflection line, domain, and plane, which are needed to develop a schema for a mapping view. I asked the PTs to reflect a circle with a yellow triangle inside over the reflection line. The reason behind this task was to see if they understood that the plane involves a set of points, and when they reflect a figure, they need to consider all points in the plane to map them under the reflection. For example, I asked, "Are there any points on the image? If yes, where do the points on the image go when you have a reflection?" "Are there any points on the white part of the plane? If yes, where do those points go when you have a reflection?"

Data Analysis

All of the data were analyzed using the APOS theoretical framework. I transcribed the audio and carefully viewed the video recordings to investigate whether the participant had an action or process structure of reflection line, domain, and plane for geometric reflection (see Table 2). During this process, I started to code based on participants' responses and interactions with the tasks to investigate which mental structures the participant had developed. Next, I created a table to identify evidence points (Arnon et al., 2014; Dubinsky, 1991). Then, I created a model for John based on my interpretation of his mental structures at the actions or processes related to reflection line, domain, and plane. In the model, I identified how a motion view of reflection line, domain and plane as an action structure evolved to a mapping view of reflection line, domain and plane as a process structure.

Table 2

Sample Data Analysis

Codes	Transcript Excerpt	Reasons And comments	Notes
Action Process	R: find the image of the triangle after performing reflection over the line, and explain how you determined where to place the figure? A PT: I selected three vertices and reflected over the reflection line. Then, I connected them together to draw triangle.	The PT' s answer shows that the PT seemed to understand the role of reflection line (e.g., mapping points over the reflection line) using perpendicularity and equidistance properties. Based on this explanation, the PT is at the process structure for reflection line. However, PT only selected vertices and sides, rather than all points in the domain to apply the reflection. At this point, the PT is at the action structure for domain.	The participant did not go through each step (e.g., mapping all points on the perimeter of the figure, inside and outside the figure) to perform a reflection. PT directly reflected vertices over the reflection line and explained the role of reflection. The participant had interiorized his action into a process structure.

After completing analyzing of data, I tested the preliminary genetic decomposition by using APOS theory. For example, my preliminary genetic decomposition indicated that John would have an action structure of the reflection line. After the first interview, I found that he had an action and a process structure of the reflection line. I also found that knowing equidistance and perpendicularity properties and the relationship between pre-image and image points of figures and the reflection line are important factors to help him to move from an action structure to a process structure. After testing the preliminary genetic decomposition, I created additional hypotheses to be tested in the subsequent interview. For instance, I hypothesized that John had an action structure of the domain. After the first interview, I found that he had an action structure of the domain because in that he considered single points of the figure in the plane to perform a geometric reflection. Therefore, I needed to create new hypotheses for the remaining interviews to test tasks again or add several tasks to ask him to move to the process structure. I repeated data analysis and testing of hypotheses after each interview.

To ensure the reliability of the data, I shared and discussed my inferences and interpretations of the participant's mental structure with a volunteer mathematics educator who was outside of the coding process but knew the APOS theoretical framework. Then, I asked him whether my inferences and interpretations were correct to code based on the participant's action and process mental structure. If our code matches, I move to the next level of analysis. If our codes are different, we have resolved all the differences by discussion. To ensure the validity of the collected data, I used triangulation. The use of transcriptions, video recordings, and observation notes allowed the collected data to be validated. This evidence was used to construct models of the PT's actions and processes for geometric reflections in terms of motion and mapping perspective.

Ethical Procedures

This study was approved with the Meeting Date and Number 13.09.2017/1703019007 by the Social and Human Sciences Ethics Committee of Purdue University/USA.

Results

A Case Study of John

The purpose of this case study is to describe one PT's (John's) understanding of geometric reflection from the perspective of APOS theory. This case elucidates how John's motion view of geometric reflection evolved into a mapping view and the factors that facilitated this development. In this narrative, the term "R" refers to "Researcher."

John was selected based on his willingness and ability to explain his thought processes. I excluded other PTs because they did not meet the threshold for verbal expression of ideas, so I would not be able to obtain sufficient external evidence to hypothesize about their mental structures. John was a 21-year-old, third-year student majoring in mathematics education. He recalled receiving formal instruction on the concept of geometric reflection in his high school geometry course, but he had not learned any concepts related to geometric reflection in his college geometry course. John described geometric reflection as "[reflecting] across the y-axis [so] that it [referring to a figure] would have to look like in a mirror" (Initial Interview). In his description, he did not refer to any properties of geometric reflection. However, when directly asked, he correctly explained the role of the reflection line (i.e., to position points symmetrically over the reflection line). Therefore, John might have understood the reflection line in practice but was unsure how to use the reflection line to define geometric reflection.

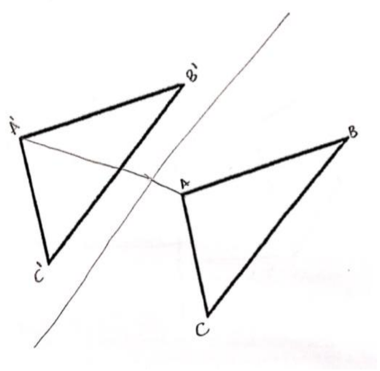
John's descriptions of performing geometric reflection suggested that he considered the domain as a single figure as he focused on points of figures rather than on all points in a plane. For example, he described a reflection line as "the line in which you are reflecting your shape" (Initial Interview). He explained the role of the reflection line as "knowing where you are going to reflect" (Initial Interview). John did demonstrate awareness of some properties of geometric reflection, stating, for example, that geometric reflection preserved the length and the angles of the original image but not the orientation. However, in describing the process, he did not talk about such properties of geometric reflection as perpendicularity in his initial interview.

John's Understanding of Reflection Line

In his first interview, John demonstrated both an action and a process structure of the reflection line. He discussed the role of the reflection line in terms of the relations between pre-image and image points of figures and used the properties of equidistance and perpendicularity for performing geometric reflection. For instance, when given a task with two figures without a reflection line and whether it was a reflection, he said it was not and drew a reflection line between the two figures and stated, "This distance [referring to the distance from point A to the reflection line] from the line [referring to reflection line that he drew] to the point [referring to the point A] is not the same as this distance from this point [referring to point A'] to the line of reflection" (Interview 1) (see Figure 3).

Figure 3

John's Drawing of a Reflection Line between Given Figures



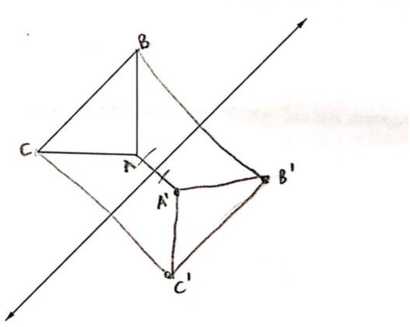
He knew that a reflection line is a useful tool for determining whether an image (A`B`C`) is the reflection of the pre-image (ABC). The drawing is also evidence that he had an action structure for the reflection line.

Later in the same interview, he was given a triangle with an oblique reflection line and asked to perform a geometric reflection (see Figure 4). John selected the closest point (referring to point A) to the line of reflection. After using an index card to measure the distance from A to the reflection line and from the reflection line to A`, he stated that the two distances would be the "same" and "perpendicular." He then reflected the remaining vertices (points B and C) and connected them to make a triangle. When asked to explain how he determined where to place the figure, John explained:

I knew that these two vertices [referring to points B and C] of the triangles were the same distance away from the line [referring to the reflection line], so I made sure that they [referring to points B and C] were even together at the same line [referring to the BC segment], or on the same line, I guess. Then I decided that this [referring to point A] is the same distance from the line of reflection (Interview 1) (see Figure 4).

Figure 4

John's Drawing of a Reflection on an Oblique Reflection Line

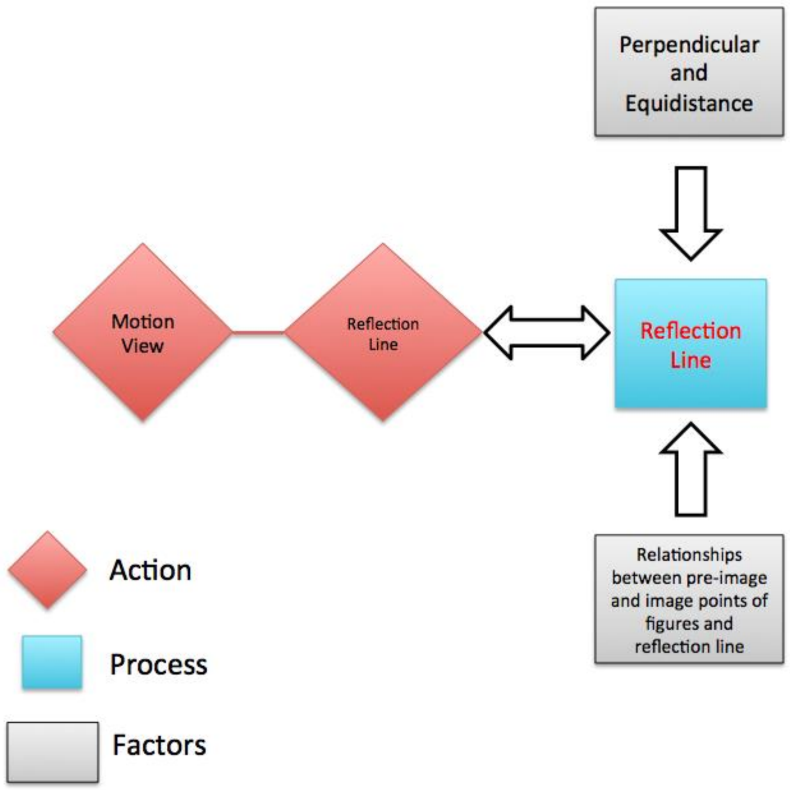


I interpreted his action of mapping the points as evidence that he had a process structure of the reflection line because he reflected the figure as a collection of parts (e.g., points, vertices, sides) rather than as a whole figure. Based on his drawing, I interpreted that he knew the relations between the pre-image and image points of the figures and the reflection line. He had progressed beyond simply identifying the reflection line as an essential component of reflection towards indicating that he needed particular points, mapped in a particular way, with several properties of geometric reflection (the equidistance and perpendicularity properties). This awareness suggested a process structure of a reflection line.

As previously mentioned in my preliminary genetic decomposition (PGD), I hypothesized that PTs begin with a motion view of the reflection line as an action structure in order to develop a mental structure for the reflection line as a process structure. My analysis identified two significant factors that facilitated John's performance of geometric reflection and his understanding: the relationships between the pre-image and image points of the figure and the reflection line, and the properties of equidistance and perpendicularity (see Figure 5). These two factors are sufficient to have a process structure of the reflection line in geometric reflection.

Figure 5

John's Mental Structures of the Reflection Line

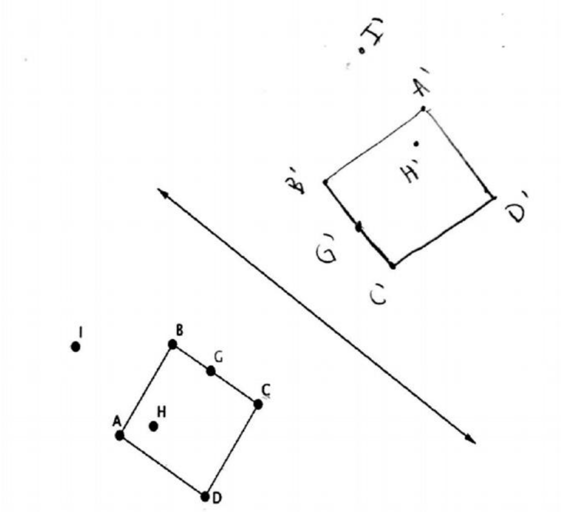


John's Understanding of Domain

The first interview revealed that John had an action structure of domain in geometric reflection because he thought of the domain as a single figure rather than all points in the domain and that geometric reflection was applied to vertices or the perimeter of the figure. He had difficulty thinking about interior and exterior points in geometric reflection, possibly because of not operating with the concept of plane. For example, John was given a square with an oblique reflection line and asked to “find the figure after performing a reflection across the line” (see Figure 6). John reflected four vertices of a square by measuring the distance between each of the vertices to the reflection line using an index card (A, B, C, D) and then connected the vertices of the reflection to make the square. He again used an index card to position other points (G, H, I) to reflect.

Figure 6

John’s Drawing of a Reflection on an Oblique Line



When asked to explain what points he reflected, he elaborated his ideas in the following dialogue:

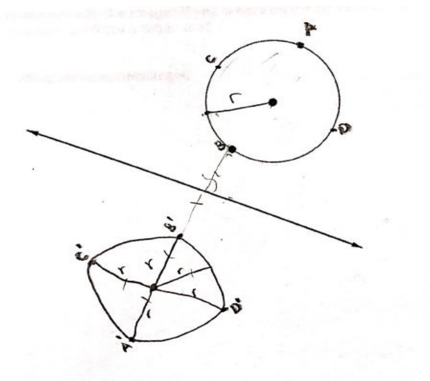
> J: I reflected A, B, C, and D, and then G came along on line BC because it was on the original line BC, and then I reflected H, and I as well.
>
> R: Okay, are there any other points being reflected beside those, like A, B, C, D, G, H, and I?
>
> J: All the other points on the lines A, B, C, [and] D.
>
> R: Okay. Are there any other points that are reflected beside [these] lines [referring to points on the segments AB, BC, CD, AD]?
>
> J: I do not believe so (Line 290-296; 09/19/2017, Interview 1).

I interpreted from his explanation that John considered only the vertices and the perimeter of the figure in performing this geometric reflection, which is evidence of an action structure of domain because he did not consider all points in the plane.

Further evidence of an action structure for domain emerged during the second interview. When I asked John to reflect a circle with an oblique reflection line (see Figure 7), he stated, “it was difficult to reflect the circle since there were no vertices to measure distance.” (Interview 2). Using an index card, he measured the distance from point B on the circle to the reflection line and then reflected point B over the reflection line. He then measured the diameter of the circle and reflected point A over the reflection line, and he applied the same process for points C and D. He then drew the circle. Thus he used the properties equidistance and perpendicularity to perform the reflection. (Interview 2).

Figure 7

John's Drawing of a Reflection on a Circle Task



The following excerpt shows how John thought about this task when asked to explain his approach.

> R: How many points did you reflect?
>
> J: I reflected all the points on the circle.
>
> R: Okay. Are there any other points being reflected beside perimeter of the circle points?
>
> J: Nope. I do not believe so. Just on the circle. Oh, well, sorry, I guess the center of the circle I reflected. Reflected this [referring to the center point of the circle of the pre-image] as well if that is a point. I guess I reflected that [again referring to the center point of the circle of pre image] across the line of reflection.
>
> R: Okay. Did you reflect only the center point?
>
> J: No. So you would reflect all of the points inside the circle. (Lines 79-87; Interview 2).

John's explanations suggest that he had started to think of reflecting points interior to the circle. John explained his reasoning as follows:

> I thought to find the center of the circle, and reflecting that point [referring to the center of the circle], and then just finding the radius and then drawing the circle. So then I started to think about ... if you actually are going to reflect the center across then it would make sense because you would obviously reflect the center and if I were going to draw the radius, then I would reflect all the points on them and the perimeter of the circle as well (Lines 113-118; Interview 2).

John's conclusions about performing geometric reflection now included points interior to the figure, at least when the figure was a circle, which seemed to be a new insight. Reflecting a circle seemed to prompt him to think about the interior points of the image. But in response to the question, "is any point outside the figure reflected?" John replied, "no." Thus, he was aware that when a figure is visible on a plane, the points on both the perimeter and interior of the figure are contained within the plane. However, he did not consider the exterior points, which were also in the plane, which was evidence of an action structure of the domain in geometric reflection.

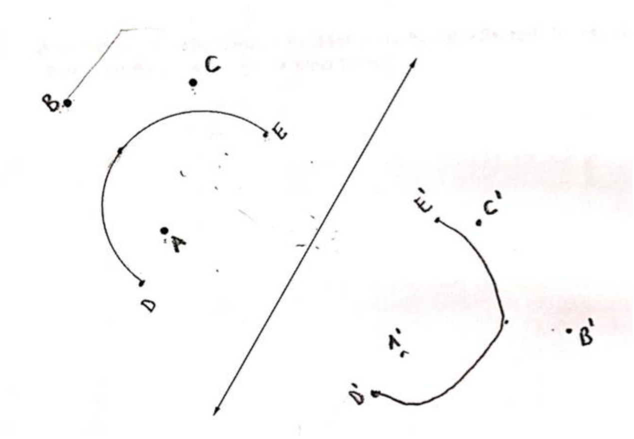
In hopes of provoking John to consider the exterior points, I posed a task using an open figure (see Figure 8). He had no difficulty using equidistance and perpendicularity properties to reflect the given figure. When I asked him what points were being reflected, he said,

> I reflected the three standing points [referring to points A, B, C] and then the end points [referring to points D, E] of this arc and then the top end point [referring to point F] of this parabola, if you think about it that way I guess. And then all of the points on this arc as well (Interview 2).

His explanation was consistent with what he did previously with other tasks.

Figure 8

John's Drawing of a Reflection on an Arc Task

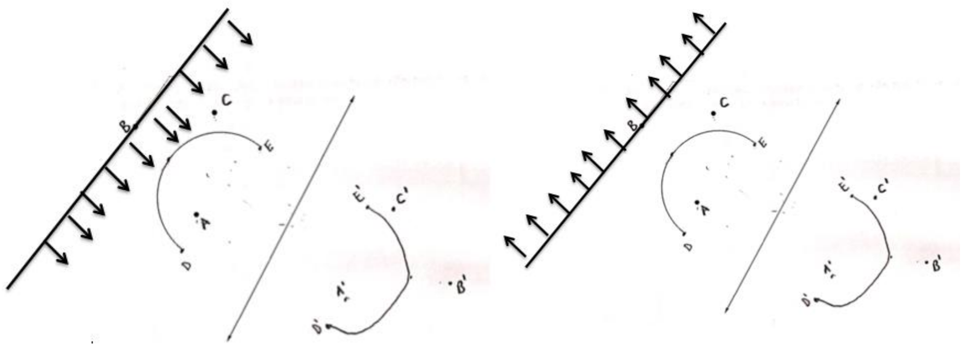


I then asked, "Are there any other points reflected beside those you mentioned?" which led to the following exchange:

> J: So when it was a circle, if this arc were complete then all the points inside the arc will be reflected but ... since this is an open figure ... so ... I would say yeah. So then I reflected all of the points from even I guess this point down [referring to point B]. And this area [referring to point B and all points below B]), I reflected across the line of reflection as well (see Figure 9a). Okay. I think ... Yeah.
>
> R: Okay. How about the other side [referring to the upper part of point B] (see Figure 9b)?
>
> J: So yeah ... Then yeah, all of these will be reflected as well, across the line of reflection then. Because this is an entire plane. That is a good way to think about it. And you are reflecting the plane onto the other side of the line of reflection.
>
> R: How did you start to think that you need to reflect entire plane?
>
> J: Because, well I saw this point [referring to point A] and it was inside my arc, and so I knew that I reflected this point [referring to point A] to this side [referring to image-figure plane], and so then I started to think about all the other points inside this arc [referring to the pre-image], and then I thought about how this point [referring to point A] could be over here [referring to the image-figure plane]. I will still reflect it [referring to point A] because it [referring to point A] was on this side [referring to the pre-image-figure plane] of the line of reflection. Then I saw that these points [referring to the points B, C] that were outside of the arc and how they were enclosed, and how they were reflected onto the other side [referring to image-figure plane] of the line of reflection as well (Interview 2).

Figure 9a and 9b

John's Drawing of a Reflection on a Semi-Circle Task

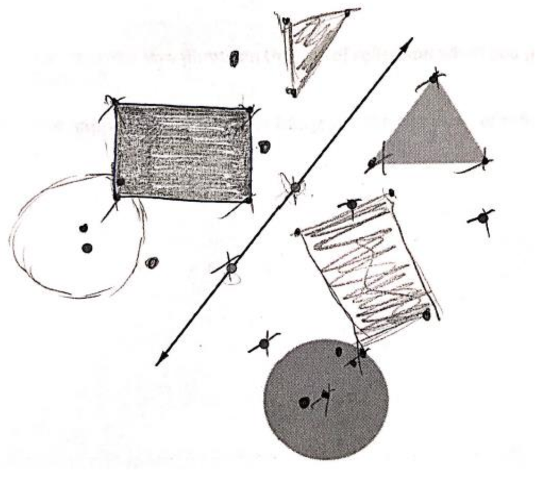


This excerpt shows that John considered and reflected more points in the plane because the figure's interior and exterior were not clearly defined for him. When he saw the two points "outside" of the arc figure (B, C), John started to think about the half plane, which I inferred to mean that he imagined that there was an infinite number of points inside and outside of the arc figure on which to perform a reflection. As evidence of this interpretation, after the arc task, he used his mathematical definition of a plane when he performed a reflection and described reflecting all the points in the pre-image plane (such as points on the perimeter, inside points and outside points of the figure). I interpreted this shift in his explanations as an indication that he had started to think of half of the plane as a non-empty space consisting of an infinite number of points when performing a geometric reflection. The task of reflecting a semi-circle seemed to trigger John to think about the exterior points of the semi-circle in the pre-image plane.

To provoke John to consider all points in the entire plane, I posed a geometric reflection task using multiple figures for both planes (see Figure 10). The task required John to reflect all points from the pre-image plane to the image plane, and all points from the image plane to the pre-image plane. John had no difficulty reflecting the given multiple figures for both planes. I then asked him to explain what points he reflected, to which he replied, "I reflected all the points in this plane [referring to pre-image plane] over to this side [referring to image plane] of the plane, and then reflected all the points from this side [referring to image plane] of the plane onto this side [referring to pre-image plane] of the plane." (Interview3).

Figure 10

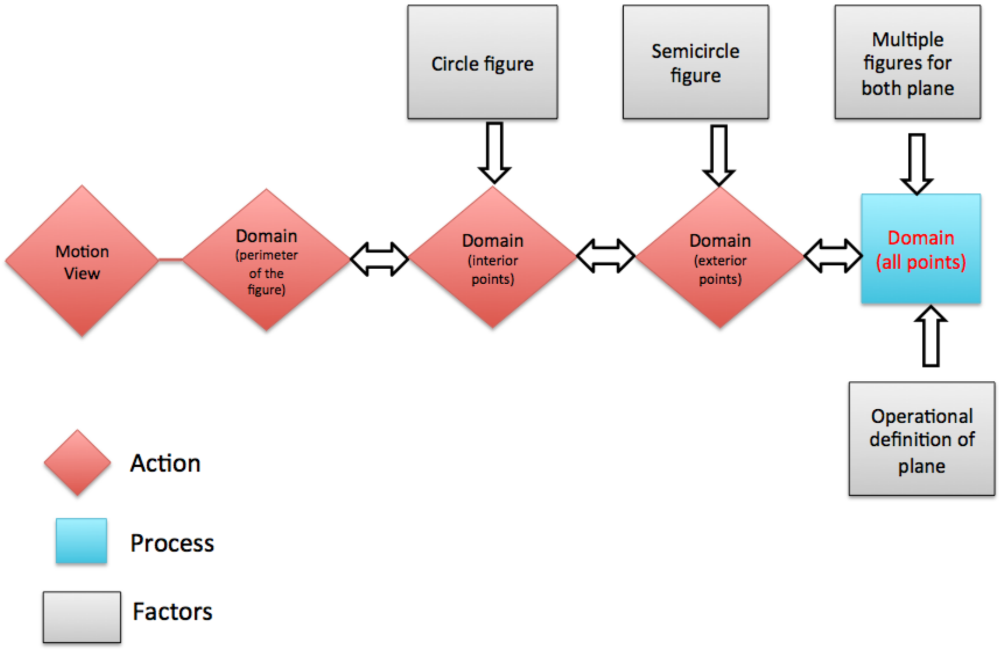
John's Drawing of Geometric Reflection for both Sides of the Plane



Thus, John had started to think about all points in the plane when performing a reflection. Although he did not consider physically reflecting unlabeled points, he showed that he understood that performing a geometric reflection requires reflecting both labeled and unlabeled points, that is, all points in the entire plane. The tasks with circle, arc, and multiple figures and my questions helped John consider "interior" and "exterior" points when performing a geometric reflection. I, therefore, inferred his explanations to mean that he had reached the process structure for the concept of domain in geometric reflection. Figure 11 shows the development of John's mental structures of domain throughout the first, second, and third interviews, illustrating how he reached a process structure of domain by unpacking his mental structures through the lens of APOS theory.

Figure 11

John's Mental Structures of Domain



As previously mentioned in relation to my PGD, I hypothesized that PTs begin with a motion view of the domain as an action structure to develop a process structure. As illustrated in John's case, my analysis identified five significant factors in the development of their performance of geometric reflection and their understanding of domain: task involving a circle figure, task involving a semi-circle figure, task involving multiple figures for both planes, operationally defining plane, and questioning. These factors are sufficient to achieve a process structure of domain in geometric reflection.

John's Understanding of the Plane

The analysis of John's initial interview demonstrated that he considered the plane and figures as collections of infinitely many points. During the first and second interviews, John consistently used the verb "move" with regard to performing geometric reflection. He considered geometric figures as moveable on the plane rather than as subsets of the plane. For instance, when performing a reflection in the circle task, John viewed points as "separated from the plane," indicating that he could move points or figures to perform a reflection. The following episode illustrates this perspective:

> R: How about the outside points? Are there any outside points that you reflected?
>
> J: I guess I didn't have to do anything to these points [referring to outside points of the circle] in putting the points inside the circle *I had to move the surrounding* points[referring to other points inside the circle] to the other side of the line, I guess is what I am trying to say. (Interview 2).

John use of the word "move" suggested that he thought that points were located on the plane, which means that they were not part of the plane (Yanik & Flores, 2009), so the points or figures were relocated to a new position relative to other points in the other plane. This perception was evidence that he had an action structure of plane for performing geometric reflection.

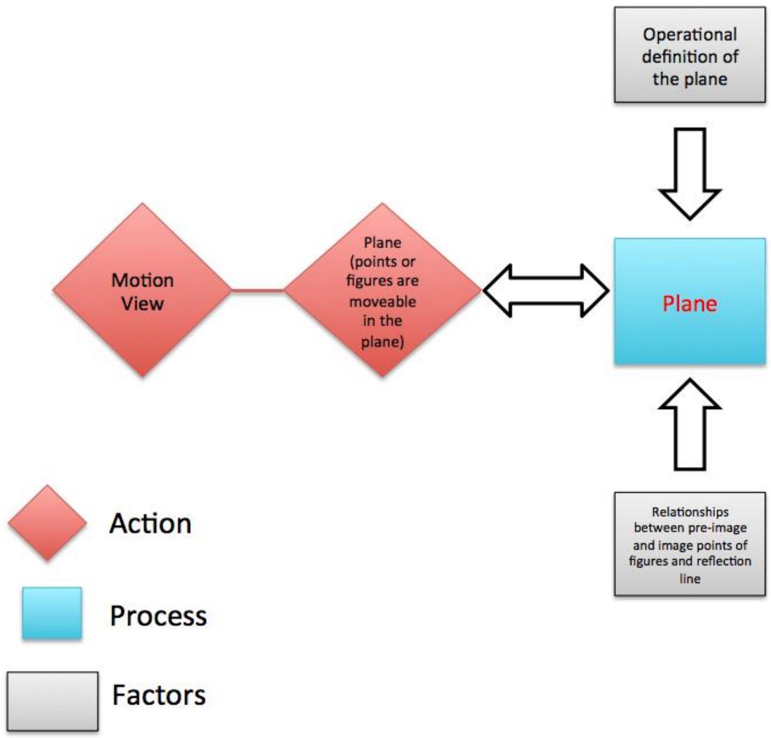
In the third interview, John was asked about the relationship between points/figures and the plane: "When you perform a reflection, is there any movement of the points or figures from one half of the plane to another half of the plane?" He explained his reasoning as follows:

> I believe that it stays on this side of the line of reflection because ... Well, it is still there now. But, it still existed. It still exists on this plane; I just reflected it over to this line. I did not pick it up and move it, but I like copied it. (Interview 3).

I interpreted from this explanation that John was now considering the points of figures to be part of the plane because when he performed a geometric reflection, there was no actual movement of points from the pre-image plane to the image plane. John's explanation demonstrated that his mathematical understanding of the relationship between the figures or points and the plane was accurate; that is, he knew that the points (and hence the figure) were embedded in the plane. Further, in practice he understood that the reflection generated a new image, although he had difficulty explaining this idea. This difficulty arose primarily because John found it easier to describe his thinking in colloquial terms that are easily relatable to how we operate with concrete objects in the real world, versus using technical mathematical language to describe the mental operation of performing reflection. Figure 12 shows John's mental structures of plane throughout the first, second, and third interviews.

Figure 12

John's Mental Structures of Plane



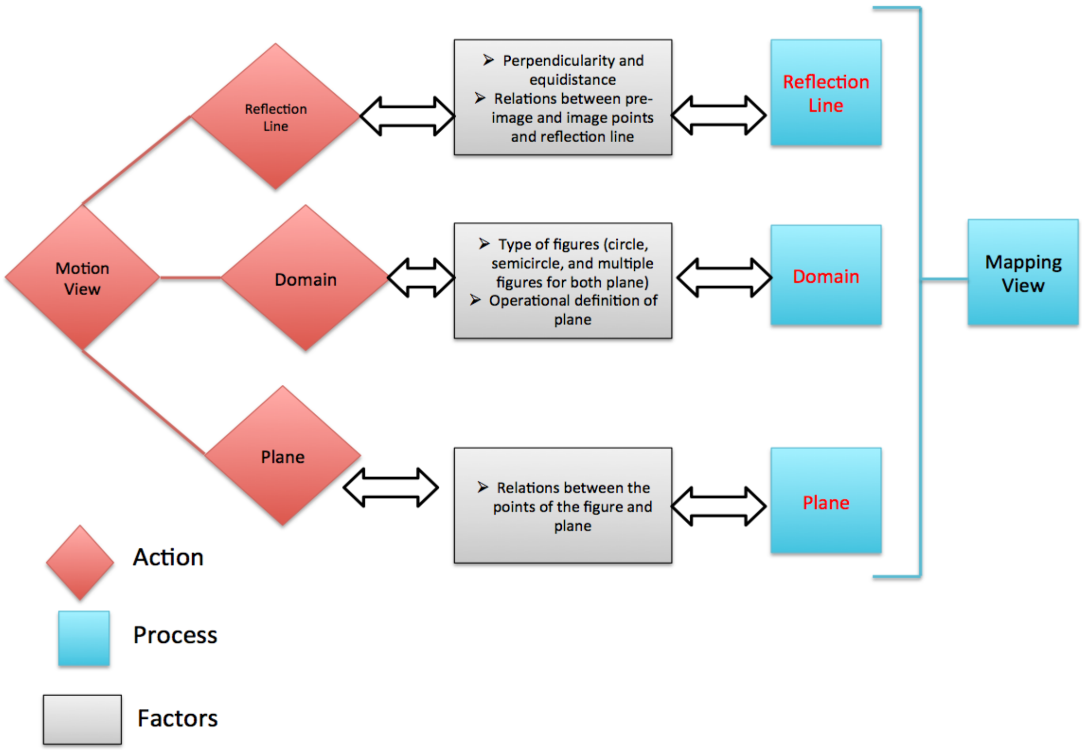
As previously mentioned in my preliminary genetic decomposition (PGD), I hypothesized that PTs begin with a motion view of plane as an action structure in order to develop a mental structure for plane as a process structure. John's model revealed that he had the process structure of plane at the end of the third interview.

Revised Genetic Decomposition

My preliminary genetic decomposition (PGD) of geometric reflection was based on the literature and inferences I drew from it. During my analysis of the data, my PGD evolved, and my final genetic decomposition (GD) is presented in Figure 13. Throughout my analysis of John's case, I looked for similarities and differences in his mental structures of three sub-concepts of geometric reflection. In the PGD, I hypothesized that to develop a mental structure for geometric reflection as a mapping view, PTs would begin with a motion view. My findings confirmed these hypotheses demonstrating that to move from a motion view of geometric reflection to a mapping view, having the process structure of reflection line, domain, and plane is sufficient.

Figure 13

PTs' Mental Structures of Plane



Briefly, in the course of the task-based interviews, John reached process structures in terms of reflection line, domain, and plane. The findings show that John knew that a reflection line is necessary to position where to place the reflected figure. He also knew that the reflection line maps every point in the plane onto itself for a geometric reflection. Additionally, the types of figures (i.e., circle, arc tasks and multiple figures for both plane) and questioning were crucial factors that helped him to think about all points in the domain rather than just a single figure. Using definitions and understanding the relations between points or figures and the plane helped him to reach a process structure of plane.

Discussion

This study aimed to describe how a motion view can develop into a mapping view and what factors can facilitate this development. The findings from the task-based interviews with a PT (John) revealed that, based on their conceptions of the three sub-concepts of reflection line, domain, and plane, John initially had a motion view of geometric reflection. After three explanatory interviews, the motion view of geometric reflection of John had evolved into a mapping view. This development was related to a variety of factors, such as the properties of equidistance and perpendicularity, the role of the reflection line, the types of figures reflected, the relation between points or figures and the plane, the operational definition of the plane, and questioning.

Understanding of the Reflection Line

Boulter and Kirby (1994), Flanagan (2001), and Yanik (2006) all found that both students and pre-service teachers (PTs) tend to have difficulty with knowing how to use a reflection line to position the points or figures in a reflection and whether or not a reflection line was necessary for performing a geometric reflection, indicating an action structure of the reflection line. Also, when they reflected points or figures, they did not use the properties of equidistance and perpendicularity. In contrast, John in the present study demonstrated understanding of the roles that the reflection line and the properties of equidistance and perpendicularity played in performing geometric reflection during the first, second and third explanatory interviews. Possible reasons for this difference from previous studies might be the educational level of the participants, the nature of the tasks, and the purpose of the initial interview in the present study.

Boulter and Kirby (1994) worked with seventh and eighth-grade students, Flanagan (2001) with tenth-grade students, and Yanik (2006) with pre-service elementary teachers (PETs). However, the researcher worked with pre-service secondary mathematics teachers (PTs), who can be expected to have better knowledge of geometric reflection than PETs and K-12 students. Another reason might be the nature of the tasks and the questions designed to provoke and well as probe thinking. For example, I asked the John to identify whether two figures constituted a reflection and justify their stance without giving them the reflection line to see whether they knew that the reflection line is necessary for performing geometric reflection. Instead of using these kinds of tasks with missing information, Boulter and Kirby (1994), Flanagan (2001) and Yanik (2006) provided reflection lines for all the tasks they used, so their participants' thinking about the role of reflection line might not have surfaced.

To summarize, John knew of the role of the reflection line and the properties of equidistance and perpendicularity in geometric reflections. My findings generated two significant commonalities as factors that facilitated their performance of geometric reflection and the progression of their understanding from an action structure to a process structure of the reflection line: the role of the reflection line and the properties of equidistance and perpendicularity. Therefore, the reflection line is shown to be a significant sub-concept for the concept of geometric reflection and essential in the move from a motion view to a mapping view.

Understanding of Domain

Hollebrands (2003) and Yanik (2006) found that students and PTs had an action structure of domain as a single figure, and they identified consideration of the domain as all points in the plane as the most challenging sub-concept to grasp in performing geometric reflection. In alignment with these two authors, I also found that considering the domain as all points in the plane was a challenging sub-concept for the John, all of whom demonstrated an action structure of domain for geometric reflection in the first interview. Although John gave a correct formal definition of a plane in the initial interview, he considered that geometric reflection was applied to the vertices or the perimeter of the figure in the first explanatory interview. One reason for this difference might be that he seemed not to use the formal definition as his operational definition when performing geometric reflection. Rather he appeared to consider the plane as an empty space and did not attend to anything except the given points or figures. In short, John's conceptual definition of a plane was inconsistent with his performance of geometric reflection.

Another reason why John performed a geometric reflection as a single figure might be the way information is presented and the content emphasized in many textbooks, with which many educators have reported dissatisfaction (Ball, 1993; Jones, 2004; Ma, 1999; Zorin, 2011). Analysis of CMP textbook showed that they support a motion view by implying that certain points or figures are reflected rather than all points in the plane. Another likelihood is that the description "a reflection line maps the figure" may result in students' understanding the reflection line as reflecting a figure as a whole rather than the points that constitute the figure. I hypothesize from the literature that the perception that a figure is being reflected as a whole supports a motion rather than a mapping view, because to have a mapping view of performing a geometric reflection, students need to consider all points in the plane rather than only given figures (Akarsu, 2018; Boulter & Kirby, 1994; Flanagan, 2001; Yanik, 2006). A close analysis of the teacher's guide for CMP textbooks found not even one explanation of performing a geometric reflection that emphasizes reflecting all points in the plane rather than a single figure. I contend that asking, "Do any points remain unchanged under the reflection" would prompt students to make the long jump to considering all points in the plane when performing a geometric reflection. Before being asked this kind of question, however, students need practice with specific tasks (e.g., circle tasks, inside and outside colored tasks, etc.) to develop a mental structure of domain and plane. Hence, I inferred that all four PTs had never learned about reflecting all points in the plane when performing reflections in their previous geometry classes.

The second interview demonstrated the importance of the type of figure provided for encouraging consideration of more points for performing geometric reflection. For example, while working on circle and semi-circle tasks, John progressed from reflecting the vertices or the perimeter of the figure to reflecting all points in the half plane. As he was thinking about the center point of the circle, he started to consider the interior points of the figure. Also, the semi-circle (open figure) task encouraged John to consider all points in the half plane because the figure's interior and exterior were not clearly defined for him. Labeling points inside and outside the semi-circle helped him consider interior and exterior points to the figure, which ultimately includes all points in the half plane, while performing geometric reflection. I inferred that John was now speculating that there could be an infinite number of points inside and outside of the semi-circle figure in the half plane on which to perform a reflection and was progressing toward conceptualizing the domain as comprising all points in the plane although he still had an action structure of domain for geometric reflection. Hence, it may be inferred that type of figure (here, half- or full-circle) is important for encouraging consideration of more points for performing geometric reflection.

By the end of the third interview, John was viewing geometric reflection as applicable to all points in the plane. Working with multiple figures in both the pre-image and image planes encouraged him to start thinking about geometric reflection as two-way, reflecting all points from the pre-image plane to the image plane and from the image plane to the pre-image plane. Working with multiple figures in both planes helped him to consider that both the image plane and the pre-image plane have an infinite number of unlabeled points and, therefore, to think about all points in the plane when performing a geometric reflection.

After all interviews, John had started to think about all points in the entire plane when performing a reflection. Although he did not physically demonstrate reflecting unlabeled points for geometric reflection, he understood that performing a reflection involves both labeled and unlabeled points, that is, all points in the plane. The tasks with circles, semi-circles, and multiple figures for both the pre-image and image planes and my questioning strategy helped John to consider all points in the plane. His operational definition of the plane in geometric reflection thus aligned with his mathematical definition of a plane, which I inferred as meaning that he had achieved a process structure for the concept of the domain in geometric reflection. Hence, considering the domain as all points in the plane was shown to be another significant sub-concept for the concept of geometric reflection and essential to move from a motion view to a mapping view.

Understanding of the Plane

Flanagan (2001) and Yanik (2006) found that students and PTs considered geometric reflection as a movement of points or figures, implying that when they performed geometric reflection, they considered the points or figures as separate from rather than as a subset of the plane, indicating an action structure of plane. In my study, in contrast, all the PTs achieved a process structure of plane. During the first and second interviews, John consistently used the verb "move" when performing geometric reflection, suggesting that he considered geometric points or figures as moveable on the plane rather than part of it. During the third interview, therefore, to ascertain his mental structures concerning the relationship between points or figures and the plane, I directly asked whether there was any movement of the points or figures from the pre-image to the image plane when he performed a reflection. In response, John indicated that he considered the points or figures as part of the plane rather than separate from it, which demonstrated that his mathematical understanding of the relationship between the figures or points and the plane was accurate and that he had a process structure of the plane.

John's example suggests that language may be a source of students' and PTs' difficulties with developing an understanding of the relationship between the points or figures and the plane as it is easier for them to describe their thinking in colloquial terms that are easily relatable to how they operate in the real world than to use technical mathematical language to describe their thought processes. Such language use can affect how they think when performing geometric reflection. Another source of students' and PTs' difficulties might be textbooks. My CMP curriculum analysis indicated that the relationships between the points or figures and plane were not explicitly emphasized in the geometric reflection unit. There is no clear explanation or example that shows all points and figures as embedded in the plane, rather than separated from the plane.

The findings of this study offer some insight into a variety of factors involved as all the participants moved from a motion view to a mapping view, such as the role of reflection line, the properties of perpendicularity and equidistance, the relationships between pre-image and image points of the figures and the reflection line, the meaning of domain, the types of figures used for solving problems, questioning strategies, the nature of the plane, the relationship between figures and plane, and the definitions of all terms.

Implications

Flanagan (2001) and Yanik (2006) found that students and pre-service elementary teachers had a motion view of geometric reflection. No researcher has explicated a mapping view of geometric transformations in general or geometric reflection specifically. Also, there has been no clear evidence documenting how a learner's motion view evolves into a mapping view. This study outlined the progression of PTs' conceptualizations from a motion view into a mapping view of geometric reflection, and the factors that facilitated this change (see Figure 13). Also, the reflection line, domain, and plane were identified as important sub-concepts and they are required in facilitating the development of PTs' motion view into a mapping view. Therefore, the results support the importance of teaching these three sub-concepts to prepare students to accurately understand geometric reflection.

The findings of this study indicate that considering the domain as all points in the plane is a challenging concept for PTs. As shown in John's case, practicing with circle, semi-circle and multiple figures encouraged them to consider more points in the plane when they were performing geometric reflection, which led to a more accurate understanding. Therefore, textbooks should provide examples of and exercises with these types of figures to help learners view the domain as comprising all points in the plane rather than as a single figure. In addition, having an understanding of the points or figures and the plane is important for developing a mapping view. In order for learners to develop a mapping view of geometric reflection, teachers should emphasize the plane and its relationship to points and figures. This study provides useful insights that can be utilized to support PTs' understanding of reflection and thus prepare them to teach this topic more effectively. There has been limited research attention to geometric reflection, and this study was the first to document how PTs move from a motion view to a mapping view.

Limitations of the Study

This study has three limitations. First, because there is lack of tasks related to domain and plane for the geometric reflection in the United States mathematics curriculum, I adapted several tasks from previous studies to emphasize these concepts of the domain and plane. Therefore, the findings were limited to the tasks that I used. Second, there were six participants for the initial interviews, and I selected four for subsequent interviews based on their willingness and ability to explain their thought processes. This criterion suggests the caveat that these four participants might be different from another sample. The third limitation is that the data consisted of verbal and nonverbal behaviors, idiosyncratic speech characteristics, gestures, and incomplete utterances, so I had to draw inferences from communications that often were not conventional, complete, and clear.

Conflicts of Interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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